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The Michael completion of a topos spread

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Abstract

We continue the investigation of the extension into the topos realm of the concepts introduced by Fox (Cahiers Top. et Géometrie Diff. Catégoriques 36 (1995) 53) and Michael (Indag. Math. 25 (1963) 629) in connection with topological singular coverings. In particular, we construct an analog of the Michael completion of a spread and compare it with the analog of the Fox completion obtained earlier by the first two named authors (J. Appl. Algebra 113 (1996) 1). Two ingredients are present in our analysis of geometric morphisms $\varphi : \mathscr{F} \to \mathscr{E}$ between toposes bounded over a base topos $\mathscr{S}$. The first is the nature of the domain of φ, which need only be assumed to be a "definable dominance" over $\mathscr{S}$, a condition that is trivially satisfied if $\mathscr{S}$ is a Boolean topos. The Heyting algebras arising from the object $\Omega_{\mathscr{S}}$ of truth values in the base topos play a special role in that they classify the definable monomorphisms in those toposes. The geometric morphisms $\mathscr{F} \to \mathscr{F}'$ over $\mathscr{E}$ which preserve these Heyting algebras (and that are not typically complete) are said to be strongly pure. The second is the nature of φ itself, which is assumed to be some kind of a spread. Applied to a spread, the (strongly pure, weakly entire) factorization obtained here gives what we call the "Michael completion" of the given spread. Whereas the Fox complete spreads over a topos $\mathscr{E}$ correspond to the $\mathscr{S}$-valued Lawvere distributions on $\mathscr{E}$ (Acta Math. 111 (1964) 14) and relate to the distribution algebras (Adv. Math.

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156 (2000) 133), the Michael complete spreads seem to correspond to some sort of "$\mathscr{S}$-additive measures" on $\mathscr{E}$ whose analysis we do not pursue here. We close the paper with several other open questions and directions for future work. © 2002 Elsevier Science B.V. All rights reserved.

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Introduction

The notion of a complete spread was introduced by Fox [10] as a common generalization of two different types of coverings with singularities (branched and folded). A different notion of a proper spread was given by Michael [22] in connection with topological cuts. In both cases, the basic idea is that of a spread, meaning a continuous map $\varphi : Y \to X$ of topological spaces, satisfying the property that the connected components (or more generally, the clopen subsets) of the $\varphi^{-1}(U)$, for U the opens of X, form a base for the topology of Y.

A *topos-theoretic* version of the notion of a spread was given in [4] as follows. A geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ between toposes bounded over a base topos $\mathscr{S}$ is said to be a spread (over $\mathscr{S}$) if there is a generating family $\alpha : F \to \varphi^*(E)$ of $\mathscr{F}$ over $\mathscr{E}$ which is a definable morphism in the sense of [1]. Notice that "complemented" is here replaced by "definable". However, over a Boolean topos $\mathscr{S}$, these two notions can be shown to agree [1].

Our original interest in (Fox complete) spreads $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ with a locally connected domain $f : \mathscr{F} \to \mathscr{S}$ arose from their correspondence, shown in [4], with the [21] $\mathscr{S}$-valued Lawvere distributions on the topos $\mathscr{E}$. The correspondence itself is given by assigning, to a pair $\langle f, \varphi \rangle$ with f locally connected, the composite $\mu = f_! \varphi^* : \mathscr{E} \to \mathscr{S}$, which is an $\mathscr{S}$-cocontinuous functor, i.e., a distribution. Its right adjoint (*comprehension*) exists and assigns, to a distribution μ on $\mathscr{E}$, a span $\langle f, \varphi \rangle$ consisting of a complete spread φ over $\mathscr{E}$ with locally connected domain f. The complete spread associated with a distribution μ is localic and its locale said to be the display locale of μ. The terminology comes from [11], where the display locale of a cosheaf is constructed.

A dual notion of distribution algebra [7] arose by considering the Heyting algebra $H = \mu_* \Omega_{\mathscr{S}}$ in $\mathscr{E}$, where $\mu_* = \varphi_* f^*$ is the right adjoint to μ, and where $\Omega_{\mathscr{S}}$ is the subobject classifier in $\mathscr{S}$.

A more general situation is that of a geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ with domain $f : \mathscr{F} \to S$ subopen in the sense of [13]. With the pair $\langle \varphi, f \rangle$ there is associated a Heyting algebra $H = \varphi_* f^* \Omega_{\mathscr{S}}$. A crucial observation made in [4] is that this H may be regarded as the sublattice of weakly complemented elements of the frame $\varphi_* \Omega_{\mathscr{F}}$, precisely in the sense that there exists an isomorphism

$$\varphi_* f^* \Omega_{\mathscr{S}} \cong \mathrm{Part}_{\Omega_{\mathscr{S}}}(\varphi_* \Omega_{\mathscr{F}})$$

compatible with the corresponding canonical subobject inclusions into the object $\varphi_* \Omega_{\mathscr{F}}$. The morphism $\varphi_* \tau : \varphi_* f^* \Omega_{\mathscr{S}} \to \varphi_* \Omega_{\mathscr{F}}$ is a monomorphism since the morphism τ:

$f^*\Omega_{\mathscr{S}} \to \Omega_{\mathscr{F}}$, defined as the characteristic morphism of $f^*\top : f^*1 \to f^*\Omega_{\mathscr{S}}$, is itself monic, since f is subopen.

We assume in Section 1 that the domain $f:\mathscr{F} \to \mathscr{S}$ is a "definable dominance", in the sense that f is subopen and the class of definable subobjects of objects in $\mathscr{F}$ is closed under composition. In this case, the Heyting algebra $H = \varphi_* f^*\Omega_{\mathscr{S}}$, although not necessarily a distribution algebra, is an $\Omega_{\mathscr{S}}$-Boolean algebra. This means first of all that H is an $\Omega_{\mathscr{S}}$-Heyting algebra, and so has an action from $e^*\Omega_{\mathscr{S}}$. In particular, the notion of an $\Omega_{\mathscr{S}}$-partition of H can be interpreted. That H is an $\Omega_{\mathscr{S}}$-Boolean algebra means that the canonical morphism $\mathrm{Part}_{\Omega_{\mathscr{S}}}(H) \to H$ which sends a partition function $\alpha : e^*\Omega_{\mathscr{S}} \to H$ to its value $\alpha(\top)$ is an isomorphism.

By an $\Omega_{\mathscr{S}}$-Stone locale in $\mathscr{E}$ (as defined in Section 2) we mean a locale of $\Omega_{\mathscr{S}}$-ideals of some $\Omega_{\mathscr{S}}$-Boolean algebra H in $\mathscr{E}$. We call weakly entire any localic morphism $\psi:\mathscr{G} \to \mathscr{E}$ over $\mathscr{S}$ defined by an $\Omega_{\mathscr{S}}$-Stone locale in $\mathscr{E}$ and strongly pure any geometric morphism $\pi:\mathscr{F} \to \mathscr{G}$ over $\mathscr{S}$ for which the unit of adjointness of $\pi^* \dashv \pi_*$ evaluated at $g^*\Omega_{\mathscr{S}}$ is an isomorphism. The terminology is purposely reminiscent of the "strongly dense" and "weakly closed" employed in [16].

We prove in Section 3 that any geometric morphism $\varphi:\mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$, whose domain $f:\mathscr{F} \to S$ is a definable dominance, factors into a strongly pure followed by a weakly entire geometric morphism (whose domain is again an definable dominance). Furthermore, if φ is a spread, then the factorization is unique. This is a relative (or fiberwise) version of the pure-entire factorization theorem given in [14].

For any (geometric) spread $\varphi:\mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ [4] whose domain $f:\mathscr{F} \to \mathscr{S}$ is a definable dominance, it is shown in Section 4 that there is an inclusion $\mathscr{F} \to \mathscr{E}^{H^{\mathrm{op}}}$ over $\mathscr{E}$, with $H = \varphi_* f^*\Omega_{\mathscr{S}}$. In Section 5 we carve out, by forcing methods [26], the largest subtopos $\mathscr{E}[H]$ of $\mathscr{E}^{H^{\mathrm{op}}}$ containing $\mathscr{F}$ (over $\mathscr{E}$) as a strongly pure subtopos. The domain of $\mathscr{E}[H] \to \mathscr{E}$ is also a definable dominance, regarded as a topos over $\mathscr{S}$. We call $\mathscr{E}[H] \to \mathscr{E}$ the "Michael completion" of the spread $\varphi:\mathscr{F} \to \mathscr{E}$, on account of the equivalence that exists, when f is a definable dominance, between the toposes $\mathrm{Sh}_{\mathscr{E}}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H))$ and $\mathscr{E}[H]$ over $\mathscr{E}$, as proven in Section 6.

For spreads with a locally connected domain we obtain, in Section 7, a comparison between the Michael completion as discussed in this paper and the (Fox) spread completion as constructed in [4]. The latter, for spreads in topos theory, coincides with Fox's spread completion [11] in topology. More generally, we compare, for a geometric morphism $\varphi:\mathscr{F} \to \mathscr{E}$ whose domain f is locally connected, and has an associated distribution μ and a distribution algebra H, the complete spread $\mathscr{E}[\mu] \to \mathscr{E}$ with the Michael complete spread $\mathscr{E}[H] \to \mathscr{E}$. In general $\mathscr{E}[\mu]$ is included in $\mathscr{E}[H]$ by means of a strongly pure inclusion which need not be an equivalence.

We end in Section 8 by listing several open questions related to this paper.

1. $\Omega_{\mathscr{S}}$-Heyting algebras

In this section we prepare the ground for a study, carried out in subsequent sections, of a geometric morphism $\varphi:\mathscr{F} \to \mathscr{E}$ between toposes defined over a base topos $\mathscr{S}$, with φ always assumed to commute with the given structure maps $f:\mathscr{F}\to\mathscr{S}$ and $e:\mathscr{E}\to\mathscr{S}$.

Given a topos $\mathscr{F}$, bounded over a base topos $\mathscr{S}$ by means of a geometric morphism $f:\mathscr{F}\to\mathscr{S}$, certain monomorphisms in $\mathscr{F}$ are said to be *definable* [1], namely, those monomorphisms $\alpha:Y\to X$ in $\mathscr{F}$ which arise by pullback from a monomorphism of the form $f^*(a):f^*(K)\to f^*(I)$, for $a:K\to I$ a monomorphism in $\mathscr{S}$, as in the diagram

$$\begin{array}{ccc} Y & \overset{\alpha}{\rightarrowtail} & X \\ \Big\downarrow & & \Big\downarrow \\ f^*K & \underset{f^*(a).}{\rightarrowtail} & f^*I \end{array}$$

Consider now a topos $\mathscr{E}$ defined over a topos $\mathscr{S}$ by means of a geometric morphism $e:\mathscr{E}\to\mathscr{S}$. In general, if we have any class of morphisms in $\mathscr{E}$ which is closed under pullback along every morphism in $\mathscr{E}$, then we may speak of a poset P in $\mathscr{E}$ that *has joins* (*meets*) *for that class* in the sense that (1) for each member $X\overset{m}{\to}Y$ of the class, composition with m, which is a morphism $P^m:P^Y\to P^X$, has a left (right) adjoint $\bigvee_m$ ($\bigwedge_m$) and (2) it is required that these left (right) adjoints satisfy the Beck–Chevalley condition (BCC) for pullback squares in $\mathscr{E}$ whose vertical arrows are in the distinguished class of morphisms. The following instance of such a notion is of particular interest to us in here.

Definition 1.1. A poset P in an $\mathscr{S}$-topos $\mathscr{E}$ with structure morphism $e:\mathscr{E}\to\mathscr{S}$ is said to have *definable joins* if it has joins for the class given by the pullback closure of the morphism

$$1\cong e^*(1)\overset{e^*(\top)}{\to}e^*(\Omega_{\mathscr{S}}).$$

Remark 1.1. 1. Since the monomorphism $0\hookrightarrow 1$ is definable in any topos $\mathscr{E}$ over $\mathscr{S}$, it follows that any poset P in $\mathscr{E}$ having definable joins also has a *least element* 0_P.

2. Notice that $e^*(\top)$ is *generic* for the class of definable monomorphisms in $\mathscr{E}$. This justifies the terminology of Definition 1.1. In [7] the definable joins were called “subterminal joins”, but this seems to conflict with other uses of the latter.

The following logical characterization should be clear (see also [7]).

Proposition 1.1. *Let $\mathscr{E}$ be a topos bounded over a topos $\mathscr{S}$ and let P be a poset in $\mathscr{E}$. Then P has definable joins if and only if*

$$\forall u\in e^*(\Omega_{\mathscr{S}})\ \forall p\in H^{\mathrm{ext}(u)}\ \exists h\in H\ (h=\sup(p))$$

is valid in $\mathscr{E}$, where $\mathrm{ext}(u)=(e^*(\top))^{-1}(u)$.

Definition 1.2. A bounded geometric morphism $f : \mathscr{F} \to S$ will be called a *definable dominance* if
1. f is subopen, and
2. the class of definable monomorphisms in $\mathscr{F}$ is closed under composition.

Remark 1.2. If $f : \mathscr{F} \to \mathscr{S}$ is subopen (i.e., the canonical morphism $\tau : f^*\Omega_{\mathscr{S}} \to \Omega_{\mathscr{F}}$ is monic) then the class of definable monomorphisms in $\mathscr{F}$ is classified by the pair $\langle f^*(\Omega_{\mathscr{S}}), f^*(\top)\rangle$. That the class of definable monos is classifiable and closed under pullback (which is automatic), and closed under composition, is a special case of the notion of a "dominance" [25]. Over a *Boolean* topos, any geometric morphism is a definable dominance. Also, any *locally connected* geometric morphism is a definable dominance, by the characterization theorem in [1].

Definition 1.3. A poset P in an $\mathscr{S}$-topos $\mathscr{E}$ with structure morphism $e : \mathscr{E} \to \mathscr{S}$ is said to be an *$\Omega_{\mathscr{S}}$-poset* if it has definable joins.

Definition 1.4. By an *$\Omega_{\mathscr{S}}$-distributive lattice* in a topos $\mathscr{E}$ defined over $\mathscr{S}$ we mean a distributive lattice P in $\mathscr{E}$ which is an $\Omega_{\mathscr{S}}$-poset such that its binary meets distribute over the definable joins. An *$\Omega_{\mathscr{S}}$-Heyting algebra* in $\mathscr{E}$ is a Heyting algebra H in $\mathscr{E}$ which is a $\Omega_{\mathscr{S}}$-poset (and so, in particular, H is an $\Omega_{\mathscr{S}}$-distributive lattice).

Remark 1.3. Any $\Omega_{\mathscr{S}}$-Heyting algebra lattice H with a top element $\mathbf{1}_H$ has an *action* from $\Omega_{\mathscr{S}}$, given by $u \cdot p = \bigvee_{x \in \mathrm{ext}(u)} p$. Writing $\|u\|$ for $u \cdot \mathbf{1}_P$, we easily see that $u \cdot p = \|u\| \wedge p$. Since H is a Heyting algebra, then H also has a *coaction* from $\Omega_{\mathscr{S}}$, given by $p^u = (\|u\| \Rightarrow p)$ [7]. The notions of action and coaction are analogous to those of *tensoring* and *cotensoring* in a different context [19].

Proposition 1.2. *Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, where the domain $f : \mathscr{F} \to \mathscr{S}$ is assumed to be definable dominance. Then $H = \varphi_* f^* \Omega_{\mathscr{S}}$ is an $\Omega_{\mathscr{S}}$-Heyting algebra in the sense of Definition* 1.4.

Proof. The required closure properties of H follow almost immediately from the assumption that f is a definable dominance. That H has definable joins is a consequence of the composability of definable monomorphisms in $\mathscr{F}$. □

Definition 1.5. An *$\Omega_{\mathscr{S}}$-ideal* of an $\Omega_{\mathscr{S}}$-Heyting algebra H is a subobject of H whose classifying map $\chi : H \to \Omega_{\mathscr{E}}$ is
1. order-reversing, in the sense that it satisfies

$$(p \leqslant q) \;\Rightarrow\; (\chi(q) \Rightarrow \chi(p)),$$

and

2. is such that the diagram

$$\begin{array}{ccc} H^X & \xrightarrow{\bigvee_\alpha} & H^Y \\ {\scriptstyle \chi^X}\downarrow & & \downarrow{\scriptstyle \chi^Y} \\ \Omega_{\mathscr{E}}^X & \xrightarrow[\bigwedge_\alpha]{} & \Omega_{\mathscr{E}}^Y \end{array}$$

commutes for any definable mono $\alpha : X \to Y$ in $\mathscr{E}$.

The object in $\mathscr{E}$ of all $\Omega_{\mathscr{S}}$-ideals of an $\Omega_{\mathscr{S}}$-Heyting algebra H will be denoted by $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$.

Remark 1.4. When condition (2) of Definition 1.5 is instantiated by $\alpha = e^*(\top)$, transposed, and applied to $p \in H$ and $u \in e^*(\Omega_{\mathscr{S}})$ (the latter regarded as the $\mathscr{S}$-definable mono $u : e^*(1) \to e^*(\Omega_{\mathscr{S}})$), then it reads

$$\chi(u \cdot p) = (\tau(u) \Rightarrow \chi(p)),$$

where $e^*(\Omega_{\mathscr{S}}) \xrightarrow{\tau} \Omega_{\mathscr{E}}$ is the canonical morphism in $\mathscr{E}$. It follows that, for 0_H the initial object of H,

$$\chi(0_H) = \chi(\bot . 0_H) = (\tau(\bot) \Rightarrow \chi(0_H)) = (\bot \Rightarrow \chi(0_H)) = \top,$$

so that an $\Omega_{\mathscr{S}}$-ideal always contains 0_H.

Proposition 1.3. *If P is an $\Omega_{\mathscr{S}}$-poset in $\mathscr{E}$, then $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is an internally cocomplete poset in $\mathscr{E}$. Moreover, the map $\downarrow : P \to \mathrm{Idl}_{\Omega_{\mathscr{S}}(P)}$ sending $x \in P$ to $\downarrow(x) = \{y \in P \mid y \leqslant x\}$ is universal among all $\Omega_{\mathscr{S}}$-cocontinuous poset morphisms to internally cocomplete posets.*

Proof. First observe that $\downarrow(x)$ is an $\Omega_{\mathscr{S}}$-ideal and that $\downarrow(-)$ preserves definable joins. In particular, for $u \in e^*(\Omega_{\mathscr{S}})$ and $x \in P$, $u \cdot \downarrow(x) = \downarrow(\|u\| \wedge x)$. Next observe that $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is a full reflective subposet of $\downarrow(P)$, the frame of downclosed subobjects of P, since $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is closed in $\downarrow(P)$ under (internal) intersections. Therefore, $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is internally cocomplete. For any $\Omega_{\mathscr{S}}$-ideal $\mathscr{I}$, we have $\mathscr{I} = \bigvee_{p \in \mathscr{I}} \downarrow(p)$, so that any internally cocontinuous map from $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is uniquely determined by its composition with $\downarrow(-)$. In particular, the internally cocontinuous extension to $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ of an $\Omega_{\mathscr{S}}$-cocontinuous poset morphism $P \xrightarrow{h} L$ must be given by $\mathscr{I} \mapsto \bigvee_{p \in \mathscr{I} h(p)}$. Moreover, this map is indeed cocontinuous since it has a right adjoint given by $x \mapsto \{p \mid h(p) \leqslant x\}$, which is clearly an $\Omega_{\mathscr{S}}$-ideal. □

Proposition 1.4. *For an $\Omega_{\mathscr{S}}$-distributive lattice P, the poset $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is a frame. Moreover, in this case, if $P \xrightarrow{h} A$ is an $\Omega_{\mathscr{S}}$-cocontinuous lattice homomorphism into* *a frame A, then the induced internally cocontinuous morphism* $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P) \to A$ *is a frame homomorphism.*

Proof. Suppose that P is an $\Omega_{\mathscr{S}}$-distributive lattice. Since $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ is closed under arbitrary meets in $\downarrow(P)$, to show that it is a frame it suffices to prove that it is closed also under exponentiation, i.e., that $[\mathscr{D} \Rightarrow \mathscr{I}] \in \mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ for any $\mathscr{D} \in \downarrow(P)$ and $\mathscr{I} \in \mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$. This in turn reduces to the case $\mathscr{D} = \downarrow(p)$ for $p \in P$, since for arbitrary $\mathscr{D}$ we have

$$[\mathscr{D} \Rightarrow \mathscr{I}] = \left[\left(\bigcup_{p\in\mathscr{D}} \downarrow(p)\right) \Rightarrow \mathscr{I}\right] = \bigcap_{p\in\mathscr{D}} [\downarrow(p) \Rightarrow \mathscr{I}].$$

We have $x \in [\downarrow(p) \Rightarrow \mathscr{I}]$ if and only if $\downarrow(x) \subseteq [\downarrow(p) \Rightarrow \mathscr{I}]$ if and only if $\downarrow(x) \cap \downarrow(p) \subseteq \mathscr{I}$ if and only if $x \wedge p \in \mathscr{I}$. It follows that $[\downarrow(p) \Rightarrow \mathscr{I}]$ is indeed an $\Omega_{\mathscr{S}}$-ideal. Moreover, $\downarrow(-)$ obviously preserves binary meets, so the extension of $P \xrightarrow{h} A$ to $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(P)$ preserves them because h does. □

2. $\Omega_{\mathscr{S}}$-Stone locales

We shall now turn to the consideration of a suitable notion of *relative Booleanness* in any topos $\mathscr{E}$ defined over a base topos $\mathscr{S}$. The notion of complemented element implicit in it is not the usual, in that it is the elements of $e^*(\Omega_{\mathscr{S}})$ (rather than those of $\mathbf{2}_{\mathscr{E}}$) that are taken as truth values. This is essentially the approach of [20] in defining the notion of a relative Boolean frame. We need to recall the notion of an $\Omega_{\mathscr{S}}$-partition of H (or of a flat function) from [4].

Definition 2.1. Let $e : \mathscr{E} \to \mathscr{S}$ be a geometric morphism and let H be an $\Omega_{\mathscr{S}}$-cocomplete Heyting algebra in $\mathscr{E}$. A *partition* $\alpha : e^*(\Omega_{\mathscr{S}}) \to H$ is a morphism in $\mathscr{E}$ that satisfies the following two conditions:

1.
$$\bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \alpha(u) = 1_H,$$

2. meaning that the join exists and the equation holds, and
$$\forall u, u' \in e^*(\Omega_{\mathscr{S}}) \quad (\alpha(u) \wedge \alpha(u') \leqslant \|u = u'\|),$$
where $(u = u') \in e^*(\Omega_{\mathscr{S}} \times \Omega_{\mathscr{S}})$ stands for $e^*(\mathrm{eq}_{\Omega_{\mathscr{S}}})(u, u')$, and where $\mathrm{eq}_{\Omega_{\mathscr{S}}} : (\Omega_{\mathscr{S}} \times \Omega_{\mathscr{S}}) \to \Omega_{\mathscr{S}}$ is the characteristic morphism of the diagonal $\delta : \Omega_{\mathscr{S}} \hookrightarrow \Omega_{\mathscr{S}} \times \Omega_{\mathscr{S}}$.

Definition 2.2. An $\Omega_{\mathscr{S}}$-Heyting algebra H in $\mathscr{E}$ is said to be an *$\Omega_{\mathscr{S}}$-Boolean algebra* in $\mathscr{E}$ if the canonical morphism

$$\sigma_H : \mathrm{Part}_{\Omega_{\mathscr{S}}}(H) \to H$$

defined by $\sigma_H(\alpha) = \alpha(\top)$ is an isomorphism.

Recall that a locale A in a topos $\mathscr{E}$ is said to be a Stone locale [14] if it is a compact and zero-dimensional locale in $\mathscr{E}$. It is shown in [15] that A is a Stone locale if and only if it is equivalent to one of the form $\mathrm{Idl}(B)$ for B a Boolean algebra. This equivalence holds because in this (the classical) case, B can be recovered as the sublattice $(\mathrm{Idl}(B))^{\mathrm{c}}$ consisting of the complemented elements (in the ordinary sense) of the locale $\mathrm{Idl}(B)$ of (ordinary) ideals of B [14]. In what follows we shall argue that in the constructive setting that arises from working relatively to an arbitrary base topos $\mathscr{S}$ instead of *Set*, this is still true.

Lemma 2.1. *Let A denote a frame in a topos $\mathscr{E}$ over $\mathscr{S}$. If two flat functions $\alpha, \beta : e^*(I) \to A$ satisfy $\alpha \leqslant \beta$* (ordered pointwise), *then $\alpha = \beta$.*

Proof. Denote by $\mathscr{F}$ the topos of sheaves on A, and by $\varphi : \mathscr{F} \to \mathscr{E}$ the corresponding localic geometric morphism. For any objects E of $\mathscr{E}$ and I in $\mathscr{S}$, there is a bijection between morphisms $\alpha : E \to \mathrm{Part}_{\mathrm{I}}(A)$ in $\mathscr{E}$ and morphisms $\alpha : E \to \varphi_* f^*(I)$ in $\mathscr{E}$, as follows from Lemma 1.12 of [4]. In turn, by adjointness, the latter morphisms are in bijection with morphisms $U : \varphi^*(E) \to f^*(I)$ in $\mathscr{F}$, i.e., with I-indexed families U of subobjects of $\varphi^*(E)$ which form a partition of $\varphi^*(E)$. Let U and V be as above, corresponding to the flat functions α and β, respectively. The assumption $\alpha \leqslant \beta$ readily implies that $U = V$, since there are no non-trivial 2-cells in $\mathscr{F}(\varphi^*(E), f^*(I))$. It follows that $\alpha = \beta$. □

Lemma 2.2. *Let $\mathscr{I} = \{\mathscr{I}_u\}$ be an $e^*(\Omega_{\mathscr{S}})$-indexed family of $\Omega_{\mathscr{S}}$-ideals of an $\Omega_{\mathscr{S}}$-Heyting algebra H in $\mathscr{E}$, and let $q \in \mathscr{I}_u$ for a given $u \in e^*(\Omega_{\mathscr{S}})$. Then for all $v \in e^*(\Omega_{\mathscr{S}})$, $(\|u = v\| \wedge q) \in \mathscr{I}_v$.*

Proof. Assume that $q \in \mathscr{I}_u$ for a given $u \in e^*(\Omega_{\mathscr{S}})$. If χ_u denotes the characteristic morphism of $\mathscr{I}_u$, this says precisely that $\chi_u(q)$ holds. Let χ_v denote the characteristic morphism of the $\Omega_{\mathscr{S}}$-ideal $\mathscr{I}_v$, for $v \in e^*(\Omega_{\mathscr{S}})$. Since $\mathscr{I}_v$ is an $\Omega_{\mathscr{S}}$-ideal, we have, for all $w \in e^*(\Omega_{\mathscr{S}})$

$$\chi_v(w \cdot q) = (\tau(w) \Rightarrow \chi_v(q)).$$

Putting $w = (u = v)$ above and using that $(u = v) \cdot q = \|u = v\| \wedge q$ and that $\tau(u = v) \Leftrightarrow (u = v)$, we get that

$$\chi_v(\|u = v\| \wedge q) = ((u = v) \Rightarrow \chi_v(q)).$$

But, as

$$((u=v)\Rightarrow\chi_v(q))\Leftrightarrow((u=v)\Rightarrow\chi_u(q))$$

and $\chi_u(q)$ holds by assumption, we conclude that $\chi(\|u=v\|\wedge q)$ holds, as claimed. □

Proposition 2.1. *Let H be an $\Omega_{\mathscr{S}}$-Heyting algebra in $\mathscr{E}$, with a top element 1_H. Then, the morphism*

$$\vartheta_H:\ \mathrm{Part}_{\Omega_{\mathscr{S}}}(H)\to\mathrm{Part}_{\Omega_{\mathscr{S}}}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H))$$

induced by $\downarrow\,:H\to\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ is an isomorphism in $\mathscr{E}$.

Proof. That the morphism ϑ is monic follows easily from $\downarrow$ being monic. To check surjectivity, we let $X\times e^*\Omega_{\mathscr{S}}\xrightarrow{\beta}\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ be the transpose of a partition function defined at X, and show that there is a partition function on H defined also at X whose transpose $X\times e^*(\Omega_{\mathscr{S}})\xrightarrow{\alpha}P$ is such that every $\Omega_{\mathscr{S}}$-ideal $\mathscr{I}_{\{u,x\}}=\beta(x)(u)$ is principal and generated by $\alpha(x)(u)$. First we establish the following. □

Claim. An Ω_S-join of $\Omega_{\mathscr{S}}$-ideals $\mathscr{I}_u$ in an $\Omega_{\mathscr{S}}$-Heyting algebra H is given by

$$\bigvee_{u\in e^*(\Omega_{\mathscr{S}})}\mathscr{I}_u=\left\{\left(\bigvee_{u\in e^*(\Omega_{\mathscr{S}})}p_u\right)\mid p\in\prod\mathscr{I}_u\right\},$$

where the joins $\bigvee_{u\in e^*(\Omega_{\mathscr{S}})}p_u$ exist (and are distributed upon by binary meets since H is a Heyting algebra). The object on the right-hand side of the above claimed equation is already down-closed. Also, since each I_u is an $\Omega_{\mathscr{S}}$-ideal and since colimits in a product are defined pointwise, it is clear that the right-hand side is an $\Omega_{\mathscr{S}}$-ideal of P. It remains to prove that each $\mathscr{I}_u$ is contained in the object on the right-hand side of the claimed equation above. Once we have this, the claim will have been shown because any $\Omega_{\mathscr{S}}$-ideal containing every $\mathscr{I}_u$ must also contain the right-hand side, so it would indeed be the smallest.

We now prove that

$$\mathscr{I}_u\subseteq\left\{\left(\bigvee_{u\in e^*(\Omega_{\mathscr{S}})}p_u\right)\mid p\in\prod\mathscr{I}_u\right\}.$$

Let $q\in\mathscr{I}_u$. We first have that

$$q=\bigvee_{v\in e^*(\Omega_{\mathscr{S}})}(\|u=v\|\wedge q).$$

Define $p(q)$ by $p(q)_v=(\|u=v\|\wedge q)$. It follows from Lemma 2.2 that $p(q)_v\in\mathscr{I}_v$. This proves the claim.

Let us now continue with the proof of the proposition. For $e^*(\Omega_{\mathscr{S}}) \to^{\beta} \mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ a partition, we have that $\bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \beta(u) = \bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \mathscr{I}_u$ exists and

$$\mathbf{1}_{\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)} = \bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \beta(u) = \bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \mathscr{I}_u,$$

where $\mathscr{I}_u = \beta(u)$, and therefore (by the claim established above) that

$$H = \left\{ \left(\bigvee_{u\in e^*(\Omega_{\mathscr{S}})} p_u \right) \mid p \in \prod \mathscr{I}_u \right\}.$$

In particular,

$$\mathbf{1}_H = \bigvee_{u\in e^*(\Omega_{\mathscr{S}})} \{r_u \mid r_u \in \beta(u)\}$$

for some family $\{r_u \mid r_u \in \beta(u)\}$. The function $\alpha(u) = \downarrow r_u$ is flat, and satisfies $\alpha \leqslant \beta$. By Lemma 2.1, we have $\alpha = \beta$, so that β factors uniquely through H. □

Definition 2.3. By an $\Omega_{\mathscr{S}}$-*Stone locale* in $\mathscr{E}$ we mean a locale in $\mathscr{E}$ which is equivalent to one of the form $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ for H an $\Omega_{\mathscr{S}}$-Boolean algebra in $\mathscr{E}$.

Let us recall some of the details concerning definable subobjects and their corresponding partitions, of use in what follows. Suppose that a definable subobject $D \hookrightarrow Y$ in a topos $f : \mathscr{F} \to \mathscr{S}$ is given by the following pullback:

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ f^*(K) & \longrightarrow & f^*(I), \end{array}$$

where $K \hookrightarrow I$ is some monomorphism in $\mathscr{S}$.

By Proposition 1.12 of [4], D may equivalently be regarded as a partition $\mathbf{D} : \Omega_{\mathscr{S}} \to f_*(\Omega_{\mathscr{F}}^Y)$, with $\mathbf{D}(\top) = D$. Indeed, in $\mathscr{F}$,

$$\mathrm{Sub}_{\mathrm{def}}(Y) \cong f_* f^* \Omega_{\mathscr{S}}^Y \cong \mathrm{Part}_{\Omega_{\mathscr{S}}}(f_*(\Omega_{\mathscr{F}}^Y)).$$

A similar analysis can be made for $K \hookrightarrow I$ in $\mathscr{S}$. In this case, where $\mathscr{S}$ is regarded as an $\mathscr{S}$-topos via the identity geometric morphism, we get a partition $\mathbf{K} : \Omega_{\mathscr{S}} \to \Omega_{\mathscr{S}}^I$ with $\mathbf{K}(\top) = K$.

From Lemma 1.13 of [4] follow that for all $u \in \Omega_{\mathscr{S}}$, $\mathbf{K}(u) = (K \Leftrightarrow u)$ and $\mathbf{D}(u) = (D \Leftrightarrow u)$, where the notation u is for the element $u \in \Omega_{\mathscr{S}}$ regarded in the respective frame in $\mathscr{S}$ via the unique frame morphism from the initial frame in $\mathscr{S}$. We use these remarks in the following result.

Lemma 2.3. *Let $D \hookrightarrow Y$ be a definable subobject of an object Y of a topos $\mathscr{F}$ over $\mathscr{S}$, where $f:\mathscr{F} \to \mathscr{S}$ is assumed subopen. Then, the corresponding partition $\mathbf{D}:\Omega_{\mathscr{S}} \to f_*(\Omega_{\mathscr{F}}^Y)$ factors uniquely as a partition through $f_*(\tau^Y):f_*(f^*(\Omega_{\mathscr{S}})^Y) \to f_*(\Omega_{\mathscr{F}}^Y)$.*

Proof. Using that $\mathbf{K}(u) = (K \Leftrightarrow u)$, $\mathbf{D}(u) = (D \Leftrightarrow u)$, the fact that f^* preserves $\Rightarrow$ since f is subopen [13], and the fact that pullbacks in a topos also preserve $\Rightarrow$, we get immediately that, for any $u \in \Omega_{\mathscr{S}}$, the diagram

$$\begin{array}{ccc} \mathbf{D}(u) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f^*(\mathbf{K}(u)) & \longrightarrow & f^*(I) \end{array}$$

is a pullback. In particular, $\mathbf{D}(u) \hookrightarrow Y$ is again a definable subobject, so that the partition $\mathbf{D}$ has a unique factorization as a partition $\alpha:\Omega_{\mathscr{S}} \to f_*(f^*(\Omega_{\mathscr{S}})^Y)$ followed by the monomorphism (again using that f is subopen) $f_*(\tau^Y):f_*(f^*(\Omega_{\mathscr{S}})^Y) \to f_*(\Omega_{\mathscr{F}}^Y)$. □

Proposition 2.2. *Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, where the domain $f:\mathscr{F} \to \mathscr{S}$ is assumed to be a definable dominance in the sense of Definition 1.2. Then $H = \varphi_* f^* \Omega_{\mathscr{S}}$ is an $\Omega_{\mathscr{S}}$-Boolean algebra in the sense of Definition 2.2.*

Proof. By Proposition 1.2, we already know that H is an $\Omega_{\mathscr{S}}$-Heyting algebra. Furthermore, from [4], we know that

$$H^E = (\varphi_* f^* \Omega_{\mathscr{S}})^E \cong \mathrm{Part}_{\Omega_{\mathscr{S}}}((\varphi_* \Omega_{\mathscr{F}})^E).$$

Now, Lemma 2.3 can be adapted to the case where the definable dominance $f:\mathscr{F} \to \mathscr{S}$ is given as in the following triangle:

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\varphi} & \mathscr{E} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle e} & \\ & \mathscr{S} & \end{array}$$

as needed here. Indeed, let $D \hookrightarrow \varphi^*(E)$ be a definable subobject in $\mathscr{F}$, with E an object of $\mathscr{E}$. We claim that the corresponding partition $\mathbf{D}:e^*\Omega_{\mathscr{S}} \to \varphi_* \Omega_{\mathscr{F}}^E$ factors uniquely (modulo canonical isomorphisms) as a partition through $\varphi_*(\tau)^E:\varphi_*(f^*\Omega_{\mathscr{S}})^E \to \varphi_*(\Omega_{\mathscr{F}})^E$. That this is the case is a consequence of Lemma 2.3, together with the easily verified fact that the transpose $\Omega_{\mathscr{S}} \to f_*\Omega_{\mathscr{F}} \cong e_*\varphi_*\Omega_{\mathscr{F}}$ of a partition $e^*\Omega_{\mathscr{S}} \to \varphi_*\Omega_{\mathscr{F}}$ in $\mathscr{E}$ is itself a partition in $\mathscr{S}$. The claim now establishes the $\Omega_{\mathscr{S}}$-Booleanness of H. □

Corollary 2.1. *Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, where the domain $f : \mathscr{F} \to S$ is assumed to be a definable dominance. Let A be the locale* $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ *in $\mathscr{E}$, where $H = \varphi_* f^* \Omega_{\mathscr{S}}$. Then A is an $\Omega_{\mathscr{S}}$-Stone locale in the sense of Definition* 2.3.

3. The weakly entire factorization

Let $\varphi : \mathscr{F} \to \mathscr{E}$ be a geometric morphism between bounded toposes over an *arbitrary* base topos $\mathscr{S}$ and assume that the domain $f : \mathscr{F} \to \mathscr{S}$ of φ is a definable dominance in the sense of Definition 1.2. For the definitions below, consider a commutative diagram

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\varphi} & \mathscr{E} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle e} & \\ & \mathscr{S} & \end{array}.$$

Definition 3.1. We shall say that a geometric morphism φ as in the commutative diagram above is *weakly entire* if it is localic and defined by an $\Omega_{\mathscr{S}}$-Stone locale, i.e., by a locale of the form $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$, for H an $\Omega_{\mathscr{S}}$-Boolean algebra in $\mathscr{E}$ in the sense of Definition 2.2.

Definition 3.2. We shall say that a geometric morphism φ as in the commutative diagram above is *strongly pure* if

$$\eta_{e^*(\Omega_{\mathscr{S}})} : e^*(\Omega_{\mathscr{S}}) \to \varphi_* \varphi^* e^*(\Omega_{\mathscr{S}}),$$

which denotes the unit of adjointness $\varphi^* \dashv \varphi_*$ evaluated at $e^*(\Omega_{\mathscr{S}})$, is an isomorphism.

Remark 3.1. In [4], the expression "$\mathscr{S}$-pure and $\mathscr{S}$-dense" was used instead of "strongly pure".

Theorem 3.1. *A geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ whose domain $f : \mathscr{F} \to \mathscr{S}$ is a definable dominance admits a* (*strongly pure*, *weakly entire*) *factorization. Moreover*, *the middle topos in the factorization is in the same class*, *that is*, *its domain is a definable dominance*.

Proof. Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, where $f : \mathscr{F} \to \mathscr{S}$ is a definable dominance. Let H denote $\varphi_* f^* \Omega_{\mathscr{S}}$ and let $\mathscr{G}$ the $\mathscr{E}$-topos of sheaves on $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$, with $\mathscr{G} \xrightarrow{\psi} \mathscr{E}$ the canonical geometric structure morphism. Since H is an $\Omega_{\mathscr{S}}$-Heyting algebra by Proposition 1.2, the latter exists. Moreover, by Proposition 2.2, H is an $\Omega_{\mathscr{S}}$-Boolean algebra. In particular, ψ is weakly entire.

By the universal property of the free frame on the $\Omega_{\mathscr{S}}$-distributive lattice H, $\varphi_*(\tau)$: $\varphi_* f^* \Omega_{\mathscr{S}} \to \varphi_* \Omega_{\mathscr{F}}$ induces a morphism

$$\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H) \to \varphi_*(\Omega_{\mathscr{F}})$$

of frames. In turn, the latter induces a geometric morphism $\mathscr{F} \xrightarrow{\pi} \mathscr{G}$ such that $\varphi \cong \psi \cdot \pi$. To see that π is strongly pure, we show (analogously as in [14]) that the unit of the adjunction $\pi^* \dashv \pi_*$ evaluated at $g^*(\Omega_{\mathscr{S}})$ is an isomorphism for *global sections* between objects regarded as sheaves in $\mathscr{E}$ for the $\Omega_{\mathscr{S}}$-cover topology. This follows from the isomorphisms:

$$\psi_* g^* \Omega_{\mathscr{S}} \cong \mathrm{Part}_{\Omega_{\mathscr{S}}}(\psi_* \Omega_{\mathscr{F}}) \cong \mathrm{Part}_{\Omega_{\mathscr{S}}}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H))$$
$$\cong \mathrm{Part}_{\Omega_{\mathscr{S}}}(H) \cong H = \varphi_* f^* \Omega_{\mathscr{S}} \cong \psi_* \pi_* \pi^* g^* \Omega_{\mathscr{S}},$$

where the first iso is from [4], the second by construction of ψ, the third by Proposition 2.1, the fourth by the $\Omega_{\mathscr{S}}$-Booleanness of H and the last by the natural isomorphisms $\varphi \cong \psi \cdot \pi$ and $f \cong g \cdot \pi$. For the general case we may localize and repeat this argument. Thus, π is strongly pure. In particular, it follows easily that the domain of ψ is also a definable dominance. □

We remark that an argument as the one employed in [14] in order to establish uniqueness is not available here since we do not know whether the weakly entire spreads are stable under arbitrary bipullback. However, the developments of the next section will give uniqueness for the factorization, if the latter is applied to a *spread* since in that case, the first part of the factorization is a (strongly pure) *inclusion*.

4. The Heyting algebra of a spread

We show next that spreads $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ with domain $f : \mathscr{F} \to \mathscr{S}$ a definable dominance can be characterized in terms of the Heyting algebra H associated with the span $\langle f, \varphi \rangle$. In what follows:

$$\gamma : P(H) =_{\mathrm{def}} \mathscr{E}^{H^{\mathrm{op}}} \to \mathscr{E}; \quad Y : H \to P(H)$$

will denote the topos of presheaves associated with a poset H in $\mathscr{E}$, with its Yoneda functor Y. A description of $P(H) \xrightarrow{\gamma} \mathscr{E}$ as a continuous fibration [24] can be given as follows. Let $\mathbf{C}$ denote the underlying small category of a subcanonical site for $\mathscr{E}$ over $\mathscr{S}$. Denote by $\mathbf{H}$ the following category. The objects of $\mathbf{H}$ are pairs (c,x), where c is an object of $\mathbf{C}$ and $c \xrightarrow{x} H$ is a morphism in $\mathscr{E}$. A morphism $(c,x) \xrightarrow{f} (d,y)$ in $\mathbf{H}$ is a morphism $c \xrightarrow{f} d$ in $\mathbf{C}$ such that $x \leqslant y \cdot f$. We define the following functors in $\mathscr{S}$:

$$F : \mathbf{H} \to \mathbf{C}; \quad T : \mathbf{C} \to \mathbf{H}.$$

The functors F and T satisfy $F(c,x) = c$ and $T(c) = (c, \top)$. T is right adjoint to F, and for every (c,x), the slice functor $F_{(c,x)}$ has a right adjoint $T_{(c,x)}$.

We now introduce a Grothendieck topology in $\mathbf{H}$. By definition, a family of morphisms $\{(c_i,x_i) \xrightarrow{f_i} (c,x)\}$ is a *covering family* if $\{c_i \xrightarrow{f_i} c\}$ is a cover, and if for every

i, we have $x \cdot f_i = x_i$. These covers generate a topology in $\mathbf{H}$ such that T is cover preserving (and flat).

Proposition 4.1. *The topos of sheaves* $\mathbf{H}^{\mathrm{op}} \to \mathscr{S}$ *for the coverage described above gives the topos* $P(H)$. *The continuous fibration* $F \dashv T$ *induces the canonical geometric morphism* $P(H) \xrightarrow{\gamma} \mathscr{E}$. *We have* $\gamma^*(E) = E \cdot F$. *If* $X \in P(H)$, *then* $\gamma_*(X) = X \cdot T$.

Consider again a geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ with domain $f : \mathscr{F} \to \mathscr{S}$ and let $H = \varphi_* \varphi^* e^*(\Omega_{\mathscr{S}})$ be its associated Heyting algebra. The adjointness $\varphi^* \dashv \varphi_*$ provides a canonical passage that takes an object (c, x) of $\mathbf{H}$ to an $\mathscr{S}$-definable subobject of $\varphi^*(c)$. Indeed, we simply form in $\mathscr{F}$ the following pullbacks:

$$\begin{array}{ccccc} X & \longrightarrow & S & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle f^*(\top)} \\ \varphi^*(c) & \xrightarrow[\varphi^* x]{} & \varphi^*(H) & \xrightarrow[\text{counit}]{} & f^*(\Omega_{\mathscr{S}}). \end{array}$$

This defines a flat functor

$$Q : \mathbf{H} \to \mathscr{F}$$

such that $Q(c, x) = X$, where X is the definable subobject of $\varphi^*(c)$ described above. The functor Q is $P(H)$-cover preserving because the condition $x \cdot f_i = x_i$ means that the following is a pullback:

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \varphi^*(c_i) & \xrightarrow[\varphi^* f_i]{} & \varphi^*(c). \end{array}$$

Since $\{c_i \xrightarrow{f_i} c\}$ is a cover (in $\mathbf{C}$) it follows that $\{X_i \to X\}$ is a cover in $\mathscr{F}$. So there is induced a geometric morphism

$$q : \mathscr{F} \to P(H)$$

such that $\gamma \cdot q \cong \varphi$.

Proposition 4.2. *A geometric morphism* $\varphi : \mathscr{F} \to \mathscr{E}$ *over* $\mathscr{S}$ *is a spread over* $\mathscr{S}$ *if and only if the induced geometric morphism* $q : \mathscr{F} \to P(H)$ *defined as above is an inclusion.*

Proof. The generic definable subobject $S \hookrightarrow \varphi^* e^* \Omega_{\mathscr{S}}$ is precisely the family that appears in the equivalent condition for a spread given in [4] Proposition 1.3.4. It is generating if and only if q is an inclusion. □

Combining the above with the results of the previous section, this suggests that we think of spreads as the *weakly zero-dimensional* geometric morphisms. This point of view is adequate in the terminology used in the theory of Stone locales [15,14], to judge from the following theorem.

It is shown in [4] that a geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ is a spread if and only if it is *localic* and furthermore the morphism

$$\varphi_*\tau : \varphi_*(f^*\Omega_{\mathscr{S}}) \to \varphi_*\Omega_{\mathscr{F}}$$

join-generates the frame $\varphi_*\Omega_{\mathscr{F}}$.

Lemma 4.1. *If $p : \mathscr{F} \to \mathscr{G}$ is a geometric morphism over $\mathscr{E}$ and is strongly pure in the sense of Definition* 3.2, *then there is a canonical isomorphism between the Heyting algebras associated*, *respectively*, *with the spans $\langle \varphi, f\rangle$ and $\langle \psi, g\rangle$.*

Proof. The isomorphism is given by

$$\psi_*(\eta_{g^*\Omega_{\mathscr{S}}}) : \psi_* g^*\Omega_{\mathscr{S}} \to \psi_* p_* p^* g^*\Omega_{\mathscr{S}} \cong \varphi_* f^*\Omega_{\mathscr{S}}. \qquad \square$$

Theorem 4.1. *Let A be a locale in $\mathscr{E}$ over $\mathscr{S}$ such that the canonical geometric morphism $\mathrm{Sh}_{\mathscr{E}}(A) \to \mathscr{S}$ is a definable dominance. Then the following are equivalent*:

(1) $\mathrm{Sh}_{\mathscr{E}}(A) \to \mathscr{E}$ *is a spread*, *and*
(2) *A is a sublocale of an $\Omega_{\mathscr{S}}$-Stone locale in $\mathscr{E}$.*

Proof. (2) $\Rightarrow$ (1). By Definition 2.3 of an $\Omega_{\mathscr{S}}$-Stone locale, weakly entire morphisms are spreads. The result now follows from the observations [4] that inclusions are spreads and that the composite of two spreads is a spread.

(1) $\Rightarrow$ (2). This follows directly from the construction of the strongly pure, weakly entire factorization Theorem 3.1. Indeed we have

$$\mathrm{Sh}_{\mathscr{E}}(A) \to \mathrm{Sh}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)) \to \mathscr{E}^{H^{\mathrm{op}}},$$

where $H = \varphi_* f^*\Omega_{\mathscr{S}}$ for φ: $\mathrm{Sh}_{\mathscr{E}}(A) \to \mathscr{E}$. If φ is a spread, then the above composite is an inclusion, hence so is $\mathrm{Sh}_{\mathscr{E}}(A) \to \mathrm{Sh}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H))$. $\square$

Corollary 4.1. *Let $\varphi : \mathscr{F} \to \mathscr{E}$ be a spread whose domain $f : \mathscr{F} \to \mathscr{S}$ is a definable dominance. Then the* (*strongly pure*, *weakly entire*) *factorization of φ which exists by Theorem* 3.1, *is unique*.

Proof. Let φ be a spread with domain $f : \mathscr{F} \to \mathscr{E}$ a definable dominance. By Theorem 3.1, φ can be factored as in the triangle

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\pi} & \mathscr{G} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathscr{E} & \end{array}$$

with π strongly pure, ψ weakly entire with domain $g:\mathscr{G} \to \mathscr{S}$ a definable dominance. Since π is strongly pure, it follows by Lemma 4.1 that $H = \varphi_* f^* \Omega_{\mathscr{S}} \cong \psi_* g^* \Omega_{\mathscr{S}}$. Therefore, as ψ is weakly entire, it is localic and defined by the $\Omega_{\mathscr{S}}$-Stone locale $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$. Since φ is a spread with associated Heyting algebra H, $\mathscr{F}$ is a subtopos of $\mathscr{E}^{H^{\mathrm{op}}}$, and therefore so is $\mathrm{Sh}_{\mathscr{E}}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H))$. In particular, π itself must be an inclusion, hence unique. □

5. A forcing construction

Consider a span

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{f}\swarrow & & \searrow^{\varphi} \\ \mathscr{S} & & \mathscr{E} \end{array}$$

consisting of a spread $\varphi:\mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ and of a *definable dominance* $f:\mathscr{F} \to \mathscr{S}$. Associated with $\langle f, \varphi \rangle$ is the *Heyting algebra* $H = H_{\langle f,\varphi \rangle} = \varphi_* f^* \Omega_{\mathscr{S}}$. Notice that H comes equipped with a morphism

$$\eta_H : e^*(\Omega_{\mathscr{S}}) \to H,$$

which we shall call the *unit for H*. This η is taken to be the unit of adjointness $\varphi^* \dashv \varphi_*$ evaluated at $e^*(\Omega_{\mathscr{S}})$.

Recall that

$$\gamma : P(H) = \mathscr{E}^{H^{\mathrm{op}}} \to \mathscr{E}; \quad Y : H \to P(H)$$

denotes the topos of presheaves associated with a poset H in $\mathscr{E}$, with its Yoneda functor Y. In this section we shall produce a topos $\mathscr{E}[H] \xrightarrow{\psi} \mathscr{E}$ that best has the property that the $\mathscr{S}$-definable subobjects of an object $\psi^*(E)$ correspond to morphisms $E \to H$ in $\mathscr{E}$. We construct $\mathscr{E}[H]$ as a subtopos of $P(H)$ in terms of a certain forcing condition.

Lemma 5.1. *For H as above*, η *be the unit for H and* Ω_H *the subobject classifier of* $P(H)$, *the inequality*

$$\begin{array}{ccc} e^*(\Omega_{\mathscr{S}}) & \xrightarrow{\eta} & H \\ \big\downarrow & \leqslant & \big\downarrow \\ \Omega_{\mathscr{E}} m & \longrightarrow & \gamma_* \Omega_H \end{array}$$

holds, where the bottom horizontal arrow in the above square denotes the unique frame morphism in $\mathscr{E}$ and the morphism $H \to \gamma_*(\Omega_H)$ sends an element of H to its down-closure.

Let us denote the transposes of the two morphisms $e^*(\Omega_{\mathscr{S}}) \to \gamma_*\Omega_H$ in the above square as follows: $\vartheta : \gamma^*\Omega_{\mathscr{S}} \xrightarrow{\gamma^*\eta} \gamma^*H \xrightarrow{\rho} \Omega_H$ and $\tau : \gamma^*\Omega_{\mathscr{S}} \to \Omega_H$ where the morphism τ is the classifying map of the subobject $\gamma^*\top : 1 \to \gamma^*\Omega_{\mathscr{S}}$.

In order to introduce the forcing condition, consider the following pullbacks in $P(H)$:

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \top \\ Z & \longrightarrow & \gamma^*\Omega_{\mathscr{S}} \, . \\ \big\downarrow & & \big\downarrow \vartheta \\ \gamma^*H & \underset{\rho}{\longrightarrow} & \Omega_H \end{array}$$

We want to ‘force’ $\mathscr{S}$ to be a definable subobject of γ^*H by forcing $t : Z \to \gamma^*H$ to be an isomorphism. Thus, we introduce in $P(H)$ the smallest topology (over $\mathscr{E}$) for which the morphism $t : Z \to \gamma^*H$ is bidense. We shall call this the *definable topology or coverage* in H. This makes H a site in $\mathscr{E}$. Let

$$\psi_H = \psi : \mathscr{E}[H] \to \mathscr{E}$$

denote the corresponding subtopos of $P(H)$. The geometric morphism ψ_H is localic since $\mathscr{E}[H]$ is a subtopos of $P(H) = \mathscr{E}^{H^{\mathrm{op}}}$.

We have defined a topos $\mathscr{E}[H] \xrightarrow{\psi} \mathscr{E}$ in which we have forced the existence of a morphism

$$\zeta : \psi^*(H) \to \Omega_{\mathscr{S}}.$$

(Here we have written $\Omega_{\mathscr{S}}$ for $\psi^*\Omega_{\mathscr{S}}$.) The morphism ζ classifies what we shall call the *generic definable subobject associated with* H: the subobject $S \hookrightarrow \psi^*(H)$ in the following diagram:

$$\begin{array}{ccccc} (E,x) & \longrightarrow & S & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \top \\ \psi^*(E) & \underset{\psi^*x}{\longrightarrow} & \psi^*(H) & \underset{\zeta}{\longrightarrow} & \Omega_{\mathscr{S}}. \end{array}$$

Then we can associate with an ‘element’ $E \xrightarrow{x} H$ the definable subobject $(E,x) \hookrightarrow \psi^*(E)$ shown above. The transpose

$$\hat{\zeta} : H \to \psi_*(\Omega_{\mathscr{S}})$$

mediates the passage from an element $E \to H$ to a definable subobject of $\psi^*(E)$. The following is immediate.

Lemma 5.2. *If a morphism $Y \to \Omega_H$ in $P(H)$ factors through $\gamma^*(H) \xrightarrow{\rho} \Omega_H$, then the subobject that it classifies is definable in $\mathscr{E}[H]$* (after $\mathscr{E}[H]$-sheafification).

Proposition 5.1. *The composite $\zeta \cdot \psi^*(\eta)$ is equal to the identity on $\Omega_{\mathscr{S}}$. Hence $\hat{\zeta}$ and the unit η compose in $\mathscr{E}$ to give the unit of $\psi^* \dashv \psi_*$ at $\Omega_{\mathscr{S}}$.*

$$\begin{array}{ccc} \Omega_{\mathscr{S}} & & \\ {\scriptstyle \eta}\downarrow & \searrow^{\text{unit}} & \\ H & \xrightarrow[\hat{\zeta}]{} & \psi_*(\Omega_{\mathscr{S}}) \end{array} .$$

Proof. Consider the pullback

$$\begin{array}{ccc} \psi^* Z & \xrightarrow{f} & \psi^* \Omega_{\mathscr{S}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi^* \vartheta} \\ \psi^* H & \xrightarrow[\psi^* \rho]{} & \psi^* \Omega_H \end{array}$$

in $\mathscr{E}[H]$. The morphism i is an isomorphism. We have $\zeta = f \cdot i^{-1}$. A diagram chase using the induced pair

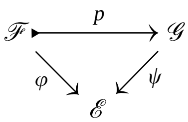
$$\langle \psi^* \eta, 1_{\Omega_{\mathscr{S}}} \rangle : \psi^* \Omega_{\mathscr{S}} \to \psi^* Z$$

gives the desired conclusion. □

6. The Michael completion of a spread

The main result of this section guarantees that the forcing topos $\mathscr{E}[H] \to \mathscr{E}$ constructed in Section 5 is non-degenerate when H arises as $\varphi_* f^* \Omega_{\mathscr{S}}$ for some span $\langle f, \varphi \rangle$, with f a definable dominance. Moreover, if φ is a spread, then $\mathscr{E}[H] \to \mathscr{E}$ will be shown to give the “Michael completion” (after Michael [22]) of φ, in the sense of the following definition.

Definition 6.1. A spread $\psi:\mathscr{G}\to\mathscr{E}$ over $\mathscr{S}$ will be said to be the *Michael completion* of a spread $\varphi:\mathscr{F}\to\mathscr{E}$ over $\mathscr{S}$ if there is a strongly pure inclusion



such that, for the Heyting algebra $H=\varphi_* f^*(\Omega_{\mathscr{S}})$, the topos $\mathscr{G}$ is the largest subtopos of the topos $\mathscr{E}^{H^{\mathrm{op}}}$ that contains $\mathscr{F}$ as a strongly pure subtopos. A spread $\psi:\mathscr{G}\to\mathscr{E}$ over $\mathscr{S}$ will be said to be a *Michael complete spread* if its Michael completion exists and is equivalent to it.

It follows from Corollary 4.1 that the Michael completion of a spread whose domain is a definable dominance exists and is given by the $\Omega_{\mathscr{S}}$-Stone locale of the $\Omega_{\mathscr{S}}$-Heyting algebra H associated with the spread. In what follows we shall prove that the Michael completion of a spread can also be obtained under the assumption that the domain of the spread be just a definable dominance in the sense of Definition 1.2 and by means of the forcing construction of Section 5.

Lemma 6.1. *Let $\varphi:\mathscr{F}\to\mathscr{E}$ be a geometric morphism over $\mathscr{S}$ and $f:\mathscr{F}\to\mathscr{S}$ a definable dominance. Let H be the Heyting algebra associated with the above span and let $q:\mathscr{F}\to P(H)$ be the geometric morphism defined in Section* 4. *Then,*

1. *The object $q_* f^*(\Omega_{\mathscr{S}})$ may be described by the formula*

$$q_* f^*(\Omega_{\mathscr{S}})(c,y)\cong\{c\xrightarrow{z}H\mid z\leqslant y\}=_{\mathrm{def}}\ \downarrow_c y,$$

where $(c,y)\in\mathbf{H}$;

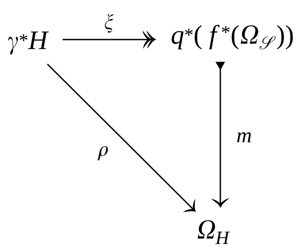
2. *The counit*

$$\xi:\gamma^*H=\gamma^*\gamma_*(q_* f^*(\Omega_{\mathscr{S}}))\to q_* f^*\Omega_{\mathscr{S}}$$

may be described as follows: for $(c,y)\in\mathbf{H}$, we have

$$\gamma^*H(c,y)=\mathscr{E}(c,H)\to\downarrow_c y;\ \ h\mapsto y\wedge h;$$

3. *The epi-mono factorization of ρ in $P(H)$ is the following:*



and therefore, the transpose $H\to\gamma_*(\Omega_H)$ of ρ is equal to $\gamma_*(m)$.

Proof. 1. Since $f:\mathscr{F}\to\mathscr{S}$ is subopen, then the elements of $q_*(f^*(\Omega_{\mathscr{S}}))(c,y)$ are in bijection with the definable subobjects of a definable subobject $Y=q^*(c,y)\hookrightarrow\varphi^*(c)$ in $\mathscr{F}$. Since in $\mathscr{F}$, definable subobjects compose, then the definable subobjects of Y coincide with the definable subobjects of $\varphi^*(c)$ under Y.

2. This describes a morphism with the appropriate universal property.

3. The morphism $q_*(f^*(\Omega_{\mathscr{S}}))\xrightarrow{m}\Omega_H$ is given at (c,x) by sending a $c\xrightarrow{h}H$ for which $h\leqslant x$ to the subobject $(c,h)\hookrightarrow(c,x)$. That the triangle commutes is immediate. If h and h' produce in this way the same subobject of (c,x), then of course they define the same *definable* subobject of $q^*(c,x)$, so that $h=h'$. Thus, m is a monomorphism. Clearly ξ is an epimorphism. $\square$

Proposition 6.1. *Let $\varphi:\mathscr{F}\to\mathscr{E}$ be a spread over $\mathscr{S}$ with f a definable dominance. Then, for $H=\varphi_*f^*(\Omega_{\mathscr{S}})$, the canonical inclusion $\mathscr{F}\xrightarrow{q}P(H)$ factors through the inclusion $\mathscr{E}[H]\to P(H)$.*

Proof. We must show that $q^*(t)$ is an isomorphism. We have in $\mathscr{F}$ the following pullback:

$$\begin{array}{ccc}
q^*(Z) & \xrightarrow{q^*r} & f^*(\Omega_{\mathscr{S}}) \\
{\scriptstyle q^*t}\downarrow & & \downarrow{\scriptstyle q^*\vartheta} \\
\varphi^*(H) & \xrightarrow[q^*\rho]{} & q^*(\Omega_H).
\end{array}$$

The counit $\delta:\varphi^*(H)\to f^*(\Omega_{\mathscr{S}})$ divides this square into two triangles, a top one and a bottom one. We claim that these two triangles commute, from which the desired conclusion follows immediately.

In order to see that the top triangle commutes, consider its transpose back in $P(H)$ depicted below, left. Recall that Z is defined by the pullback below, right.

$$\begin{array}{ccc}
Z & \xrightarrow{r} & \gamma^*e^*(\Omega_{\mathscr{S}}) \\
{\scriptstyle t}\downarrow & & \downarrow \\
\gamma^*(H) & \xrightarrow[\xi]{} & q_*(\Omega_{\mathscr{S}})
\end{array}
\qquad
\begin{array}{ccc}
Z & \xrightarrow{r} & \gamma^*e^*(\Omega_{\mathscr{S}}) \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \vartheta} \\
\gamma^*(H) & \xrightarrow[\rho]{} & \Omega_H
\end{array}$$

ξ is the counit for γ, whose description is given in Lemma 6.1. Observe that $\gamma^*\eta$ makes the following triangle commute. The morphism depicted vertically is a unit.

$$\begin{array}{ccc}
 & & \gamma^*e^*(\Omega_{\mathscr{S}}) \\
 & {\scriptstyle \gamma^*(\eta)}\swarrow & \downarrow \\
\gamma^*(H) & \xrightarrow[\xi]{} & q_*(\Omega_{\mathscr{S}})
\end{array} \quad .$$

Thus we have only to show that $\xi \cdot \gamma^*(\eta) \cdot r$ and $\xi \cdot t$ are equal. We have

$$m \cdot \xi \cdot \gamma^*(\eta) \cdot r = \rho \cdot \gamma^*(\eta) \cdot r = \vartheta \cdot r = \rho \cdot t = m \cdot \xi \cdot t.$$

But m is a monomorphism (Lemma 6.1).

To see that the bottom triangle

$$\begin{array}{ccc} & & f^*(\Omega_{\mathscr{S}}) \\ & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle q^*\vartheta} \\ \varphi^*H & \underset{q^*\rho}{\longrightarrow} & q^*(\Omega_H) \end{array}$$

commutes recall that ρ factors through the subobject $q_*(f^*(\Omega_{\mathscr{S}}))$ of Ω_H. Recalling also that ϑ denotes $\rho \cdot \gamma^*(\eta)$, we see that it must be shown that

$$\varphi^*(H) \underset{1}{\overset{\varphi^*\eta\cdot\delta}{\rightrightarrows}} \varphi^*(H) \xrightarrow{q^*\xi} q^*q_*\gamma^*e^*(\Omega_{\mathscr{S}})$$

commutes. The transpose to $\mathscr{E}$ of this diagram is the following:

$$H \underset{\eta_H}{\overset{\varphi_*\varphi^*\eta}{\rightrightarrows}} \varphi_*\varphi^*(H) \xrightarrow{\varphi_*q^*\xi} \varphi_*q^*q_*\gamma^*e^*(\Omega_{\mathscr{S}}) \ .$$

Since φ is a *spread*, it is easy to see that this diagram commutes. Indeed, by Proposition 4.2, $\mathscr{F} \xrightarrow{q} P(H)$ is an inclusion. In particular, the counit $q^*q_*e^*(\Omega_{\mathscr{S}}) \to e^*(\Omega_{\mathscr{S}})$ is an isomorphism. The result of applying φ_* to this counit and then composing with $\varphi_*q^*\xi$ is equal to the counit $\varphi_*\varphi^*H \to H$. This counit coequalizes η_H and $\varphi_*\varphi^*\eta$, so that $\varphi_*q^*\xi$ must do so as well. □

Theorem 6.1. *Let $\langle f : \mathscr{F} \to \mathscr{S}, \varphi : \mathscr{F} \to \mathscr{E} \rangle$ be a span with f a definable dominance and φ a spread over $\mathscr{S}$. Consider the factorization*

$$\mathscr{F} \xrightarrow{p} \mathscr{E}[H] \xrightarrow{\psi} \mathscr{E}$$

of φ, where $H = \varphi_*(f^*(\Omega_{\mathscr{S}}))$. Then $\mathscr{E}[H] \xrightarrow{\psi} \mathscr{E}$ is the Michael completion of the spread $\varphi : \mathscr{F} \to \mathscr{E}$.

Proof. We first observe that there is a morphism $p_*(\Omega_{\mathscr{S}}) \xrightarrow{\alpha} \Omega_{\mathscr{S}}$ in $\mathscr{E}[H]$ such that $p_*(\Omega_{\mathscr{S}}) \xrightarrow{\alpha} \Omega_{\mathscr{S}} \xrightarrow{\beta} p_*(\Omega_{\mathscr{S}})$ is equal to the identity, where β denotes the unit of $p^* \dashv p_*$. In order to define α, we use the description of $p_*(\Omega_{\mathscr{S}}) = q_*(\Omega_{\mathscr{S}})$ provided by

Lemma 6.1. Given $c \xrightarrow{z} H$ such that $z \leqslant y$, we pass to the following composite morphism:

$$(c, y) \to \psi^*(c) \xrightarrow{\psi^* z} \psi^*(H) \xrightarrow{\zeta} \Omega_{\mathscr{S}}.$$

This morphism classifies the definable subobject $(c, z) \hookrightarrow (c, y)$ in $\mathscr{E}[H]$. It is clear that $\beta \cdot \alpha = 1$ holds.

To show that $\alpha \cdot \beta = 1$ also holds, it will be enough to show that the square

$$\begin{array}{ccc} \Omega_{\mathscr{S}} & \xrightarrow{\psi^*\eta} & \psi^*(H) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\zeta} \\ p_*(\Omega_{\mathscr{S}}) & \xrightarrow[\alpha]{} & \Omega_{\mathscr{S}} \end{array}$$

commutes in $\mathscr{E}[H]$, since $\zeta \cdot \psi^*(\eta) = 1_{\Omega_{\mathscr{S}}}$, which we now show. Consider the counit of $\psi^* \dashv \psi_*$:

$$\varepsilon : \psi^*(H) = \psi^*(\psi_*(p_*(\Omega_{\mathscr{S}}))) \to p_*(\Omega_{\mathscr{S}}).$$

That $\varepsilon \cdot \psi^*(\eta) = \beta$ holds we leave as an exercise. To see that $\alpha \cdot \varepsilon = \zeta$ holds we only have to remind ourselves how these three morphisms are described in terms of the $\mathscr{S}$-site **H**. For example, the counit ε is derived (after $\mathscr{E}[H]$-sheafifying) from the counit ξ for $\gamma^* \dashv \gamma_*$, which is described in Lemma 6.1. We leave the (routine) verifications to the reader.

By construction, $\mathscr{E}[H]$ is indeed the largest subtopos of $\mathscr{E}^{H^{\mathrm{op}}}$ containing $\mathscr{F}$ as a strongly pure subtopos. □

Theorem 6.2. *Let $\varphi : \mathscr{F} \to \mathscr{E}$ be a spread over $\mathscr{S}$ whose domain $f : \mathscr{F} \to \mathscr{S}$ is a definable dominance. Let $H = \varphi_* f^* \Omega_{\mathscr{S}}$. Associated with H are, on the one hand, the Michael complete spread $\psi : \mathscr{E}[H] \to \mathscr{E}$, and on the other hand, the weakly entire geometric morphism ϱ: $\mathrm{Sh}_{\mathscr{E}}(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)) \to \mathscr{E}$.*

There exists an equivalence represented by a horizontal arrow in the following commutative triangle:

$$\begin{array}{ccc} Sh_{\mathscr{E}}(Idl_{\Omega_{\mathscr{S}}}(H)) & \longrightarrow & \mathscr{E}[H] \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathscr{E} & \end{array}.$$

In particular, for a spread whose domain is a definable dominance, its Michael completion is given by the weakly entire part of its (strongly pure, weakly entire) unique factorization.

Proof. We wish to show that the topos $\mathscr{G}$ of sheaves on the frame of $\Omega_{\mathscr{S}}$-ideals of H has the universal property defining $\mathscr{E}[H]$. By the universal property of the free frame on the $\Omega_{\mathscr{S}}$-cocomplete distributive lattice H, $\varphi_* \tau : \varphi_* f^* \Omega_{\mathscr{S}} \to \varphi_* \Omega_{\mathscr{F}}$ induces a unique frame morphism

$$\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H) \to \varphi_* \Omega_{\mathscr{F}}$$

such that restricted to $\downarrow : H \to \mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ agrees with $\varphi_* \tau$. In turn, this induces a geometric morphism $\mathscr{F} \xrightarrow{\upsilon} \mathscr{G}$ over $\mathscr{E}$ (i.e., such that $\varphi \cong \rho \cdot \upsilon$) with

$$\mathscr{F} \xrightarrow{\upsilon} \mathscr{G} \xrightarrow{r} P(H) \cong \mathscr{F} \xrightarrow{q} P(H).$$

To see now that υ is strongly pure, observe first that the triangle

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\quad p \quad} & \mathscr{E}[H] \\ & {\scriptstyle \upsilon} \searrow \quad \nearrow {\scriptstyle \hat{r}} & \\ & Sh_{\mathscr{E}}(Idl_{\Omega_{\mathscr{S}}}(H)) & \end{array}$$

commutes. To see that this, in turn, is the case, we post-compose with the inclusion $\pi : \mathscr{E}[H] \to P(H)$ and observe that both composites agree. But we have that

$$\pi \cdot \hat{r} \cdot \upsilon \cong r \cdot \upsilon \cong q$$

and that $\pi \cdot p \cong q$, both by construction. In the above commutative triangle, the composite p is strongly pure by Theorem 6.1 and the second morphism $\hat{r}$ in the composition is an inclusion since $r \cong \pi \cdot \hat{r}$ and r is an inclusion and since $\rho : \mathscr{G} \to \mathscr{E}$ is a spread. The strong purity of υ now follows from [5] (Proposition 1.2). □

7. Quasi-components and $\mathscr{S}$-additive measures

Associated with a span

$$\begin{array}{ccc} & \mathscr{F} & \\ {\scriptstyle f} \swarrow & & \searrow {\scriptstyle \varphi} \\ \mathscr{S} & & \mathscr{E} \end{array}$$

consisting of a geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ and of a *definable dominance* $f : \mathscr{F} \to \mathscr{S}$, there is both a Heyting algebra $H = \varphi_* f^*(\Omega_{\mathscr{S}})$ and an $\mathscr{S}$-coproducts preserving $\mathscr{S}$-indexed functor $\mu : \mathscr{E} \to \mathscr{S}$ which we now introduce, after some preliminary considerations. We will think of μ as an *$\mathscr{S}$-additive measure* on $\mathscr{E}$, in the spirit of measure theory.

Proposition 7.1. *For $f:\mathscr{F}\to\mathscr{S}$ a definable dominance, and Y an object of $\mathscr{F}$, the canonical order-preserving map*

$$\tau^Y:(f^*\Omega_{\mathscr{S}})^Y\to\Omega_{\mathscr{F}}^Y$$

preserves finite meets.

Proof. Since f is subopen, the pair $\langle f^*(\Omega_{\mathscr{S}}),f^*(\top)\rangle$ classifies definable subobjects in $\mathscr{F}$. We know from [1] that definable monos are pullback stable in $\mathscr{F}$; as f is a definable dominance, they also compose. The diagonal with codomain Y in the following pullback square represents the meet of two given subobjects of Y:

$$\begin{array}{ccc} A\wedge B & \rightarrowtail & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & Y \end{array}$$

and gives $f_*((f^*(\Omega_{\mathscr{S}}))^Y)\cong\mathrm{Sub}_{\mathrm{def}}(Y)$ (for f a definable dominance) the structure of a sub meet-semilattice of $f_*((\Omega_{\mathscr{F}})^Y)$. It is easily shown that the subobject $\mathrm{id}:Y\hookrightarrow Y$ is definable. □

Recall the notion of a *completely* (*or strongly*) *prime filter* Q of a meet-semilattice P (with 0 and 1) in a topos $\mathscr{S}$. It consists of an upclosed subobject Q of P that contains 1 but not 0, is closed under binary meets, and has the further property that if $(\bigvee_{i\in I}p_i)\in Q$ for some $I\in S$, then for some $i\in I$ one has that $p_i\in Q$. This gives an $\mathscr{S}$-valued functor CPF defined on the category of meet-semilattices in $\mathscr{S}$. This functor can be $\mathscr{S}$-indexed by interpreting "completely prime filter" internally in the slice toposes $\mathscr{S}/I$.

The assignment $Y\mapsto f_\#(Y)=\mathrm{CPF}(\mathrm{Sub}_{\mathrm{def}}(Y))$ is functorial: given $\alpha:Y\to Z$ in $\mathscr{F}$ and $Q\in f_\#(Y)$, define

$$f_\#(\alpha)(Q)=\{R\in\mathrm{Sub}_{\mathrm{def}}(Z)\mid\alpha^*(R)\in Q\},$$

easily shown to be a completely prime filter of the meet semilattice $\mathrm{Sub}_{\mathrm{def}}(Z)$. Notice that the functor $f_\#$ can be $\mathscr{S}$-indexed since all the components in its definition are $\mathscr{S}$-indexed. We refer to it as the *quasi-components functor* associated with the definable dominance f, motivated by the topological notion of quasi-component of a space.

Proposition 7.2. *Assume that $f:\mathscr{F}\to\mathscr{S}$ is locally connected. Then there is a natural equivalence $f_\#\cong f_!$ of $\mathscr{S}$-indexed functors.*

Proof. For any object I of $\mathscr{S}$ there is an isomorphism (as in [18, p. 29])

$$\sigma_I:\ \mathrm{CPF}((\Omega_{\mathscr{S}})^I)\cong I$$

natural in I. It follows from Proposition 7.1 that, since f is a definable dominance, for any object Y of $\mathscr{F}$, $f_\#(Y)$ exists. We also have the object $f_!(Y)$ where $f_!$ is the $\mathscr{S}$-indexed left adjoint to f^* that exists on account of local connectedness of f. It follows that:

$$f_!(Y) \cong \mathrm{CPF}((\Omega_{\mathscr{S}})^{f!(Y)}) \cong \mathrm{CPF}(f_*(f^*(\Omega_{\mathscr{S}})^Y)) = f_\#(Y)$$

naturally in Y. □

Given any span $\langle f, \varphi \rangle$ with φ arbitrary and f a definable dominance, let $\mu = \mu_{\langle f,\varphi \rangle} = f_\# \varphi^* : \mathscr{E} \to \mathscr{S}$. First, we need a lemma saying that there is a "counit" relating $f_\#$ and f^*, although in general (meaning f a definable dominance) it need not be the case that $f_\#$ is left adjoint to f^*.

Lemma 7.1. *Let $f : \mathscr{F} \to \mathscr{S}$ be a definable dominance and let $f_\# : \mathscr{F} \to \mathscr{S}$ be the associated quasi-components functor. Then, there is a canonical natural transformation $\varepsilon : f_\# f^* \to \mathrm{id}_{\mathscr{S}}$.*

Proof. Since f^* is lex, there is given, for each $I, J \in \mathscr{S}$, a canonical morphism $\varrho_{\langle I,J \rangle} : f^*(J^I) \to f^*(J)^{f^*(I)}$ defined as the transpose of the composite $f^*(I) \times f^*(J) \cong f^*(I \times J) \to f^*(J)$. From these we obtain the morphisms

$$f_*(\varrho_{\langle I,\Omega_{\mathscr{S}} \rangle}) : f_*(f^*(\Omega^I_{\mathscr{S}})) \to f_*((f^*(\Omega_{\mathscr{S}}))^{f^*(J)}).$$

For any $U \in \Omega^I_{\mathscr{S}}$ (defined at K), composing $f_* f^*(U)$ with $f_*(\varrho_{\langle I,\Omega_{\mathscr{S}} \rangle})$ gives

$$\hat{U} = f_*(\varrho_{\langle I,\Omega_{\mathscr{S}} \rangle}) \cdot f_* f^*(U) \in f_*((f^*(\Omega_{\mathscr{S}}))^{f^*(I)})$$

(defined at K).

Let $I \in S$ and let $Q \in f_\#(f^*(I))$. It is easy to verify (since Q is a quasi-component of I) that $\{U \in \Omega^I_{\mathscr{S}} \mid \hat{U} \in Q\}$ is a quasi-component of $\Omega^I_{\mathscr{S}}$ and therefore, that applying the isomorphism $\sigma_I : \mathrm{CPF}((\Omega_{\mathscr{S}})^I) \cong I$ to it gives an (generalized) element of I. We may now define the component of ε at I to be the morphism

$$\varepsilon_I(Q) = \sigma_I(\{U \in \Omega^I_{\mathscr{S}} \mid \hat{U} \in Q\})$$

and easily check naturality in I. □

In the "general case" of a span $\langle f, \varphi \rangle$, with f a definable dominance, we can still think of $\mu = \mu_{\langle f,\varphi \rangle} = f_\# \varphi^*$ as a measure on $\mathscr{E}$ that is "$\mathscr{S}$-additive" in the sense of the proposition below.

Proposition 7.3. *Let $\langle f, \varphi \rangle$ be a span, with $f : \mathscr{F} \to \mathscr{S}$ a definable dominance and $\varphi : \mathscr{F} \to \mathscr{E}$ an arbitrary geometric morphism over $\mathscr{S}$. Let $\mu = f_\# \varphi^* : \mathscr{E} \to \mathscr{S}$, with*

$f_{\#}:\mathscr{F} \to \mathscr{S}$ the quasi-components functor associated with the definable dominance f. Then, μ preserves $\mathscr{S}$-coproducts.

Proof. Since φ^* preserves all $\mathscr{S}$-colimits, it is enough to verify that $f_{\#}$ preserves $\mathscr{S}$-coproducts. For any morphisms $a:I \to J$ and $\alpha:Y \to f^*(J)$ in $\mathscr{S}$, there is a canonical morphism

$$\theta: f_{\#}(f^*(I) \times_{f^*(J)} Y) \to I \times_J f_{\#}(Y),$$

where $I \times_J f_{\#}(Y)$ is the pullback of the composite $\varepsilon_J \cdot f_{\#}(\alpha)$ along $a:I \to J$. The induced morphism θ arises from the naturality of ε. That θ is an isomorphism can be verified directly in terms of the behavior of definable subobjects and the CPF-functor with respect to coproducts of families indexed by an object in $\mathscr{S}$. We leave this to the reader. □

With a span

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{f}\swarrow & & \searrow^{\varphi} \\ \mathscr{S} & & \mathscr{E} \end{array}$$

consisting of a geometric morphism $\varphi:\mathscr{F} \to \mathscr{E}$ over $\mathscr{S}$ and of a *locally connected* $f:\mathscr{F} \to \mathscr{S}$, there is then associated both a Heyting algebra $H = \varphi_* f^* \Omega_{\mathscr{S}}$ and a measure $\mu:\mathscr{E} \to \mathscr{S}$ as before, since f is a definable dominance. But in this case where f is locally connected and not just a definable dominance, μ is *distribution* (since it preserves all $\mathscr{S}$-indexed colimits) and H is a *distribution algebra* in the sense of [7].

In this particular case we can compare the *complete spread* $\mathscr{E}[\mu] \to \mathscr{E}$ associated with the distribution μ [4], with the *Michael complete spread* $\mathscr{E}[H] \to \mathscr{E}$ (see Sections 5 and 6), where H is the distribution algebra corresponding to μ as in [7]. Recall that the domain of the complete spread $\mathscr{E}[\mu] \to \mathscr{E}$ is locally connected, in particular a definable dominance (even an $\Omega_{\mathscr{S}}$-definable dominance). Therefore, the complete spread $\mathscr{E}[\mu] \to \mathscr{E}$ admits a Michael completion and the latter is given by $\mathscr{E}[H] \to \mathscr{E}$.

A comparison thus exists as given by a strongly pure inclusion p over $\mathscr{E}$ as in the following triangle:

$$\begin{array}{ccc} \mathscr{E}[\mu] & \xrightarrow{\quad p \quad} & \mathscr{E}[H] \\ {}_{\varphi_\mu}\searrow & & \swarrow_{\varphi_H} \\ & \mathscr{E} & \end{array} .$$

Theorem 7.1. *Let $\varphi : \mathscr{F} \to \mathscr{E}$ be an $\mathscr{S}$-spread with locally connected domain $f : \mathscr{F} \to \mathscr{S}$ and let $\varphi_\mu : \mathscr{E}[\mu] \to \mathscr{E}$ be its spread completion, with $f_\mu : \mathscr{E}[\mu] \to \mathscr{S}$ locally connected by construction. Consider now the strongly pure inclusion p from $\mathscr{E}[\mu]$ to its Michael completion (which must be equivalent to) $\mathscr{E}[H]$, where H is the distribution algebra corresponding to μ. Then the following are equivalent*:
1. *p_* preserves $\mathscr{S}$-coproducts.*
2. *φ_H has locally connected domain.*
3. *φ_μ and φ_H are equivalent over $\mathscr{E}$.*

Proof. 1. (1) $\Rightarrow$ (2). If p_* preserves $\mathscr{S}$-coproducts then, as $\mathscr{E}[\mu]$ is locally connected over $\mathscr{S}$, then also $\mathscr{E}[H]$ must be locally connected over $\mathscr{S}$, by Bunge and Funk [4] Proposition 2.7.

2. (2) $\Rightarrow$ (3). If the domain of φ_H is locally connected, then it follows from [4] Theorem 2.15 that there is a unique spread completion of it over $\mathscr{E}$. Since strongly pure morphisms compose, first of all we deduce that the original spread φ and its spread completion have equivalent weakly proper completions. It follows that (the unique strongly pure morphism) p is an equivalence.

3. That (3) $\Rightarrow$ (1) is immediate. $\square$

8. Final questions and remarks

1. It may be interesting to explore a topos-theoretic definition of a *cut* following Michael [22] and to relate it to the definition of a *branched cover* given in [9,12] and which involves the (Fox) spread completion (rather than the Michael completion) of a geometric spread.

2. Knowing that a Stone locale [15] is a compact, zero-dimensional locale, suggests that an $\Omega_{\mathscr{S}}$-Stone locale, which is certainly weakly zero-dimensional (in the sense that the structure map of its topos of sheaves is a spread, as observed in Section 3), could perhaps be alternatively characterized by adding a suitable notion of "weak compactness" similar to that studied by Vermeulen [27] but relative to an arbitrary base topos.

3. The notion of an $\mathscr{S}$-additive measure on a topos $\mathscr{E}$ defined over $\mathscr{S}$ warrants further investigation, as does the matter of the completion of a spread "without local connectedness" in the spirit of [23], that is, in terms of *quasi-components* rather than of components.

4. The matter of the stability under bipullback of the strongly pure and of the weakly entire geometric morphisms defined over a base topos $\mathscr{S}$ is closely connected with the possibility of giving a constructive version also of the uniqueness part in the pure-entire factorization of Johnstone [14]. It deserves further investigation (beyond [6]).

5. In our investigations concerning distributions, a particular role was played by the *symmetric topos* [2–6] which is the topos classifier of distributions on any $\mathscr{S}$-bounded topos $\mathscr{E}$, equivalently, a classifier of the complete spreads over $\mathscr{E}$ with a locally connected domain. It would perhaps be interesting to investigate the question of the existence and properties of a topos classifier for the Michael complete spreads over $\mathscr{E}$ with

domain a definable dominance; equivalently, a classifier for the $\mathscr{S}$-additive measures on $\mathscr{E}$.

6. The question of resolving a left exact monad **T** on a topos $\mathscr{E}$ by a geometric morphism is investigated and partially answered in [17]. Not every left exact monad on an arbitrary topos $\mathscr{E}$ will admit a resolution in the form of a (Michael) complete spread. However, our work here is relevant in that the key is in an analysis of the structure of $T(\Omega_s)$, where T is the left exact functor that is part of a monad **T**. This matter will be pursued elsewhere [8].

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