

An algebraic approach to subframe logics. Intuitionistic case

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Abstract

We develop duality between nuclei on Heyting algebras and certain binary relations on Heyting spaces. We show that these binary relations are in 1–1 correspondence with subframes of Heyting spaces. We introduce the notions of nuclear and dense nuclear varieties of Heyting algebras, and prove that a variety of Heyting algebras is nuclear iff it is a subframe variety, and that it is dense nuclear iff it is a cofinal subframe variety. We give an alternative proof that every (cofinal) subframe variety of Heyting algebras is generated by its finite members.

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1. Introduction

It was shown by Diego [8] that the variety of Hilbert algebras is locally finite. Using this result McKay [20] showed that every superintuitionistic logic axiomatizable by $(\perp, \wedge, \rightarrow)$ -formulas has the finite model property (fmp, for short). On the other hand, it follows from Zakharyashev [25,26] (see also [7]) that a superintuitionistic logic L is axiomatizable by \rightarrow -formulas iff L is axiomatizable by (\wedge, \rightarrow) -formulas iff L is a subframe logic, and that L is axiomatizable by $(\perp, \wedge, \rightarrow)$ -formulas iff L is a cofinal subframe logic. Consequently, every cofinal subframe superintuitionistic logic has the fmp. In the modal case, Fine [11] proved that every subframe logic over $\mathbf{K4}$ has the fmp, and Zakharyashev [26] extended Fine's result by showing that every cofinal subframe logic over $\mathbf{K4}$ has the fmp. That there exist subframe logics over \mathbf{K} without the fmp, and even incomplete, was shown by Wolter [23].

Most of the results mentioned above were obtained using model-theoretic tools. The aim of this paper is to give an algebraic insight into subframe logics. Our main tools will be *nuclei*, also known as *local operators*, which play a central role in topos theory (see, e.g., a comprehensive textbook [17]). Nuclei have also been studied from a lattice-theoretic point of view, both in connection with locale theory (see, e.g., [16,15]), as well as operators on Heyting algebras [3,19,13,4,5,12].

In this paper we connect nuclei on Heyting algebras with subframes of Heyting spaces by first developing duality between nuclei on Heyting algebras and certain binary relations on Heyting spaces, and then establishing that these

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binary relations are in 1–1 correspondence with subframes of Heyting spaces. We introduce the notions of nuclear and dense nuclear varieties of Heyting algebras, and prove that a variety of Heyting algebras is nuclear iff it is a subframe variety, and that it is dense nuclear iff it is a cofinal subframe variety. We also give an alternative proof that every (cofinal) subframe variety of Heyting algebras is generated by its finite members.

Our approach extends to the case of (cofinal) subframe varieties of **K4**-algebras, but since the extension requires further non-trivial insight, it will form the subject of another paper.

The paper is organized as follows. In Section 2 we recall basics of the duality between Heyting algebras and Heyting spaces, as well as the notion of a subframe of a Heyting space. In Section 3 we recall the definition and basic facts about nuclei, and define nuclear and dense nuclear varieties of Heyting algebras. In Section 4 we develop duality between nuclei on Heyting algebras and certain binary relations on Heyting spaces. In Section 5 we connect the binary relations on Heyting spaces with subframes of Heyting spaces, thus obtaining a 1–1 correspondence between nuclei on Heyting algebras and subframes of their dual spaces. As a result, we obtain that a variety of Heyting algebras is nuclear iff it is a subframe variety, and that it is dense nuclear iff it is a cofinal subframe variety. In Section 6 we obtain rather simple dual proofs of several useful results about the lattice of nuclei of a given Heyting algebra. In Section 7 we prove that every (dense) nuclear variety is generated by its finite members, which gives an alternative proof of the known fact that every (cofinal) subframe variety is generated by its finite members. Finally, in Section 8 we give an alternative proof of the axiomatization of subframe and cofinal subframe varieties by (\wedge, \rightarrow) -identities and $(\perp, \wedge, \rightarrow)$ -identities, respectively.

2. Subframe varieties

We recall that a *meet semilattice* is a commutative idempotent semigroup $\langle S, \cdot \rangle$. For a given meet semilattice $\langle S, \cdot \rangle$, we denote \cdot by \wedge , and define a partial order \leq on S by $a \leq b$ iff $a = a \wedge b$. Then $a \wedge b$ becomes the greatest lower bound of $\{a, b\}$ and $\langle S, \wedge \rangle$ can be characterized as a partially ordered set $\langle S, \leq \rangle$ such that every finite subset of S has a greatest lower bound. Below we will be interested in meet semilattices with the greatest element \top , i.e., in commutative idempotent monoids $\langle M, \wedge, \top \rangle$. Let \mathbb{M} denote the category of meet semilattices with \top and meet semilattice homomorphisms preserving \top ; that is \mathbb{M} is the category of commutative idempotent monoids and monoid homomorphisms. We recall that $M \in \mathbb{M}$ is called an *implicative semilattice* if for every $a \in M$ the order-preserving map $a \wedge (-) : M \rightarrow M$ has a right adjoint, denoted by $a \rightarrow (-)$. If in addition M is a bounded lattice, then M is called a *Heyting algebra*. We observe that every Heyting algebra is a distributive lattice since $a \wedge (-)$, as a left adjoint to $a \rightarrow (-)$, preserves all existing right limits. Let \mathbb{HA} denote the category of Heyting algebras and Heyting algebra homomorphisms.

In order to describe the dual category of \mathbb{HA} we recall that a topological space X is a *Stone space* if X is compact, Hausdorff, and 0-dimensional. We call a subset A of X *clopen* if it is closed and open. Let $\mathcal{CO}(X)$ denote the set of all clopens of X . For a partial order \leq on X and $A \subseteq X$ let

$$\uparrow A = \{x \in X : \exists a \in A \text{ with } a \leq x\}$$

and

$$\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}.$$

We call A an *upset* of X if $A = \uparrow A$, and a *downset* of X if $A = \downarrow A$. We call X a *Heyting space* if X is a Stone space and \leq is a partial order on X such that (i) $\uparrow x$ is closed for each $x \in X$ and (ii) $A \in \mathcal{CO}(X)$ implies $\downarrow A \in \mathcal{CO}(X)$. For two Heyting spaces X and Y , a map $f : X \rightarrow Y$ is called a *Heyting space morphism* if f is a continuous p -morphism; that is f is continuous, order-preserving, and $f(x) \leq y$ implies there exists z with $x \leq z$ and $f(z) = y$. Let \mathbb{HS} denote the category of Heyting spaces and Heyting space morphisms.

Theorem 1 ([9, P. 149, Thm. 3]). \mathbb{HA} is dually equivalent to \mathbb{HS} .

Proof (Sketch). For $A \in \mathbb{HA}$ let A_* denote the set of prime filters of A ordered by inclusion. For $a \in A$ let $\varphi(a) = \{x \in A_* : a \in x\}$. We define topology on A_* by letting $\{\varphi(a), -\varphi(a)\}_{a \in A}$ be a basis, where we denote by $-\varphi(a)$ the set-theoretic complement of $\varphi(a)$ in A_* . For $h \in \text{Hom}(A, B)$ we define $h_* : B_* \rightarrow A_*$ by $h_*(x) = h^{-1}(x)$. Then $A_* \in \mathbb{HS}$, $h_* \in \text{Hom}(B_*, A_*)$, and $(-)_* : \mathbb{HA} \rightarrow \mathbb{HS}$ is a well-defined contravariant functor. For $X \in \mathbb{HS}$ let X^* denote the set of clopen upsets of X . For $U, V \in X^*$ let $U \rightarrow V = -\downarrow(U - V)$. Then $X^* \in \mathbb{HA}$. Also, for

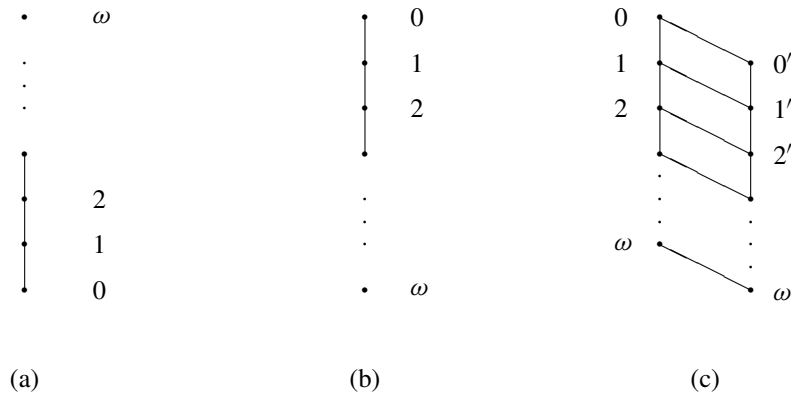


Fig. 1.

$f \in \text{Hom}(X, Y)$ we define $f^* : Y^* \rightarrow X^*$ by $f^*(U) = f^{-1}(U)$. Then $f^* \in \text{Hom}(Y^*, X^*)$ and $(-)^* : \mathbb{HS} \rightarrow \mathbb{HA}$ is a well-defined contravariant functor. Moreover, $(-)_*$ and $(-)^*$ yield a dual equivalence between \mathbb{HA} and \mathbb{HS} . \square

Suppose X is a Heyting space. It is well known [9, P. 148, Thm. 1] that if C is closed in X , then $\uparrow C$ is also closed in X . We will freely use this fact throughout the paper.

Let X and Y be Heyting spaces. We recall from [7, P. 289] that a partial map $f : X \rightarrow Y$ is a *subreduction* of X to Y if (i) f is a p -morphism from the domain of f to Y and (ii) A a clopen downset of Y implies $\downarrow f^{-1}(A)$ is a clopen downset of X . Then Y is called a *subframe* of X if (i) Y is a subspace of X , (ii) \leq_Y is the restriction of \leq_X to Y , and (iii) the partial identity map $i : X \rightarrow Y$ is a subreduction.

Lemma 2. *Suppose X is a Heyting space. A subspace Y of X is a subframe of X iff Y is a closed subspace of X and $A \in \mathcal{CO}(Y)$ implies $\downarrow A \in \mathcal{CO}(X)$.*

Proof. First suppose that Y is a subframe of X . Since Y is a Heyting space, it is a compact subspace of X ; and as X is Hausdorff, we have that Y is a closed subspace of X . Let i denote the partial identity map from X to Y and let $A \in \mathcal{CO}(Y)$. Because Y is a Heyting space, $\downarrow A \cap Y$ is a clopen downset of Y . Moreover, $\downarrow A = \downarrow(\downarrow A \cap Y)$. So $\downarrow A = \downarrow(\downarrow A \cap Y) = \downarrow i^{-1}(\downarrow A \cap Y) \in \mathcal{CO}(X)$, as i is a subreduction.

Now suppose that Y is a closed subspace of X and $A \in \mathcal{CO}(Y)$ implies $\downarrow A \in \mathcal{CO}(X)$. Then Y with the subspace topology is a Stone space. Moreover, if \leq_Y denotes the restriction of \leq to Y , then $\uparrow_Y y = \uparrow y \cap Y$ is closed in Y , and $A \in \mathcal{CO}(Y)$ implies that $\downarrow_Y A = \downarrow A \cap Y \in \mathcal{CO}(Y)$, as $\downarrow A \in \mathcal{CO}(X)$. Thus, Y is a Heyting space. Moreover, for a clopen downset A of Y we have $\downarrow i^{-1}(A) = \downarrow A$, which is a clopen downset of X . Therefore, Y is a subframe of X . \square

Remark 3. It is an immediate consequence of Lemma 2 that every clopen subspace of a Heyting space X is a subframe of X . However, there exist subframes of X that are not clopen. Let X be the ordinal $\omega + 1$ with its usual order and topology (see Fig. 1(a)). Then X is a Heyting space and $Y = \{\omega\}$ is a non-clopen subframe of X . We also note that if Y is a subframe of X , then Y is a Heyting space, but that not every closed subspace of X that is a Heyting space is a subframe of X . Let X be the dual of $\omega + 1$ (see Fig. 1(b)). Then $Y = \{\omega\}$ is a closed subspace of X that trivially is a Heyting space, but Y is not a subframe of X because Y is clopen in Y , but $\downarrow Y = Y = \{\omega\}$ is not open in X . In fact, there even exist closed subspaces of X that are not Heyting spaces. Let X be the space shown in Fig. 1(c). Then X is the union of two disjoint copies of the dual of the ordinal $\omega + 1$, and the order on X is defined as shown in Fig. 1(c). So X is a Heyting space, $Y = \{\omega, 0', 1', 2', \dots, \omega'\}$ is a closed subspace of X , but Y is not a Heyting space because $\{\omega\}$ is a clopen subset of Y , but $\downarrow\{\omega\} = \{\omega, \omega'\}$ is not open in Y .

Let X be a Heyting space and $Y \subseteq X$. We say that $x \in Y$ is a *maximal point* of Y if $x \leq y$ implies $x = y$ for each $y \in Y$. Let $\max Y$ denote the set of all maximal points of Y . It follows from [10, P. 54, Thm. 2.1] that if Y is a closed subset of X , then for each $x \in Y$ there exists $y \in \max Y$ such that $x \leq y$.

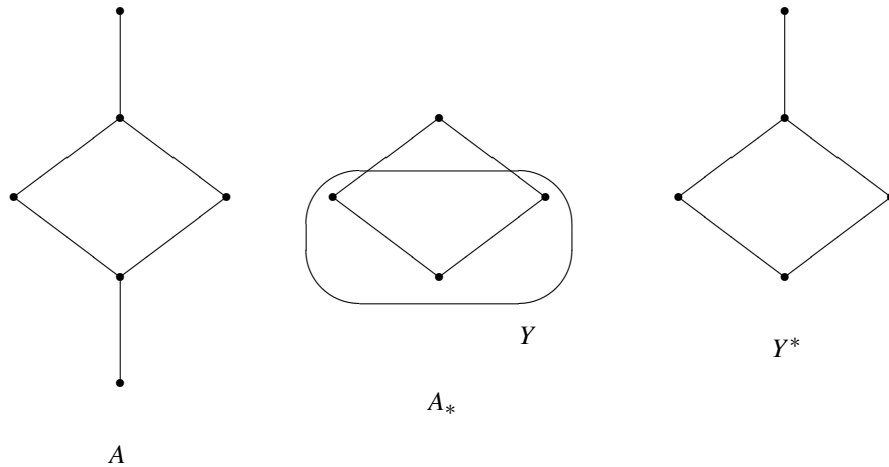


Fig. 2.

Following [7, P. 295], we call a subframe Y of a Heyting space X *cofinal* if $\uparrow Y \subseteq \downarrow Y$. It is obvious that Y is a cofinal subframe of X iff $\max(\uparrow Y) \subseteq Y$.

Definition 4. Suppose \mathbb{V} is a variety of Heyting algebras.

- (i) \mathbb{V} is called a subframe variety if $A \in \mathbb{V}$ and X a subframe of A_* imply that $X^* \in \mathbb{V}$.
- (ii) \mathbb{V} is called a cofinal subframe variety if $A \in \mathbb{V}$ and X a cofinal subframe of A_* imply that $X^* \in \mathbb{V}$.

It is obvious that if \mathbb{V} is a subframe variety, then it is a cofinal subframe variety. The converse, however, is not true in general as is witnessed, e.g., by

$$\mathbb{H}\mathbb{A} + (\neg a \vee \neg\neg a = \top).$$

Indeed, it follows from [7, P. 317] that $\mathbb{H}\mathbb{A} + (\neg a \vee \neg\neg a = \top)$ is a cofinal subframe variety. To see that $\mathbb{H}\mathbb{A} + (\neg a \vee \neg\neg a = \top)$ is not a subframe variety observe that the Heyting algebra A shown in Fig. 2 belongs to $\mathbb{H}\mathbb{A} + (\neg a \vee \neg\neg a = \top)$, that Y is a subframe of A_* , but that the Heyting algebra Y^* does not belong to $\mathbb{H}\mathbb{A} + (\neg a \vee \neg\neg a = \top)$.

3. Nuclear varieties

Suppose $M \in \mathbb{M}$. We recall that a *nucleus* on M is a unary operator $j : M \rightarrow M$ satisfying the following conditions:

- (i) $a \leq j(a)$;
- (ii) $j(j(a)) \leq j(a)$;
- (iii) $j(a \wedge b) = j(a) \wedge j(b)$.

If in addition M has the least element \perp , then a nucleus $j : M \rightarrow M$ is called *dense* if $j(\perp) = \perp$. Note that if j is dense, then

$$j(\neg j(\perp)) = j(\neg\perp) = j(\top) = \top.$$

Definition 5. We call a nucleus j on M *locally dense* if $j(\neg j(\perp)) = \top$.

Clearly every dense nucleus is locally dense. To see an example of a locally dense nucleus that is not dense, let A be the Heyting algebra shown in Fig. 3. Define j on A by $j(\perp) = j(a) = a$ and $j(b) = j(a \vee b) = j(\top) = \top$. It is routine to verify that j is a nucleus. Moreover, $j(\perp) \neq \perp$, so j is not dense, but $j(\neg j(\perp)) = j(\neg a) = j(b) = \top$, so j is locally dense.

For $M \in \mathbb{M}$ and a nucleus j on M let $M_j = \{j(a) \mid a \in M\}$. Then it is easy to see that $M_j = \{a \mid j(a) = a\}$, and that M_j is a submonoid of M .

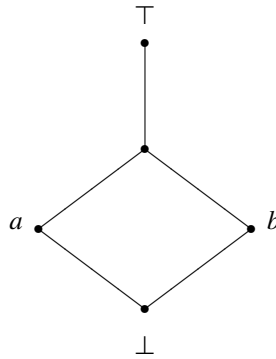


Fig. 3.

Localizations for left exact categories have been extensively studied (see, e.g., [6] for connections to factorization systems); in our simple context, a *localization* of a monoid M is a triple (L, i, l) , where L is a submonoid of M and the inclusion $i : L \hookrightarrow M$ has a left exact left adjoint $l : M \rightarrow L$; that is $l \in \text{Hom}(M, L)$ and for every $a \in M$ and $b \in L$ we have $l(a) \leq b$ iff $a \leq i(b)$. Observe that if (L, i, l) is a localization of M , then for every $a \in L$ we have $i(l(i(a))) = i(a)$. Now since i is 1–1, it follows that $l(i(a)) = a$. Therefore, l is a retract of the inclusion i . In fact, there is a 1–1 correspondence between localizations of M and retracts r of the inclusion i with the property $a \leq r(a)$ for every $a \in M$. Nuclei and localizations are in 1–1 correspondence for locales. This is parallel to the 1–1 correspondence between local operators and subtopoi in an elementary topos [17, P. 201, A.4.4.8] (see also [2] for related topics). The same 1–1 correspondence also works for semilattices and Heyting algebras, as stated below.

Proposition 6. *For a given $M \in \mathbb{M}$ there exists a 1–1 correspondence between nuclei on M and localizations of M .*

Proof (Sketch). Given a nucleus j , we have that M_j is a submonoid of M and the inclusion $M_j \hookrightarrow M$ has j as a left exact left adjoint. Conversely, given a localization (L, i, l) , we have that $i \circ l$ is a nucleus. Moreover, the two correspondences are inverse to each other. \square

Proposition 7. *Suppose j is a nucleus on an implicative semilattice M . Then M_j is an implicative subsemilattice of M . If in addition M is a Heyting algebra, then M_j is also a Heyting algebra.*

Proof. We show that $j(j(a) \rightarrow j(b)) = j(a) \rightarrow j(b)$. The \geq side is obvious. For the \leq side observe that $j(a) \wedge j(j(a) \rightarrow j(b)) = j(j(a)) \wedge j(j(a) \rightarrow j(b)) = j(j(a) \wedge (j(a) \rightarrow j(b))) \leq j(j(b)) = j(b)$. Thus, the equality. It follows that M_j is closed with respect to \wedge and \rightarrow , and so is an implicative subsemilattice of M .

Suppose M is a Heyting algebra. We denote by \vee the join in M . Let also \perp denote the least element of M . We show that $j(a \vee b) = j(j(a) \vee j(b))$. The \leq side is obvious. For the \geq side, since j is order-preserving, from $a, b \leq a \vee b$ it follows that $j(a), j(b) \leq j(a \vee b)$. Therefore, $j(a) \vee j(b) \leq j(a \vee b)$, and so $j(j(a) \vee j(b)) \leq j(j(a \vee b)) = j(a \vee b)$. Thus, the equality. Now define \vee_j and \perp_j on M_j by $a \vee_j b = j(a \vee b)$ and $\perp_j = j(\perp)$. Then $\langle M_j, \wedge, \rightarrow, \top \rangle$ becomes an implicative semilattice, while $\langle M_j, \wedge, \vee_j, \perp_j, \top \rangle$ becomes a bounded lattice. Therefore, $\langle M_j, \wedge, \vee_j, \rightarrow, \perp_j, \top \rangle$ is a Heyting algebra. \square

Remark 8. For an implicative semilattice M and a nucleus j on M we point out that M_j is not only closed under \rightarrow , but it satisfies a stronger condition; namely, $a \in M$ and $b \in M_j$ imply $a \rightarrow b \in M_j$. Indeed, $j(a \rightarrow b) \leq j(a) \rightarrow j(b) = j(a) \rightarrow b \leq a \rightarrow b$. Thus, $j(a \rightarrow b) \leq a \rightarrow b$, and so the equality. Implicative subsemilattices of M satisfying this extra condition are called *total* [18]. As a result we obtain that M_j is a total implicative subsemilattice of M . However, there exist (infinite) total implicative subsemilattices of M that do not give rise to nuclei on M . In fact, a total implicative subsemilattice T of M gives rise to a nucleus on M iff for each $a \in M$, the set $\{t \in T : a \leq t\}$ has a least element [19, P. 12, Lem. 2.6].

Definition 9.

- (i) We call a variety \mathbb{V} of Heyting algebras nuclear if whenever $A \in \mathbb{V}$ and j is a nucleus on A , then $A_j \in \mathbb{V}$.
- (ii) We call \mathbb{V} a dense nuclear variety if whenever $A \in \mathbb{V}$ and j is a dense nucleus on A , then $A_j \in \mathbb{V}$.
- (iii) We call \mathbb{V} a locally dense nuclear variety if whenever $A \in \mathbb{V}$ and j is a locally dense nucleus on A , then $A_j \in \mathbb{V}$.

As we pointed out earlier, every dense nucleus is locally dense, but not the other way around. Nevertheless, we have the following result.

Theorem 10. *A variety of Heyting algebras is dense nuclear iff it is locally dense nuclear.*

Proof. Suppose \mathbb{V} is a variety of Heyting algebras. It is clear that if \mathbb{V} is locally dense nuclear, then \mathbb{V} is dense nuclear. Conversely, suppose \mathbb{V} is dense nuclear, $A \in \mathbb{V}$, and j is a locally dense nucleus on A . We need to show that $A_j \in \mathbb{V}$. Let $B = \downarrow(\neg j(\perp))$. Define $h : A \rightarrow B$ by $h(a) = a \wedge \neg j(\perp)$. Then it is well known (see, e.g., [22, P. 138, Thm. 8.2]) that h is an onto Heyting algebra homomorphism. Thus, B is a homomorphic image of $A \in \mathbb{V}$, and so $B \in \mathbb{V}$. Define j_B on B by $j_B(b) = j(b) \wedge \neg j(\perp)$ for each $b \in B$. Then it is routine to verify that j_B is a nucleus on B . Moreover, $j_B(\perp) = j(\perp) \wedge \neg j(\perp) = \perp$. Thus, j_B is dense. For $a \in A$ we have

$$\begin{aligned} j_B(h(a)) &= j_B(a \wedge \neg j(\perp)) = j(a \wedge \neg j(\perp)) \wedge \neg j(\perp) \\ &= j(a) \wedge j(\neg j(\perp)) \wedge \neg j(\perp) = j(a) \wedge \neg j(\perp) \\ &= h(j(a)). \end{aligned}$$

Therefore, h commutes with j . Moreover, for $j(a), j(b) \in A_j$ we have $h(j(a)) = h(j(b))$ implies $j(a) \wedge \neg j(\perp) = j(b) \wedge \neg j(\perp)$. Thus, $j(j(a) \wedge \neg j(\perp)) = j(j(b) \wedge \neg j(\perp))$. So $j(a) \wedge j(\neg j(\perp)) = j(b) \wedge j(\neg j(\perp))$, and as j is locally dense, we obtain that $j(a) = j(b)$. Therefore, A_j is isomorphic to B_{j_B} . Now $B_{j_B} \in \mathbb{V}$ since $B \in \mathbb{V}$, j_B is a dense nucleus on B , and \mathbb{V} is dense nuclear. Consequently, $A_j \in \mathbb{V}$, implying that \mathbb{V} is locally dense nuclear. \square

It is also clear that every nuclear variety is dense nuclear. The converse though is not true. We already observed at the end of Section 2 that there are cofinal subframe varieties that are not subframe. On the other hand, it is one of our main goals to show that a variety \mathbb{V} of Heyting algebras is a subframe variety iff \mathbb{V} is nuclear, and that \mathbb{V} is a cofinal subframe variety iff \mathbb{V} is dense nuclear, and so the result follows.

4. Duality for Heyting algebras with nuclei

For a nucleus j on a Heyting algebra A and S a subset of A , let

$$j^*(S) = \{a \in A : j(a) \in S\}.$$

Let \mathcal{F} denote the (complete) lattice of filters of A .

Lemma 11. *j^* is a closure operator on \mathcal{F} .*

Proof. First we show that j^* is well-defined. If $F \in \mathcal{F}$ and $a, b \in j^*(F)$, then $j(a), j(b) \in F$. So $j(a \wedge b) = j(a) \wedge j(b) \in F$, and so $a \wedge b \in j^*(F)$. Also, if $a \in j^*(F)$ and $a \leq b$, then $j(a) \in F$ and $j(a) \leq j(b)$. Therefore, $j(b) \in F$, so $b \in j^*(F)$, and so j^* is well-defined. To see that j^* is increasing, $a \in F$ implies $j(a) \in F$. So $a \in j^*(F)$, and so $F \subseteq j^*(F)$. To see that j^* is idempotent, $a \in j^*(j^*(F))$ implies $j(a) \in j^*(F)$. So $j(j(a)) = j(a) \in F$, and so $a \in j^*(F)$. Therefore, $j^*(j^*(F)) \subseteq j^*(F)$. Finally, to see that j^* is order-preserving, if $F \subseteq G$ and $a \in j^*(F)$, then $j(a) \in F \subseteq G$. So $a \in j^*(G)$, and so $F \subseteq G$ implies $j^*(F) \subseteq j^*(G)$. \square

Let \mathcal{F}^* denote the (complete) lattice of fixed points of j^* ; that is $\mathcal{F}^* = \{F \in \mathcal{F} : j^*(F) = F\}$. Let also \mathcal{F}_j denote the (complete) lattice of filters of A_j .

Theorem 12. *\mathcal{F}^* is isomorphic to \mathcal{F}_j .*

Proof. Define $f : \mathcal{F}^* \rightarrow \mathcal{F}_j$ by $f(F) = F \cap A_j$. It is obvious that f is a well-defined order-preserving map. Now define $g : \mathcal{F}_j \rightarrow \mathcal{F}^*$ by $g(G) = j^*(G)$. The same argument as in Lemma 11 implies that g is a well-defined order-preserving map. Moreover, for $F \in \mathcal{F}^*$ and $a \in A$, we have $a \in g(f(F))$ iff $j(a) \in f(F)$ iff $j(a) \in F \cap A_j$ iff $j(a) \in F$ iff $a \in j^*(F) = F$. Therefore, $g(f(F)) = F$ for each $F \in \mathcal{F}^*$. Furthermore, for $G \in \mathcal{F}_j$ and $b \in A_j$ we have $b \in f(g(G))$ iff $b \in g(G) \cap A_j$ iff $b \in g(G)$ iff $b = j(b) \in G$. Thus, $f(g(G)) = G$ for each $G \in \mathcal{F}_j$. It follows that \mathcal{F}^* is isomorphic to \mathcal{F}_j . \square

Let $\mathcal{PF}^* = \{F \in \mathcal{F} : F \text{ is prime and } j^*(F) = F\}$. Let also \mathcal{PF}_j denote the set of prime filters of A_j . Obviously $\mathcal{PF}^* \subseteq \mathcal{F}^*$, $\mathcal{PF}_j \subseteq \mathcal{F}_j$, and both \mathcal{PF}^* and \mathcal{PF}_j are ordered sets.

Corollary 13. \mathcal{PF}^* is isomorphic to \mathcal{PF}_j .

Proof. By Theorem 12 it is sufficient to show that if $F \in \mathcal{PF}^*$, then $f(F) \in \mathcal{PF}_j$, and that if $G \in \mathcal{PF}_j$, then $g(G) \in \mathcal{PF}^*$. For $F \in \mathcal{PF}^*$ and $a, b \in A_j$, if $a \vee_j b \in f(F)$, then $j(a \vee b) \in F \cap A_j$. Therefore, $a \vee b \in j^*(F) = F$, and as F is prime in A , either $a \in F$ or $b \in F$. Thus, either $a \in f(F)$ or $b \in f(F)$, and so $f(F) \in \mathcal{PF}_j$. For $G \in \mathcal{PF}_j$ and $a, b \in A$, if $a \vee b \in g(G)$, then $j(a \vee b) \in G$. As $j(a \vee b) = j(j(a) \vee j(b))$ (see the proof of Proposition 7), $j(a \vee b) = j(a) \vee_j j(b)$. So $j(a \vee b) \in G$ implies $j(a) \vee_j j(b) \in G$, and since G is prime in A_j , either $j(a) \in G$ or $j(b) \in G$. Thus, either $a \in g(G)$ or $b \in g(G)$, and so $g(G) \in \mathcal{PF}^*$. \square

Let \mathbb{NA} denote the category whose objects are (A, j) pairs, where A is a Heyting algebra and j is a nucleus on A , and whose objects are Heyting algebra homomorphisms that commute with j .

We will construct a category dually equivalent to \mathbb{NA} . For a binary relation R on X and $A \subseteq X$ let

$$R[A] = \{x \in X : \exists a \in A \text{ with } aRx\}$$

and

$$R^{-1}[A] = \{x \in X : \exists a \in A \text{ with } xRa\}.$$

If R is a binary relation on X and Q is a binary relation on Y , then a map $f : X \rightarrow Y$ is called a p -morphism if (i) xRy implies $f(x)Qf(y)$ and (ii) $f(x)Qz$ implies there exists y with xRy and $f(y) = z$.

Definition 14. Let \mathbb{NS} denote the category whose objects are (X, \leq, R) triples, where (X, \leq) is a Heyting space and R is a binary relation on X such that

- (1) xRy iff $(\exists z \in X)(zRz \ \& \ x \leq z \leq y)$,
- (2) $R[x]$ is closed for each $x \in X$,
- (3) $A \in \mathcal{CO}(X)$ implies $R^{-1}[A] \in \mathcal{CO}(X)$,

and whose morphisms are continuous p -morphisms with respect to both \leq and R .

Lemma 15. Suppose (X, \leq, R) is an object of \mathbb{NS} . Then the following hold:

- (1) xRy iff $(\exists z)(xRz \ \& \ z \leq y)$ iff $(\exists z)(x \leq z \ \& \ zRy)$.
- (2) xRy implies $x \leq y$.

Proof. (1) We show that xRy iff $(\exists z)(xRz \ \& \ z \leq y)$; that xRy iff $(\exists z)(x \leq z \ \& \ zRy)$ can be shown similarly. If xRy , then $(\exists z \in X)(zRz \ \& \ x \leq z \leq y)$. So $xRz \leq y$. Conversely, suppose $(\exists z)(xRz \ \& \ z \leq y)$. Then $(\exists u)(uRu \ \& \ x \leq u \leq z)$. So $(\exists u)(uRu \ \& \ x \leq u \leq y)$, implying that xRy .

(2) Obvious. \square

Theorem 16. \mathbb{NA} is dually equivalent to \mathbb{NS} .

Proof. For $(A, j) \in \mathbb{NA}$ let $(A, j)_* = (X, \leq, R)$, where $(X, \leq) = A_*$ and xRy iff $j^*(x) \subseteq y$.

Claim 17. $(A, j)_* \in \mathbb{NS}$.

Proof. First we show that xRy iff $(\exists z \in X)(zRz \ \& \ x \leq z \leq y)$. If $(\exists z \in X)(zRz \ \& \ x \leq z \leq y)$, then $j^*(z) = z$ and $x \subseteq z \subseteq y$. So $j^*(x) \subseteq j^*(z) = z \subseteq y$, and so xRy . Conversely, suppose that xRy . Then $j^*(x) \subseteq y$. Let $x_j = x \cap A_j$ and I denote the ideal in A_j generated by $\{j(a) : a \notin y\}$. If there exist $j(a_1), \dots, j(a_n) \in \{j(a) : a \notin y\}$ such that $j(j(a_1) \vee \dots \vee j(a_n)) = j(a_1 \vee \dots \vee a_n) \in x$, then $a_1 \vee \dots \vee a_n \in y$, and as y is prime, one of the a_i 's belongs to y , which is a contradiction. Therefore, $x_j \cap I = \emptyset$. So there exists a prime filter ∇ in A_j such that $x_j \subseteq \nabla$ and $\nabla \cap I = \emptyset$. Let $z = j^*(\nabla)$. By Corollary 13, z is a prime filter of A . Moreover, $x \subseteq j^*(x) = j^*(x_j) \subseteq z = j^*(z) \subseteq y$. So $(\exists z \in X)(zRz \ \& \ x \leq z \leq y)$.

Furthermore, $y \in R[x]$ iff xRy iff $j^*(x) \subseteq y$ iff (for each $a \in A$) $(j(a) \in x \text{ implies } a \in y)$. Therefore, $R[x] = \bigcap \{\varphi(a) : x \in \varphi(j(a))\}$, implying that $R[x]$ is closed for each $x \in X$.

Next we show that $\varphi(j(a)) = -R^{-1} - \varphi(a)$. If $x \in \varphi(j(a))$, then $j(a) \in x$. So xRy implies $a \in y$. Therefore, xRy implies $y \in \varphi(a)$, and so $x \in -R^{-1} - \varphi(a)$. Conversely, if $x \notin \varphi(j(a))$, then $j(a) \notin x$. So $a \notin j^*(x)$. Therefore,

there exists a prime filter y such that $j^*(x) \subseteq y$ and $a \notin y$. Thus, there exists $y \in X$ such that xRy and $y \notin \varphi(a)$, implying that $y \notin -R^{-1} - \varphi(a)$.

Finally, since by Lemma 15 we have $R^{-1}[Y] = R^{-1} \downarrow Y$ for each $Y \subseteq X$, then for $a, b \in A$, we obtain

$$\begin{aligned} R^{-1}[\varphi(a) - \varphi(b)] &= R^{-1} \downarrow (\varphi(a) - \varphi(b)) = R^{-1} - \downarrow (\varphi(a) - \varphi(b)) \\ &= R^{-1} - (\varphi(a) \rightarrow \varphi(b)) = - - R^{-1} - \varphi(a \rightarrow b) \\ &= -\varphi(j(a \rightarrow b)). \end{aligned}$$

Now, if U is clopen, then $U = \bigcup_{k=1}^n (\varphi(a_k) - \varphi(b_k))$ for some a_k 's and b_k 's in A . So

$$\begin{aligned} R^{-1}[U] &= R^{-1} \left[\bigcup_{k=1}^n (\varphi(a_k) - \varphi(b_k)) \right] \\ &= \bigcup_{k=1}^n R^{-1}[\varphi(a_k) - \varphi(b_k)] \\ &= \bigcup_{k=1}^n -\varphi(j(a_k \rightarrow b_k)), \end{aligned}$$

and so $R^{-1}[U]$ is clopen. \square

For $h \in \text{Hom}((A_1, j_1), (A_2, j_2))$ we define $h_* : (A_2)_* \rightarrow (A_1)_*$ by $h_*(x) = h^{-1}(x)$.

Claim 18. $h_* \in \text{Hom}((A_2, j_2)_*, (A_1, j_1)_*)$.

Proof. That $h_* \in \text{Hom}((A_2)_*, (A_1)_*)$ follows from Theorem 1. It is left to be shown that h_* is a p -morphism with respect to R . If xR_2y , then $j_2^*(x) \subseteq y$. So $j_1^*(h^{-1}(x)) = h^{-1}(j_2^*(x)) \subseteq h^{-1}(y)$, and so $h_*(x)R_1h_*(y)$. If $h_*(x)R_1z$, then $j_1^*(h^{-1}(x)) \subseteq z$. Let F be the filter generated by $j_2^*(x) \cup h(z)$ and I be the ideal generated by $A_2 - h(z)$. If $F \cap I \neq \emptyset$, then there exist $a \in j_2^*(x)$, $b \in z$, and $c \notin z$ such that $a \wedge h(b) \leq h(c)$. Therefore, $a \leq h(b \rightarrow c)$, and so $j_2(a) \leq j_2(h(b \rightarrow c)) = h(j_1(b \rightarrow c))$. It follows that $h(j_1(b \rightarrow c)) \in x$, so $j_1(b \rightarrow c) \in h^{-1}(x)$, and so $b \rightarrow c \in j_1^*(h^{-1}(x)) \subseteq z$. This yields $c \in z$, which is a contradiction. Consequently, F and I are disjoint. Therefore, there exists a prime filter y of A_2 with $F \subseteq y$ and $y \cap I = \emptyset$. For this $y \in (A_2)_*$, we have that xR_2y and $h_*(y) = z$. \square

It follows that $(-)_* : \mathbb{NA} \rightarrow \mathbb{NS}$ is a well-defined contravariant functor. Now for $(X, \leq, R) \in \mathbb{NS}$ let $(X, \leq, R)^* = (X^*, -R^{-1}-)$.

Claim 19. $(X, \leq, R)^* \in \mathbb{NA}$.

Proof. That X^* is a Heyting algebra follows from Theorem 1, and that $-R^{-1}-$ is a nucleus on X^* follows from Definition 14 and Lemma 15. \square

For $f \in \text{Hom}((X_1, \leq_1, R_1), (X_2, \leq_2, R_2))$ we define $f^* : (X_2)^* \rightarrow (X_1)^*$ by $f^*(U) = f^{-1}(U)$.

Claim 20. $f^* \in \text{Hom}((X_2, \leq_2, R_2)^*, (X_1, \leq_1, R_1)^*)$.

Proof. That $f^* \in \text{Hom}(X_2^*, X_1^*)$ follows from Theorem 1, and that f^* preserves $-R^{-1}-$ follows from f being a p -morphism with respect to R . \square

It follows that $(-)^* : \mathbb{NS} \rightarrow \mathbb{NA}$ is a well-defined contravariant functor.

Claim 21. $(-)_*$ and $(-)^*$ yield a dual equivalence between \mathbb{NA} and \mathbb{NS} .

Proof. For $(A, j) \in \mathbb{NA}$, it follows from Theorem 1 that $\varphi : A \rightarrow A_*^*$ is a \mathbb{HA} -isomorphism, and it follows from the proof of Claim 17 that φ preserves j . Thus, (A, j) is isomorphic to $(A, j)_*^*$. For $(X, \leq, R) \in \mathbb{NS}$, it follows from Theorem 1 that $\psi : X \rightarrow X_*^*$ defined by $\psi(x) = \{U \in X^* : x \in U\}$ is a \mathbb{HS} -isomorphism. We show that xRy iff $\psi(x)R_*^*\psi(y)$. Indeed, $\psi(x)R_*^*\psi(y)$ iff $\{U \in X^* : -R^{-1}-U \in \psi(x)\} \subseteq \psi(y)$ iff $\{U \in X^* : x \in -R^{-1}-U\} \subseteq \{U \in X^* : y \in U\}$. We show that the last condition is equivalent to xRy . If xRy and

$x \in -R^{-1} - U$, then clearly $y \in U$. Conversely, if xRy , then $y \notin R[x]$. So $R[x]$ is a closed upset not containing y , and so there exists $U \in X^*$ with $R[x] \subseteq U$ and $y \notin U$. Therefore, there exists $U \in X^*$ with $x \in -R^{-1} - U$ and $y \notin U$. Thus, xRy iff $\psi(x)R^*_*\psi(y)$, and so (X, \leq, R) is NS-isomorphic to $(X, \leq, R)^*_*$. \square

This finishes the proof. \square

Let \mathbb{DNA} denote the subcategory of \mathbb{NA} of those (A, j) pairs, where j is a dense nucleus on A , and let \mathbb{LDNA} denote the subcategory of \mathbb{NA} of those (A, j) , where j is a locally dense nucleus on A . Clearly \mathbb{DNA} is a proper subcategory of \mathbb{LDNA} .

Let also \mathbb{DNS} denote the subcategory of \mathbb{NS} of those (X, \leq, R) triples, where for each $x \in X$ there exists $y \in X$ with $x \leq yRy$, and let \mathbb{LDNS} denote the subcategory of \mathbb{NS} of those (X, \leq, R) triples, where for each $x, y \in X$ with $xRx \leq y$ there exists $z \in X$ with $y \leq zRz$. Clearly \mathbb{DNS} is a proper subcategory of \mathbb{LDNS} .

Lemma 22. *Let $(X, \leq, R) \in \mathbb{NA}$.*

(1) *The following conditions are equivalent:*

- (i) $(\forall x \in X)(\exists y \in X)(x \leq yRy)$.
- (ii) $(\forall x \in \max X)(xRx)$.
- (iii) $R^{-1}[X] = X$.

(2) *The following conditions are equivalent:*

- (i) $(\forall x, y \in X)(xRx \leq y \rightarrow \exists z \in X : y \leq zRz)$.
- (ii) $(\forall x, y \in X)(xRx \leq y \rightarrow \exists z \in \max X : y \leq zRz)$.
- (iii) $-R^{-1} - R^{-1}[X] = X$.

Proof. (1) (i) \Rightarrow (ii) Suppose $x \in \max X$. By (i), there exists $y \in X$ with $x \leq yRy$. But $x \in \max X$ implies $x = y$. Thus, xRx . (ii) \Rightarrow (iii) Clearly $R^{-1}[X] \subseteq X$. Suppose $x \in X$. Then there exists $y \in \max X$ with $x \leq y$. By (ii), yRy . Thus, xRy , and so $x \in R^{-1}[\max X] \subseteq R^{-1}[X]$. (iii) \Rightarrow (i) Suppose $x \in X$. By (iii), there exists $z \in X$ with xRz . Since $(X, \leq, R) \in \mathbb{NA}$, there exists $y \in X$ with $x \leq yRy \leq z$.

(2) (i) \Rightarrow (ii) Suppose $x, y \in X$ with $xRx \leq y$. Then there exists $z \in \max X$ with $y \leq z$. So $xRx \leq z$, and by (i), there exists $w \in X$ with $z \leq wRw$. But $z = w$ as $z \in \max X$. Therefore, $y \leq zRz$. (ii) \Rightarrow (iii) Clearly $-R^{-1} - R^{-1}[X] \subseteq X$. Suppose $x \in X$ and xRy . Then there exists $z \in X$ with $x \leq zRz \leq y$. From $zRz \leq y$, by (ii), it follows that there exists $w \in \max X$ with $y \leq wRw$. Thus, $y \in R^{-1}[\max X] \subseteq R^{-1}[X]$, and so $x \in -R^{-1} - R^{-1}[X]$. (iii) \Rightarrow (i) Suppose $x, y \in X$ with $xRx \leq y$. Then $x \in X$ and xRy , so by (iii), $y \in R^{-1}[X]$. Therefore, there exists $w \in X$ with yRw . Thus, there exists $z \in X$ with $y \leq zRz \leq w$. \square

Theorem 23.

(1) \mathbb{DNA} is dually equivalent to \mathbb{DNS} .

(2) \mathbb{LDNA} is dually equivalent to \mathbb{LDNS} .

Proof. (1) It is sufficient to show that from $(A, j) \in \mathbb{DNA}$ it follows that $(A, j)^*_* \in \mathbb{DNS}$, and that from $(X, \leq, R) \in \mathbb{DNS}$ it follows that $(X, \leq, R)^* \in \mathbb{DNA}$. If $(A, j) \in \mathbb{DNA}$, then $j(\perp) = \perp$. Let $(A, j)^*_* = (X, \leq, R)$. Then

$$\begin{aligned} R^{-1}[X] &= R^{-1}[-\emptyset] = R^{-1} - \varphi(\perp) = - - R^{-1} - \varphi(\perp) \\ &= -\varphi(j(\perp)) = -\varphi(\perp) = -\emptyset = X. \end{aligned}$$

Thus, $(A, j)^*_* \in \mathbb{DNS}$ by Lemma 22(1). Conversely, if $(X, \leq, R) \in \mathbb{DNS}$, then again by Lemma 22(1),

$$-R^{-1} - \emptyset = -R^{-1}[X] = -X = \emptyset.$$

So $(X, \leq, R)^* \in \mathbb{DNA}$.

(2) It is sufficient to show that from $(A, j) \in \mathbb{LDNA}$ it follows that $(A, j)_* \in \mathbb{LDNS}$, and that from $(X, \leq, R) \in \mathbb{LDNS}$ it follows that $(X, \leq, R)^* \in \mathbb{LDNA}$. If $(A, j) \in \mathbb{LDNA}$, then $j(\neg j(\perp)) = \top$. Let $(A, j)_* = (X, \leq, R)$. Since $R^{-1}[Y] = R^{-1} \downarrow Y$ for each $Y \subseteq X$, we have

$$\begin{aligned} -R^{-1} - R^{-1}[X] &= -R^{-1} - R^{-1}[-\emptyset] = -R^{-1} - R^{-1} - \varphi(\perp) \\ &= -R^{-1}[\varphi(j(\perp))] = -R^{-1} \downarrow \varphi(j(\perp)) \\ &= -R^{-1} - - \downarrow \varphi(j(\perp)) = -R^{-1} - \varphi(\neg j(\perp)) \\ &= \varphi(j(\neg j(\perp))) = \varphi(\top) = X. \end{aligned}$$

Thus, $(A, j)_* \in \mathbb{LDNS}$ by Lemma 22(2). Conversely, suppose $(X, \leq, R) \in \mathbb{LDNS}$. Then again by Lemma 22(2),

$$\begin{aligned} \varphi(j(\neg j(\perp))) &= -R^{-1} - \varphi(\neg j(\perp)) = -R^{-1} - - \downarrow \varphi(j(\perp)) \\ &= -R^{-1} \downarrow \varphi(j(\perp)) = -R^{-1} \varphi(j(\perp)) \\ &= -R^{-1} - R^{-1} - \varphi(\perp) = -R^{-1} - R^{-1}[-\emptyset] \\ &= -R^{-1} - R^{-1}[X] = X = \varphi(\top). \end{aligned}$$

So $(X, \leq, R)^* \in \mathbb{LDNA}$. \square

Remark 24. We conclude this section by comparing our triples (X, \leq, R) to Goldblatt frames [13]. A *Goldblatt frame* is a triple (X, \leq, R) such that (X, \leq) is a poset and R is a binary relation on X with (i) $x \leq yRz \Rightarrow xRz$, (ii) $xRy \Rightarrow x \leq y$, and (iii) $xRy \Rightarrow (\exists z \in X)(xRzRy)$. It follows from Definition 14 and Lemma 15 that our triples are Goldblatt frames. The converse however is not true in general. Let \mathbf{Q} denote the set of rational numbers. Then $(\mathbf{Q}, \leq, <)$ is a Goldblatt frame, but it does not satisfy (1) of Definition 14 because for no $r \in \mathbf{Q}$ we have that $r < r$. On the other hand, if every Goldblatt frame is a Heyting space that satisfies (2) and (3) of Definition 14, then it is one of our triples. To see this it is sufficient to verify (1) of Definition 14. Let xRy . We show that there exists z with $x \leq zRz \leq y$. If xRx , then we let $z = x$, and by (ii) obtain that $x \leq xRx \leq y$. Suppose $x \not R x$, and consider a maximal R -chain C in the interval $[x, y]$. Then the family $\{R[x], R^{-1}[c]\}_{c \in C}$ has the finite intersection property, and by compactness of X , there exists $z \in R[x] \cap \bigcap_{c \in C} R^{-1}[c]$. Obviously $xRzRy$. By (iii), there exists w such that $xRwRz$. Since C is a maximal chain and $x \not R x$, we have $w = z$. Thus, zRz and by (ii) we have $x \leq zRz \leq y$.

5. Subframe = nuclear

For $(A, j) \in \mathbb{NA}$ let X denote the dual space of A , X_j denote the dual space of A_j , and $X^* = \{x \in X : j^*(x) = x\}$. We view X^* as a subspace of X with the subspace topology.

Lemma 25. X^* is homeomorphic to X_j .

Proof. Let $f : X^* \rightarrow X_j$ and $g : X_j \rightarrow X^*$ be the same as in Theorem 12. It follows from Corollary 13 that f and g establish a 1–1 correspondence between X^* and X_j . It remains to show that both f and g are continuous. For $a \in A_j$ let $\varphi_j(a) = \{x \in X_j : a \in x\}$.

Claim 26. For each $a \in A$ we have that $f^{-1}(\varphi_j(j(a))) = \varphi(a) \cap X^*$.

Proof. Since $j^*(x) = x$ for each $x \in X^*$, we have:

$$\begin{aligned} x \in f^{-1}(\varphi_j(j(a))) &\quad \text{iff} \quad f(x) \in \varphi_j(j(a)) \text{ iff } j(a) \in x \cap A_j \\ &\quad \text{iff} \quad a \in x \text{ iff } x \in \varphi(a) \cap X^*. \quad \square \end{aligned}$$

Let U be a basic open of X_j . Then $U = \bigcup_{k=1}^n (\varphi_j(j(a_k)) - \varphi_j(j(b_k)))$ for some a_k 's and b_k 's in A . Therefore,

$$\begin{aligned}
f^{-1}(U) &= f^{-1}\left(\bigcup_{k=1}^n (\varphi_j(j(a_k)) - \varphi_j(j(b_k)))\right) \\
&= \bigcup_{k=1}^n f^{-1}(\varphi_j(j(a_k)) - \varphi_j(j(b_k))) \\
&= \bigcup_{k=1}^n (f^{-1}(\varphi_j(j(a_k))) - f^{-1}(\varphi_j(j(b_k)))) \\
&= \bigcup_{k=1}^n ((\varphi(a_k) \cap X^*) - (\varphi(b_k) \cap X^*)),
\end{aligned}$$

which is open in X^* . Thus, f is continuous.

Claim 27. For each $a \in A$ we have that $g^{-1}(\varphi(a) \cap X^*) = \varphi_j(j(a))$ and that $g^{-1}(X^* - \varphi(a)) = X_j - \varphi_j(j(a))$.

Proof. For $y \in X_j$ we have that

$$\begin{aligned}
y \in g^{-1}(\varphi(a) \cap X^*) &\quad \text{iff} \quad g(y) \in \varphi(a) \cap X^* \text{ iff } j^*(y) \in \varphi(a) \cap X^* \\
&\quad \text{iff} \quad a \in j^*(y) \text{ iff } j(a) \in y \text{ iff } y \in \varphi_j(j(a)),
\end{aligned}$$

and that

$$\begin{aligned}
y \in g^{-1}(X^* - \varphi(a)) &\quad \text{iff} \quad g(y) \in X^* - \varphi(a) \text{ iff } j^*(y) \notin \varphi(a) \\
&\quad \text{iff} \quad a \notin j^*(y) \text{ iff } j(a) \notin y \text{ iff } y \notin \varphi_j(j(a)) \\
&\quad \text{iff} \quad y \in X_j - \varphi_j(j(a)). \quad \square
\end{aligned}$$

Now let V be a basic open of X^* . Then there exist a_k 's and b_k 's in A such that

$$V = \bigcup_{k=1}^n (\varphi(a_k) - \varphi(b_k)) \cap X^* = \bigcup_{k=1}^n ((\varphi(a_k) \cap X^*) \cap (X^* - \varphi(b_k))).$$

Therefore,

$$\begin{aligned}
g^{-1}(V) &= g^{-1}\left(\bigcup_{k=1}^n ((\varphi(a_k) \cap X^*) \cap (X^* - \varphi(b_k)))\right) \\
&= \bigcup_{k=1}^n g^{-1}((\varphi(a_k) \cap X^*) \cap (X^* - \varphi(b_k))) \\
&= \bigcup_{k=1}^n (g^{-1}(\varphi(a_k) \cap X^*) \cap g^{-1}(X^* - \varphi(b_k))) \\
&= \bigcup_{k=1}^n (\varphi_j(j(a_k)) \cap (X_j - \varphi_j(j(b_k)))) \\
&= \bigcup_{k=1}^n (\varphi_j(j(a_k)) - \varphi_j(j(b_k))),
\end{aligned}$$

which is open in X_j . Thus, g is continuous. \square

Theorem 28. Suppose X is a Heyting space.

- (1) There is a 1–1 correspondence between subframes of X and binary relations $R \subseteq X^2$ such that $(X, \leq, R) \in \mathbf{NS}$.
- (2) There is a 1–1 correspondence between cofinal subframes of X and $R \subseteq X^2$ such that $(X, \leq, R) \in \mathbf{LDNS}$.

Proof. (1) For a subframe S of X we define R_S on X by xR_Sy iff $(\exists s \in S)(x \leq s \leq y)$. We show that $(X, \leq, R_S) \in \mathbb{NS}$. Since $s \in S$ iff sR_Ss , we have that xR_Sy iff $(\exists s \in X)(x \leq sR_Ss \leq y)$. Moreover, $y \in R_S[x]$ iff xR_Sy iff $(\exists s \in S)(x \leq s \leq y)$ iff $(\exists s \in S)(s \in \uparrow x \ \& \ y \in \uparrow s)$ iff $y \in \uparrow(S \cap \uparrow x)$. Therefore, $R_S[x] = \uparrow(S \cap \uparrow x)$, and so $R_S[x]$ is closed for each $x \in X$. Furthermore, for $A \in \mathcal{CO}(X)$, $x \in R_S^{-1}[A]$ iff $(\exists y \in A)(xR_Sy)$ iff $(\exists y \in A)(\exists s \in S)(x \leq s \leq y)$ iff $x \in \downarrow(S \cap \downarrow A)$. Therefore, $R_S^{-1}[A] = \downarrow(S \cap \downarrow A)$, and as $S \cap \downarrow A$ is a clopen downset of S , we obtain that $R_S^{-1}[A] \in \mathcal{CO}(X)$. It follows that $(X, \leq, R_S) \in \mathbb{NS}$.

For $(X, \leq, R) \in \mathbb{NS}$ let $(A, j) = (X, \leq, R)^*$ and let X_j be the dual space of A_j . We set $S_R = \{x \in X : xR_Sx\}$, and equip S_R with the subspace topology. It follows from Lemma 25 that S_R is homeomorphic to X_j . Thus, S_R is a closed subspace of X . Moreover, since S_R is a Heyting space, $A \in \mathcal{CO}(S_R)$ implies $\downarrow A \cap S_R \in \mathcal{CO}(S_R)$. Let B be a clopen downset of X such that $\downarrow A \cap S_R = B \cap S_R$. Then $\downarrow A = \downarrow(\downarrow A \cap S_R) = \downarrow(B \cap S_R) = \downarrow(\downarrow B \cap S_R) = R^{-1}[B] \in \mathcal{CO}(X)$. Thus, by Lemma 2, S_R is a subframe of X .

For $x, y \in X$, $xR_{S_R}y$ iff $(\exists s \in S_R)(x \leq s \leq y)$ iff $(\exists s \in X)(x \leq sR_Ss \leq y)$ iff xR_Sy . Therefore, $R = R_{S_R}$. Finally, $x \in S_{R_S}$ iff xR_Sx iff $(\exists s \in S)(x \leq s \leq x)$ iff $(\exists s \in S)(x = s)$ iff $x \in S$. Thus, $S = S_{R_S}$. This completes the proof of (1).

(2) It is sufficient to show that if S is a cofinal subframe of X , then $(X, \leq, R_S) \in \mathbb{LDNS}$, and that if $(X, \leq, R) \in \mathbb{LDNS}$, then S_R is cofinal in X . For the former, suppose S is a cofinal subframe of X and $xR_Sx \leq y$. Then $x \in S$ and $x \leq y$. Since S is cofinal, there exists $z \in S$ with $y \leq z$. So $y \leq zR_Sz$, and so $(X, \leq, R_S) \in \mathbb{LDNS}$. For the latter, suppose $x \in S_R$ and $x \leq y$. Then $xR_Sx \leq y$. Since $(X, \leq, R) \in \mathbb{LDNS}$, there exists $z \in X$ with $y \leq zRz$. Because zRz , we have $z \in S_R$. Therefore, there exists $z \in S_R$ with $y \leq z$, implying that S_R is cofinal in X . \square

Corollary 29. Suppose $\mathbb{V} \subseteq \mathbb{HA}$ is a variety of Heyting algebras.

- (1) \mathbb{V} is subframe iff \mathbb{V} is nuclear.
- (2) The following conditions are equivalent:
 - (i) \mathbb{V} is cofinal subframe.
 - (ii) \mathbb{V} is locally dense nuclear.
 - (iii) \mathbb{V} is dense nuclear.

Proof. (1) For a Heyting algebra A and a nucleus j on A , let X denote the dual space of A and X_j denote the dual space of A_j . Also, for $R \subseteq X^2$ such that $(X, \leq, R) \in \mathbb{NS}$, let j_R denote the corresponding nucleus on A . Then using Theorems 16 and 28(1) we obtain that \mathbb{V} is subframe iff for each $A \in \mathbb{V}$ and each subframe S of X we have $S^* \in \mathbb{V}$ iff for each $A \in \mathbb{V}$ and each $R \subseteq X^2$ such that $(X, \leq, R) \in \mathbb{NS}$ we have $A_{j_R} \in \mathbb{V}$ iff for each $A \in \mathbb{V}$ and each nucleus j on A we have $A_j \in \mathbb{V}$ iff \mathbb{V} is nuclear.

(2) That (i) is equivalent to (ii) follows along the same lines as (1), but uses Theorems 23(2) and 28(2) instead of Theorems 16 and 28(1), respectively. That (ii) is equivalent to (iii) follows from Theorem 10. \square

6. The lattice $N(A)$

In this section we investigate the lattice $N(A)$ of nuclei of a Heyting algebra A using the duality developed in the previous two sections. We obtain rather simple dual proofs of several useful theorems about $N(A)$. Some of these are well known from the literature, but our approach appears to be novel.

Let A be a Heyting algebra and $N(A)$ denote the set of all nuclei on A . We partially order $N(A)$ by

$$j \leq k \quad \text{iff} \quad j(a) \leq k(a) \quad \text{for each } a \in A.$$

Then $N(A)$ is a bounded meet semilattice, where $(j \wedge k)(a) = j(a) \wedge k(a)$, $\top(a) = \top$, and $\perp(a) = a$ for each $a \in A$ (see [3, Section 3] and [19, Section 2]).

Let X be the dual space of A and let $SF(X)$ denote the set of all subframes of X . Obviously $SF(X)$ is partially ordered by set inclusion.

Theorem 30. $N(A)$ is dually isomorphic to $SF(X)$.

Proof. For each $j \in N(A)$, let $S_j = \{x \in X : j^*(x) = x\}$. By Theorem 28(1), $S_j \in SF(X)$ and $j \mapsto S_j$ establishes a 1–1 correspondence between $N(A)$ and $SF(X)$. To see that this 1–1 correspondence is a dual isomorphism, let $j \leq k$ and $x \in S_k$. Then $k^*(x) = x$. If $a \in j^*(x)$, then $j(a) \in x$. So $k(a) \in x$, and so $a \in x$. Therefore, $j^*(x) = x$,

implying that $x \in S_j$. Thus, $S_k \subseteq S_j$. Conversely, suppose $S_k \subseteq S_j$. We identify A with X^* . Then for $Y \in X^*$ we have $S_k \cap \downarrow - Y \subseteq S_j \cap \downarrow - Y$. So $\downarrow(S_k \cap \downarrow - Y) \subseteq \downarrow(S_j \cap \downarrow - Y)$, and so $-\downarrow(S_j \cap \downarrow - Y) \subseteq -\downarrow(S_k \cap \downarrow - Y)$. Thus, $j \leq k$, and $N(A)$ is dually isomorphic to $SF(X)$. \square

Corollary 31. *Under the above dual isomorphism we have:*

- (i) $S_{j \wedge k} = S_j \cup S_k$.
- (ii) $S_{\top} = \emptyset$.
- (iii) $S_{\perp} = X$.

Proof. (i) It is easy to verify that $(j \wedge k)^*(x) = j^*(x) \cap k^*(x)$ for each $x \in X$. Therefore, $x \in S_{j \wedge k}$ iff $(j \wedge k)^*(x) = x$ iff $j^*(x) \cap k^*(x) = x$ iff $j^*(x) = x$ or $k^*(x) = x$ iff $x \in S_j \cup S_k$. Thus, $S_{j \wedge k} = S_j \cup S_k$.

(ii) That $S_{\top} = \emptyset$ follows from $\top^*(x) \neq x$ for each $x \in X$.

(iii) That $S_{\perp} = X$ follows from $\perp^*(x) = x$ for each $x \in X$. \square

Theorem 32. *Under the above dual isomorphism, clopens of X correspond to complemented elements of $N(A)$.*

Proof. Let S be a clopen subset of X . Then $-S$ is also clopen. We consider the nuclei j_S and j_{-S} on X^* . If $j_T \subseteq j_S$, j_{-S} , then $S, -S \subseteq T$, so $T = X$, and so j_T corresponds to \perp ; and if $j_S, j_{-S} \subseteq j_T$, then $T \subseteq S, -S$, so $T = \emptyset$, and so j_T corresponds to \top . Thus, j_S and j_{-S} correspond to complemented elements of $N(A)$. \square

As an immediate consequence of [Theorem 32](#) we obtain the following corollary.

Corollary 33.

- (1) $N(A)$ is a Boolean algebra iff $SF(X) = \mathcal{CO}(X)$.¹
- (2) If A is finite, then $N(A)$ is a Boolean algebra.²

Proof. (1) $N(A)$ is a Boolean algebra iff every element of $N(A)$ is complemented iff every subframe of X is clopen iff $SF(X) = \mathcal{CO}(X)$.

(2) If A is finite, then $SF(X) = \mathcal{CO}(X)$, and so $N(A)$ is a Boolean algebra by (1). \square

The nuclei $u_a = a \vee (-)$, $v_a = a \rightarrow (-)$, and $w_a = ((-) \rightarrow a) \rightarrow a$ play an important role in the algebraic theory of nuclei [3,19,16]. In the next theorem we give their dual characterization.

Theorem 34.

- (1) $S_{u_a} = -\varphi(a)$.
- (2) $S_{v_a} = \varphi(a)$.
- (3) u_a and v_a are complemented elements of $N(A)$.³
- (4) $S_{w_a} = \max(-\varphi(a))$.

Proof. (1) Observe that $x \in S_{u_a}$ iff $u_a^*(x) = x$ iff $(\forall b)(a \vee b \in x \Rightarrow b \in x)$ iff $a \notin x$ iff $x \in -\varphi(a)$. Therefore, $S_{u_a} = -\varphi(a)$.

(2) Observe that $x \in S_{v_a}$ iff $v_a^*(x) = x$ iff $(\forall b)(a \rightarrow b \in x \Rightarrow b \in x)$ iff $a \in x$ iff $x \in \varphi(a)$. Therefore, $S_{v_a} = \varphi(a)$.

(3) Follows immediately from (1), (2), and [Theorem 32](#).

(4) Observe that $x \in S_{w_a}$ iff $w_a^*(x) = x$ iff $(\forall b)((b \rightarrow a) \rightarrow a \in x \Rightarrow b \in x)$. We show that the last condition is equivalent to $x \in \max(-\varphi(a))$. Suppose $x \in \max(-\varphi(a))$ and $(b \rightarrow a) \rightarrow a \in x$. We need to show that $b \in x$. For each $y \in X$, from $x \leq y$ and $y \in \varphi(b \rightarrow a)$ it follows that $y \in \varphi(a)$. Thus, $x \notin \varphi(a)$ implies $x \notin \varphi(b \rightarrow a)$. Therefore, there exists $z \in X$ with $x \leq z$, $z \in \varphi(b)$, and $z \notin \varphi(a)$. From $x \in \max(-\varphi(a))$ it follows that $x = z$. Thus, $b \in x$. Conversely, suppose $x \notin \max(-\varphi(a))$. We need to find $b \in A$ such that $(b \rightarrow a) \rightarrow a \in x$, but $b \notin x$.

¹ For complete A , a purely algebraic criterion for $N(A)$ to be a Boolean algebra can be found in [3, P. 7, Thm. 2] (see also [19, P. 17, Thm. 3.9]). However, that criterion cannot be generalized to arbitrary Heyting algebras.

² For a purely algebraic proof see [19, P. 11, Cor. 2].

³ For an algebraic proof see [3, P. 5, Lem. 8].

If $x \in \varphi(a)$, then $(\perp \rightarrow a) \rightarrow a = a \in x$, but $\perp \notin x$. Therefore, we can assume that $x \in -\varphi(a)$. But then since $x \notin \max(-\varphi(a))$, there exists $b \in A$ such that $x \notin \varphi(b)$ and $\max(-\varphi(a)) \subseteq \varphi(b)$. We claim $x \in \varphi((b \rightarrow a) \rightarrow a)$. Suppose $x \leq y$ and $y \in \varphi(b \rightarrow a)$. If $y \notin \varphi(a)$, then there exists $z \in \max(-\varphi(a))$ with $y \leq z$. So $z \in \varphi(b)$ and $z \notin \varphi(a)$, which implies that $y \notin \varphi(b \rightarrow a)$, a contradiction. Therefore, $y \in \varphi(a)$. So $x \in \varphi((b \rightarrow a) \rightarrow a)$ and $x \notin \varphi(b)$. Thus, there exists $b \in A$ with $(b \rightarrow a) \rightarrow a \in x$, but $b \notin x$. Consequently, $x \in S_{w_a}$ iff $x \in \max(-\varphi(a))$, and so $S_{w_a} = \max(-\varphi(a))$. \square

We conclude this section by giving simple dual proofs of several well-known theorems about nuclei on Boolean algebras. For algebraic proofs see [3, P. 5, Cor. 1] and [19, P. 8, Cor. 1].

Theorem 35. *Let B be a Boolean algebra.*

- (1) $N(B)$ is isomorphic to B .
- (2) Every nucleus on B is of the form $u_{j(\perp)}$.
- (3) The only dense nucleus on B is the identity map.

Proof. (1) Let B be a Boolean algebra with the dual Stone space X , and let j be a nucleus on B . Since \leq is simply $=$ on X , we have that $SF(X) = \mathcal{CO}(X)$. Therefore, as B is isomorphic to $\mathcal{CO}(X)$ and $N(B)$ is isomorphic to $SF(X)$, we obtain that $N(B)$ is isomorphic to B .

(2) Since $R \subseteq \{(x, x) : x \in X\}$, we have $S_R = R^{-1}[X]$, and so

$$-R^{-1} - \emptyset = -R^{-1}[X] = -S_R.$$

Thus, for $a \in B$ we have

$$\begin{aligned} \varphi(j(a)) &= -R^{-1} - \varphi(a) = -\downarrow(S_R \cap \downarrow - \varphi(a)) \\ &= -(S_R \cap -\varphi(a)) = -S_R \cup \varphi(a) = -R^{-1} - \emptyset \cup \varphi(a) \\ &= -R^{-1} - \varphi(\perp) \cup \varphi(a) = \varphi(j(\perp)) \cup \varphi(a) = \varphi(j(\perp) \vee a). \end{aligned}$$

Therefore, $j(a) = a \vee j(\perp)$, and so every nucleus on B is of the form $u_{j(\perp)}$.

(3) If j is dense, then by (2), $j(a) = a \vee j(\perp) = a \vee \perp = a$. \square

7. The finite model property

In this section we show that every nuclear and dense nuclear variety is generated by its finite members. It will follow that every subframe and cofinal subframe variety is generated by its finite members.

We already know that if M is an implicative semilattice and $a \in M$, then $w_a = ((-) \rightarrow a) \rightarrow a$ is a nucleus on M , and that if j and k are two nuclei on $M \in \mathbb{M}$, then $j \wedge k$ is also a nucleus on M . Moreover, if both j and k are dense, then $(j \wedge k)(\perp) = j(\perp) \wedge k(\perp) = \perp$. So $j \wedge k$ is also dense.

Proposition 36. *Let A be a Heyting algebra and let $M \hookrightarrow A$ be a finite implicative subsemilattice of A . Define $j : A \rightarrow A$ by*

$$j(a) = \bigwedge_{m \in M} ((a \rightarrow m) \rightarrow m) = \bigwedge_{m \in M} w_m(a).$$

Then (i) j is a nucleus on A ; (ii) $j(m) = m$ for each $m \in M$; (iii) j is dense iff $\perp \in M$; (iv) M is a Heyting subalgebra of A_j .

Proof. (i) and (ii) are obvious; for (iii), $j(\perp) = \perp$ iff $\bigwedge M = \perp$ iff $\perp \in M$; for (iv), it follows from (ii) that $M \subseteq A_j$. Since M is an implicative semilattice, $\top \in M$ and M is closed under \wedge and \rightarrow . For $a, b \in M$ observe that

$$\begin{aligned} j(a \vee b) &= \bigwedge_{m \in M} (((a \vee b) \rightarrow m) \rightarrow m) \\ &= \bigwedge_{m \in M} (((a \rightarrow m) \wedge (b \rightarrow m)) \rightarrow m) \in M \quad \text{as } a \rightarrow m, b \rightarrow m \in M. \end{aligned}$$

So M is closed under \vee_j . Also, $j(\perp) = \bigwedge M \in M$. Therefore, $\perp_j \in M$, and so M is a Heyting subalgebra of A_j . \square

Corollary 37.

- (1) If \mathbb{V} is a nuclear variety, $A \in \mathbb{V}$, and M is a finite implicative subsemilattice of A , then $M \in \mathbb{V}$.
 (2) If \mathbb{V} is a dense nuclear variety, $A \in \mathbb{V}$, and M is a finite implicative subsemilattice of A such that $\perp \in M$, then $M \in \mathbb{V}$.

Proof. (1) By Proposition 36, there is a nucleus j on A such that M is a Heyting subalgebra of A_j . Since \mathbb{V} is nuclear, $A_j \in \mathbb{V}$. Therefore, $M \in \mathbb{V}$.

(2) Let j be the same as above. Since $\perp \in M$, by Proposition 36, j is a dense nucleus on A and M is a Heyting subalgebra of A_j . Since \mathbb{V} is dense nuclear, $A_j \in \mathbb{V}$. Therefore, $M \in \mathbb{V}$. \square

Theorem 38.

- (1) If \mathbb{V} is a nuclear variety, then \mathbb{V} is generated by its finite members.
 (2) If \mathbb{V} is a dense nuclear variety, then \mathbb{V} is generated by its finite members.

Proof. (1) is a consequence of (2).

(2) Suppose $p(t_1, \dots, t_n) = q(s_1, \dots, s_m)$ is not an identity of \mathbb{V} . Then there exist $A \in \mathbb{V}$ and an assignment ν of t_1, \dots, t_n and s_1, \dots, s_m into A such that $\nu(p(t_1, \dots, t_n)) \neq \nu(q(s_1, \dots, s_m))$. So

$$p^A(\nu(t_1), \dots, \nu(t_n)) \neq q^A(\nu(s_1), \dots, \nu(s_m)).$$

Let M denote the implicative subsemilattice of A generated by \perp and all the subpolynomials of $p^A(\nu(t_1), \dots, \nu(t_n))$ and $q^A(\nu(s_1), \dots, \nu(s_m))$. By Diego's Theorem, M is a finite implicative subsemilattice of A such that $\perp \in M$. By Corollary 37(2), $M \in \mathbb{V}$. Moreover, if for $a, b \in M$ we have $a \vee b \in M$, then by Proposition 36,

$$a \vee_M b = a \vee_j b = j(a \vee b) = a \vee b.$$

Therefore, $p^M(\nu(t_1), \dots, \nu(t_n)) \neq q^M(\nu(s_1), \dots, \nu(s_m))$, and so $p(t_1, \dots, t_n) = q(s_1, \dots, s_m)$ is refuted on a finite member of \mathbb{V} . \square

Corollary 39 ([11,25]).

- (1) If \mathbb{V} is a subframe variety, then \mathbb{V} is generated by its finite members.
 (2) If \mathbb{V} is a cofinal subframe variety, then \mathbb{V} is generated by its finite members.

Proof. Apply Corollary 29 and Theorem 38. \square

8. Another characterization

In this section we give an algebraic proof of Zakharyashev's theorem that $\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a subframe variety iff \mathbb{V} is axiomatized by (\wedge, \rightarrow) -identities, and that \mathbb{V} is a cofinal subframe variety iff \mathbb{V} is axiomatized by $(\perp, \wedge, \rightarrow)$ -identities.

For $K \subseteq \mathbb{H}\mathbb{A}$ a class of Heyting algebras, let $\mathbf{H}(K)$ denote the class of all homomorphic images and $\mathbf{S}(K)$ denote the class of all subalgebras of the members of K . Also, let $\mathbf{H}^{(\wedge, \rightarrow)}(K)$ and $\mathbf{S}^{(\wedge, \rightarrow)}(K)$ denote the classes of all (\wedge, \rightarrow) -homomorphic images and (\wedge, \rightarrow) -subalgebras of the members of K , and $\mathbf{H}^{(\perp, \wedge, \rightarrow)}(K)$ and $\mathbf{S}^{(\perp, \wedge, \rightarrow)}(K)$ denote the classes of all $(\perp, \wedge, \rightarrow)$ -homomorphic images and $(\perp, \wedge, \rightarrow)$ -subalgebras of the members of K . Since (\wedge, \rightarrow) -homomorphic images of implicative semilattices are determined by filters [21,18], we have that $\mathbf{H}(K) = \mathbf{H}^{(\wedge, \rightarrow)}(K) = \mathbf{H}^{(\perp, \wedge, \rightarrow)}(K)$.

Now we give a slight generalization of the Jankov formulas [14,24]. Let A be a Heyting algebra. It is well known (see, e.g., [1, P. 179, Thm. 5]) that A is subdirectly irreducible iff $A - \{\top\}$ has a greatest element, denoted by c . Suppose A is a finite subdirectly irreducible Heyting algebra. With each $a \in A$ we associate a propositional letter p_a , and define

$$\chi(A) = \left[\bigwedge_{a,b \in A} (p_{a \wedge b} \leftrightarrow (p_a \wedge p_b)) \wedge \bigwedge_{a,b \in A} (p_{a \rightarrow b} \leftrightarrow (p_a \rightarrow p_b)) \right] \rightarrow p_c$$

and

$$\chi(A, \perp) = \left[\bigwedge_{a,b \in A} (p_a \wedge b \leftrightarrow (p_a \wedge p_b)) \wedge \bigwedge_{a,b \in A} (p_a \rightarrow b \leftrightarrow (p_a \rightarrow p_b)) \wedge \bigwedge_{a \in A} (p_{\neg a} \leftrightarrow \neg p_a) \right] \rightarrow p_c$$

The following is a straightforward generalization of [14,24].

Proposition 40. *Suppose A is a finite subdirectly irreducible Heyting algebra and B is an arbitrary Heyting algebra. Then:*

- (1) *The identity $\chi(A) = \top$ is satisfied on B iff $A \notin \mathbf{S}^{(\wedge, \rightarrow)}\mathbf{H}(B)$.*
- (2) *The identity $\chi(A, \perp) = \top$ is satisfied on B iff $A \notin \mathbf{S}^{(\perp, \wedge, \rightarrow)}\mathbf{H}(B)$.*

Theorem 41.

- (1) *$\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a nuclear variety iff \mathbb{V} is axiomatized by (\wedge, \rightarrow) -identities.*
- (2) *\mathbb{V} is a dense nuclear variety iff \mathbb{V} is axiomatized by $(\perp, \wedge, \rightarrow)$ -identities.*

Proof. (1) Suppose \mathbb{V} is axiomatized by (\wedge, \rightarrow) -identities, $A \in \mathbb{V}$, and j is a nucleus on A . Let Φ denote the set of (\wedge, \rightarrow) -identities axiomatizing \mathbb{V} . Then Φ is satisfied on A . By Proposition 7, A_j is an implicative subsemilattice of A . Therefore, Φ is satisfied on A_j . Thus, $A_j \in \mathbb{V}$, and so \mathbb{V} is nuclear. Conversely, suppose \mathbb{V} is a nuclear variety. Let K denote the class of all finite non-isomorphic subdirectly irreducible Heyting algebras that are not in \mathbb{V} . For A a finite Heyting algebra and B an arbitrary Heyting algebra, since \mathbb{V} is nuclear, from $B \in \mathbb{V}$ and $A \in \mathbf{S}^{(\wedge, \rightarrow)}\mathbf{H}(B)$ it follows that $A \in \mathbb{V}$ (see Corollary 37). Therefore, by Proposition 40, $B \in \mathbb{V}$ implies that $\{\chi(A) = \top : A \in K\}$ is satisfied on B . Thus, $\mathbb{V} \subseteq \mathbb{H}\mathbb{A} + \{\chi(A) = \top : A \in K\}$. Moreover, the finite subdirectly irreducible algebras of \mathbb{V} and $\mathbb{H}\mathbb{A} + \{\chi(A) = \top : A \in K\}$ coincide. Since we already showed that varieties axiomatized by (\wedge, \rightarrow) -identities are nuclear, it follows from Theorem 38 that $\mathbb{H}\mathbb{A} + \{\chi(A) = \top : A \in K\}$ is generated by its finite members.⁴ Therefore, $\mathbb{V} = \mathbb{H}\mathbb{A} + \{\chi(A) = \top : A \in K\}$, which implies that \mathbb{V} is axiomatized by (\wedge, \rightarrow) -identities.

(2) is proved similarly. \square

Corollary 42.

- (1) *The following conditions are equivalent:*
 - (i) *$\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a subframe variety.*
 - (ii) *$\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a nuclear variety.*
 - (iii) *\mathbb{V} is axiomatized by (\wedge, \rightarrow) -identities.*
- (2) *The following conditions are equivalent:*
 - (i) *$\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a cofinal subframe variety.*
 - (ii) *$\mathbb{V} \subseteq \mathbb{H}\mathbb{A}$ is a dense nuclear variety.*
 - (iii) *\mathbb{V} is axiomatized by $(\perp, \wedge, \rightarrow)$ -identities.*

We point out that the equivalence between the model-theoretic condition (i) and the logical condition (iii) is well known [25,26]; the further equivalence with condition (ii) provides a purely algebraic characterization of subframe and cofinal subframe logics.

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References

- [1] Raymond Balbes, Philip Dwinger, Distributive Lattices, University of Missouri Press, Columbia, MO, 1974.
- [2] Michael Barr, On categories with effective unions, in: Categorical Algebra and its Applications, Louvain-La-Neuve, 1987, in: Lecture Notes in Math., vol. 1348, Springer, Berlin, 1988, pp. 19–35.

⁴ Instead of Theorem 38 one can alternatively use [20, P. 261, Thm. 2.2].

- [3] R. Beazer, D.S. Macnab, Modal extensions of Heyting algebras, *Colloq. Math.* 41 (1) (1979) 1–12.
- [4] Silvio Bozzi, Gian Carlo Meloni, Completezza proposizionale per l'operatore modale "accade localmente che", *Boll. Unione Mat. Ital. A* (5) 14 (3) (1977) 489–497 (in Italian).
- [5] Silvio Bozzi, Gian Carlo Meloni, Representation of Heyting algebras with covering and propositional intuitionistic logic with local operator, *Boll. Unione Mat. Ital. A* (5) 17 (3) (1980) 436–442.
- [6] C. Cassidy, M. Hébert, G.M. Kelly, Reflective subcategories, localizations and factorization systems, *J. Aust. Math. Soc. Ser. A* 38 (3) (1985) 287–329.
- [7] Alexander Chagrov, Michael Zakharyashev, *Modal Logic*, in: *Oxford Logic Guides*, vol. 35, Clarendon Press, Oxford, 1997.
- [8] Antonio Diego, Sur les algèbres de Hilbert (Translated from the Spanish by Luisa Iturrioz), in: *Collection de Logique Mathématique, Sér. A, Fasc. XXI*, Gauthier-Villars, Paris, 1966.
- [9] L.L. Esakia, Topological Kripke models, *Soviet Math. Dokl.* 15 (1974) 147–151.
- [10] L.L. Esakia, Heyting algebras. I. Duality theory, Metsniereba, Tbilisi, 1985 (in Russian).
- [11] Kit Fine, Logics containing K4. II, *J. Symbolic Logic* 50 (3) (1985) 619–651.
- [12] Silvio Ghilardi, Gian Carlo Meloni, Models with coverings for intuitionistic and local propositional logic, in: *Proceedings of the Conference on Mathematical Logic, Vol. 2*, Siena, 1983–1984, Univ. Siena, Siena, 1985, pp. 333–338.
- [13] Robert I. Goldblatt, Grothendieck topology as geometric modality, *Z. Math. Logik Grundlag. Math.* 27 (6) (1981) 495–529.
- [14] V.A. Jankov, On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures, *Dokl. Akad. Nauk SSSR* 151 (1963) 1293–1294 (in Russian).
- [15] André Joyal, Myles Tierney, An extension of the Galois theory of Grothendieck, in: *Mem. Amer. Math. Soc.*, vol. 51, 1984.
- [16] Peter T. Johnstone, Stone spaces, in: *Cambridge Studies in Advanced Mathematics*, vol. 3, Cambridge University Press, Cambridge, 1982.
- [17] Peter T. Johnstone, Sketches of an elephant: A topos theory compendium, Vol. 1, in: *Oxford Logic Guides*, vol. 43, The Clarendon Press Oxford University Press, New York, 2002.
- [18] Peter Köhler, Brouwerian semilattices, *Trans. Amer. Math. Soc.* 268 (1) (1981) 103–126.
- [19] D.S. Macnab, Modal operators on Heyting algebras, *Algebra Universalis* 12 (1) (1981) 5–29.
- [20] C.G. McKay, The decidability of certain intermediate propositional logics, *J. Symbolic Logic* 33 (1968) 258–264.
- [21] William C. Nemitz, Implicative semi-lattices, *Trans. Amer. Math. Soc.* 117 (1965) 128–142.
- [22] H. Rasiowa, R. Sikorski, *The mathematics of metamathematics*, in: *Monografie Matematyczne, Tom 41*, Państwowe Wydawnictwo Naukowe, Warsaw, 1963.
- [23] Frank Wolter, The structure of lattices of subframe logics, *Ann. Pure Appl. Logic* 86 (1) (1997) 47–100.
- [24] Andrzej Wroński, Intermediate logics and the disjunction property, *Rep. Math. Logic* 1 (1973) 39–51.
- [25] Michael V. Zakharyashev, Syntax and semantics of superintuitionistic logics, *Algebra Logic* 28 (1989) 262–282.
- [26] Michael V. Zakharyashev, Canonical formulas for K4. II. Cofinal subframe logics, *J. Symbolic Logic* 61 (2) (1996) 421–449.