Proceedings of A. Razmadze
Mathematical Institute
Vol. **133** (2003), 21–35

EXTERIOR PROBLEMS WITH FRICTION IN THE COUPLE-STRESS ELASTICITY

R. GACHECHILADZE

Abstract. The exterior problem of statics with friction in the couple-stress theory of elasticity for homogeneous isotropic media is considered. The question on the existence and uniqueness of weak solutions of that problem is investigated, when the friction effect is taken into account on the whole or on some part of the boundary.

რეზიუმე. განხილულია დრეკადობის მომენტური თეორიის სტატიკის გარე ამოცანა ხახუნის გათვალისწინებით ერთგვაროვანი იზოტროპული სხეულებისათვის. შესწავლილია ამ ამოცანის სუსტი ამონახსნების არსებობისა და ერთადერთობის საკითხი, როდესაც დრეკადი სხეულის მთელ საზღვარზე, ან საზღვრის გარკვეულ ნაწილზე გათვალისწინებულია ხახუნის ეფექტი.

In this paper we consider the deformation of an elastic body on whose whole or a part of the boundary are imposed the conditions of friction which are described by Coulomb's law. Such kind of problems of the classical theory of elasticity in bounded regions have been studied mainly in [1]–[4], while statical and dynamical problems in the couple-stress elasticity have been investigated in [5]. In the present paper we first reduce equivalently the exterior problem to the variational inequality at the boundary of an elastic medium and then investigate the question on the existence and uniqueness of a solution of that inequality.

Let $\Omega \subset \mathbb{R}^3$ be a bounded region with the boundary $\Gamma (\Gamma \in C^\infty)$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$, $\nu(x)$ be a unit vector of the exterior (with respect to Ω^-) normal at the point $x \in \Gamma$; suppose

$$\mathcal{M}(\partial) = \left\| \begin{array}{ccc} \mathcal{M}^{(1)}(\partial) & \vdots & \mathcal{M}^{(2)}(\partial) \\ \cdots & \cdots & \cdots \\ \mathcal{M}^{(3)}(\partial) & \vdots & \mathcal{M}^{(4)}(\partial) \end{array} \right\|_{6\times 6}$$

2000 *Mathematics Subject Classification.* 73T05, 35J85.
Key words and phrases. Couple-stress elasticity, statics, friction, variational inequality, isotropic homogeneous medium.

is the matrix differential operator of statics of the couple-stress elasticity, where (see, for e.g., [5] and [6])

$$\mathcal{M}^{(1)}(\partial)=\|\mathcal{M}^{(1)}_{jk}(\partial)\|_{3\times3},\quad \mathcal{M}^{(1)}_{jk}(\partial)=a_{ijlk}\,\frac{\partial^2}{\partial x_i\partial x_l},$$

$$\mathcal{M}^{(2)}(\partial)=\|\mathcal{M}^{(2)}_{jk}(\partial)\|_{3\times3},\quad \mathcal{M}^{(2)}_{jk}(\partial)=b_{ijlk}\,\frac{\partial^2}{\partial x_i\partial x_l}-\varepsilon_{lrk}a_{ijlr}\,\frac{\partial}{\partial x_i},$$

$$\mathcal{M}^{(3)}(\partial)=\|\mathcal{M}^{(3)}_{jk}(\partial)\|_{3\times3},\quad \mathcal{M}^{(3)}_{jk}(\partial)=b_{lkij}\,\frac{\partial^2}{\partial x_i\partial x_l}+\varepsilon_{irj}a_{irlk}\,\frac{\partial}{\partial x_l},$$

$$\mathcal{M}^{(4)}(\partial)=\|\mathcal{M}^{(4)}_{jk}(\partial)\|_{3\times3},$$

$$\mathcal{M}^{(4)}_{jk}(\partial)=c_{ijlk}\,\frac{\partial^2}{\partial x_i\partial x_l}-\varepsilon_{lrk}b_{lrij}\,\frac{\partial}{\partial x_i}+\varepsilon_{irj}b_{irlk}\,\frac{\partial}{\partial x_l}-\varepsilon_{ipj}\varepsilon^{(1)}_{lrk}a_{iplr},$$

and

$$\mathcal{N}(\partial,\nu)=\begin{Vmatrix}\mathcal{N}^{(1)}(\partial,\nu) & \vdots & \mathcal{N}^{(2)}(\partial,\nu)\\ \cdots & \cdots & \cdots\\ \mathcal{N}^{(3)}(\partial,\nu) & \vdots & \mathcal{N}^{(4)}(\partial,\nu)\end{Vmatrix}_{6\times6}$$

is the matrix differential operator of stress, where

$$\mathcal{N}^{(1)}(\partial,\nu)=\|\mathcal{N}^{(1)}_{jk}(\partial,\nu)\|_{3\times3},\quad \mathcal{N}^{(1)}_{jk}(\partial,\nu)=a_{ijlk}\nu_i\,\frac{\partial}{\partial x_l},$$

$$\mathcal{N}^{(2)}(\partial,\nu)=\|\mathcal{N}^{(2)}_{jk}(\partial,\nu)\|_{3\times3},\quad \mathcal{N}^{(2)}_{jk}(\partial,\nu)=b_{ijlk}\nu_i\,\frac{\partial}{\partial x_l}-a_{ijlr}\varepsilon_{lrk}\nu_i,$$

$$\mathcal{N}^{(3)}(\partial,\nu)=\|\mathcal{N}^{(3)}_{jk}(\partial,\nu)\|_{3\times3},\quad \mathcal{N}^{(3)}_{jk}(\partial,\nu)=b_{lkij}\nu_i\,\frac{\partial}{\partial x_l},$$

$$\mathcal{N}^{(4)}(\partial,\nu)=\|\mathcal{N}^{(4)}_{jk}(\partial,\nu)\|_{3\times3},\quad \mathcal{N}^{(4)}_{jk}(\partial,\nu)=c_{ijlk}\nu_i\,\frac{\partial}{\partial x_l}-b_{lrij}\varepsilon_{lrk}\nu_i,$$

ε_{ijk} is the Lewy–Civita symbol (here and throughout the paper, repetition of the index denotes summation with respect to that index from 1 to 3). $U=(u,\omega)$, $u=(u_1,u_2,u_3)$ is the displacement vector, $\omega=(\omega_1,\omega_2,\omega_3)$ is the rotation vector, $\sigma U=\mathcal{N}^{(1)}u+\mathcal{N}^{(2)}\omega$ is a force stress vector, $\mu U=\mathcal{N}^{(3)}u+\mathcal{N}^{(4)}\omega$ is a couple-stress vector; a_T and a_N denote, respectively, the tangential and normal components of the vector $a\in\mathbb{R}^3$.

Elastic constants a_{ijlk}, b_{ijlk}, c_{ijlk} involved in the definition of the operator $\mathcal{M}(\partial)$ satisfy the conditions

$$a_{ijlk}=a_{lkij},\quad c_{ijlk}=c_{lkij}$$

and $\exists\,\alpha_0>0$, $\forall\,\xi_{ij}$, $\eta_{ij}\in\mathbb{R}$:

$$a_{ijlk}\xi_{ij}\xi_{lk}+2b_{ijlk}\xi_{ij}\eta_{lk}+c_{ijlk}\eta_{ij}\eta_{lk}\geq\alpha_0(\xi_{ij}\xi_{ij}+\eta_{lk}\eta_{lk}).\tag{1}$$

If
$$\begin{aligned} a_{ijlk} &= \lambda\delta_{ij}\delta_{lk} + (\mu-\alpha)\delta_{ik}\delta_{jl} + (\mu+\alpha)\delta_{il}\delta_{jk},\\ c_{ijlk} &= \varepsilon\delta_{ij}\delta_{lk} + (\nu-\beta)\delta_{ik}\delta_{jl} + (\nu+\beta)\delta_{il}\delta_{jk},\\ b_{ijlk} &= 0, \end{aligned} \tag{2}$$
then the medium is homogeneous and isotropic (λ, μ, α, ε, ν, β are elastic constants, and δ_{ij} is the Kronecker symbol), and condition (1) is reduced to the following one (see [6]):
$$\mu>0,\ \ 3\lambda+2\mu>0,\ \ \alpha>0,\ \ \nu>0,\ \ 3\varepsilon+2\nu>0,\ \ \beta>0.$$

By $H^S(\Omega)$, $H^S_{\rm loc}(\Omega^-)$ and $H^S(\Gamma)$ we denote Sobolev-Slobodetskii spaces ($S\in\mathbb{R}$) whose definition and basic properties can be found in [7] (see also [8]). It will also be admitted that $\omega\in X^m$, if every component of the vector $\omega=(\omega_1,\dots,\omega_m)$ belongs to some space X.

Let $U=(u,\omega)\in(H^1_{\rm loc}(\Omega^-))^6$, $V=(v,w)\in(H^1_{\rm loc}(\Omega^-))^6$, and in the neighborhood of $|x|=\infty$ the conditions
$$u,v=O(|x|^{-1}),\quad \frac{\partial u_i}{\partial x_j},\ \ \frac{\partial v_i}{\partial x_j},\ \ \omega,\ \ w=O(|x|^{-2}). \tag{3}$$
are satisfied.

Then the bilinear form $\mathcal{B}(U,V)$ is defined by the formula
$$\mathcal{B}(U,V)=a_{ijlk}\int\limits_{\Omega^-}\xi_{ij}(U)\xi_{lk}(V)\,dx+c_{ijlk}\int\limits_{\Omega^-}\eta_{ij}(U)\eta_{lk}(V)\,dx,$$
where a_{ijlk} and c_{ijlk} are defined by formulas (2), $\xi_{ij}(U)=\frac{\partial u_i}{\partial x_j}-\varepsilon_{ijk}\omega_k$ and $\eta_{ij}(U)=\frac{\partial\omega_i}{\partial x_j}$.

Definition. The vector function $U\in(H^1_{\rm loc}(\Omega^-))^6$ is a weak solution of the equation
$$\mathcal{M}(\partial)U(x)+\mathcal{G}(x)=0\quad\big(\mathcal{G}\in(L^2_{\rm loc}(\Omega^-))^6\big),$$
if
$$\mathcal{B}(U,\Phi)=(\mathcal{G},\Phi)_{0,\Omega^-}\quad\bigg(\big(\varphi,\psi\big)_{0,\Omega^-}=\int\limits_{\Omega^-}\varphi\overline{\psi}\,dx\bigg),$$
for all $\Phi\in(C_0^\infty(\Omega^-))^6$.

It should be noted that if $U=(u,\omega)\in(H^1_{\rm loc}(\Omega^-))^6$, u and ω in the neighborhood of $|x|=\infty$ satisfy conditions (3), and $\mathcal{M}U\in(L^2(\Omega^-))^6$, then we can define $\mathcal{N}U\big|_\Gamma$ as the functional of the class $(H^{-1/2}(\Gamma))^6$ by the formula
$$\big\langle\mathcal{N}U\big|_\Gamma,V\big|_\Gamma\big\rangle=\mathcal{B}(U,V)+(\mathcal{M}U,V)_{0,\Omega^-},\ \ \forall\,V=(v,w)\in(H^1(\Omega^-))^6,$$
v and w satisfy conditions (3); here $\langle\cdot,\cdot\rangle$ denotes the duality relation between the dual pairs $(H^{-1/2}(\Gamma))^6$ and $(H^{1/2}(\Gamma))^6$.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \varnothing$ and $\mathcal{G} \in (L^2_{\rm loc}(\Omega^-))^6$, $\chi \in (H^{1/2}(\Gamma_1))^6$, $F_N \in L^\infty(\Gamma_2)$, $\psi \in (L^\infty(\Gamma_2))^3$, $\mathcal{F} \in L^\infty(\Gamma_2)$, $\mathcal{F} \geq 0$, $g = \mathcal{F}|F_N|$.

We consider the following problem

Problem $(I)^-$**.** Find the vector function $U = (u,\omega) \in (H^1_{\rm loc}(\Omega^-))^6$ which is a weak solution of the equation

$$\mathcal{M}(\partial)U(x) + \mathcal{G}(x) = 0, \quad x \in \Omega^-, \tag{4}$$

U satisfies at infinity conditions (3) and $U = \chi$ on Γ_1, while on Γ_2 the conditions are fulfilled:

$$\sigma_T(U) \in (L^\infty(\Gamma_2))^3, \ \ \sigma_N(U) = F_N, \ \ \mu(U) = \psi.$$

If $|\sigma_T(U)| \leq g$, then $u_T = 0$, and if $|\sigma_T(U)| = g$, then $\exists \gamma \geq 0 : u_T = -\gamma\sigma_T(U)$.

Let $U_0 = (u_0,\omega_0) \in (H^1_{\rm loc}(\Omega^-))^6$ be a weak solution of equation (4), satisfying conditions (3) and $U_0\big|_{\Gamma_1} = \chi$, $\mathcal{N}U_0\big|_{\Gamma_2} = 0$ (as is known, this problem has the unique solution). Then for the vector function $V = U - U_0$ (instead of V we again write U) we obtain the following problem.

Problem $(F)^-$**.** Find the vector function $U \in (H^1_{\rm loc}(\Omega^-))^6$ which is a weak solution of the equation

$$\mathcal{M}(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{5}$$

satisfies in the neighborhood of $|x| = \infty$ conditions (3) and the condition $U = 0$ on Γ_1, while on Γ_2 satisfies the conditions

$$\sigma_T(U) \in (L^\infty(\Gamma_2))^3, \quad \mu U = \psi, \quad \sigma_N(U) = F_N.$$

If $|\sigma_T(U)| < g$, then $u_T = \varphi_T$, but if $|\sigma_T(U)| = g$, then $\exists \gamma \geq 0 : u_T = \varphi_T - \gamma\sigma_T(U)$, where $\varphi_T = -u_{0T}\big|_\Gamma \in (H^{1/2}(\Gamma))^3$.

To reduce the problem to the variational inequality, we have first to construct Green's operator for the Dirichlet problem.

Let $h \in (H^{1/2}(\Gamma))^6$, and find the vector function $U \in (H^1_{\rm loc}(\Omega^-))^6$ which is a weak solution of equation (5) satisfying in the neighborhood of $|x| = \infty$ conditions (3) and $U = h$ on Γ. It is known that this problem has the unique solution which is given in terms of a simple layer potential

$$U(x) = \int\limits_\Gamma \Psi(x-y)\big(\mathcal{H}^{-1}(h)\big)(y)\, d_yS, \ \ x \in \Omega^-, \tag{6}$$

where Ψ is the fundamental solution of the differential operator $\mathcal{M}(\partial)$ (see [6]), and the operator

$$\mathcal{H}(h)(x) = \lim_{\Omega^- \ni z \to x \in \Gamma} \int\limits_\Gamma \Psi(z-y)h(y)\, d_yS.$$

As is known (see [9], [10]), the operator $\mathcal{H}$ is invertible, and

$$\begin{aligned} \mathcal{H} &: \big(H^S(\Gamma)\big)^6 \longrightarrow \big(H^{S+1}(\Gamma)\big)^6, \\ \mathcal{H}^{-1} &: \big(H^S(\Gamma)\big)^6 \longrightarrow \big(H^{S-1}(\Gamma)\big)^6, \ \ \forall\, S \in \mathbb{R}, \end{aligned} \tag{7}$$

but a simple layer operator itself maps continuously the space $(H^S(\Gamma))^6$ into the space $(H^{S+1+\frac{1}{2}}_{\rm loc}(\Omega^-))^6$.

The Green's operator G^- for the first exterior problem is defined by formula (6), i.e.,

$$\begin{gathered} \mathcal{M}(\partial)(G^- h)(x) = 0, \quad x \in \Omega^-, \\ G^- h\big|_\Gamma = h \end{gathered}$$

for all $h \in (H^{1/2}(\Gamma))^6$, and in the neighborhood of $|x| = \infty$ the conditions:

$$G^- h = (\xi, \eta), \ \ \xi = O(|x|^{-1}), \ \ \frac{\partial \xi_i}{\partial x_j}, \ \ \frac{\partial \eta_i}{\partial x_j}, \ \ \eta = O(|x|^{-2}). \tag{8}$$

are satisfied.

We introduce the following operator:

$$\begin{gathered} S^- : \big(H^{1/2}(\Gamma)\big)^6 \longrightarrow \big(H^{-1/2}(\Gamma)\big)^6, \\ \forall\, h \in \big(H^{1/2}(\Gamma)\big)^6 : S^- h = \big\{\mathcal{N}(\partial, \nu)(G^- h)(x)\big\}^-_\Gamma \end{gathered}$$

(note that the operator S^- is defined correctly because $G^- h \in (H^1_{\rm loc}(\Omega^-))^6$ satisfies at infinity conditions (8) and $\mathcal{M}(G^- h) = O \in (L^2(\Omega^-))^6$).

Taking into account the properties of the operator G^-, from Green's formula we have

$$\begin{gathered} \forall\, h,\, g \in \big(H^{1/2}(\Gamma)\big)^6 : \ \ \langle S^- h, g\rangle = \mathcal{B}(G^- h, G^- g) = \\ = a_{ijek} \int\limits_{\Omega^-} \xi_{ij}(G^- h)\xi_{ek}(G^- g)\, dx + c_{ijek} \int\limits_{\Omega^-} \eta_{ij}(G^- h)\eta_{ek}(G^- g)\, dx. \end{gathered}$$

To reduce Problem $(F)^-$ to the variational inequality, we consider the convex closed set

$$\mathcal{K} = \Big\{ h = (\xi, \eta) \in (H^{1/2}(\Gamma))^6 : h\big|_{\Gamma_1} = O \Big\},$$

the continuous convex functional

$$j(\xi) = \int\limits_{\Gamma_2} g|\xi_T - \varphi_T|\, ds$$

and the following variational inequality:

Find $h_0 = (\xi_0, \eta_0) \in \mathcal{K}$ such that

$$\langle S^- h_0, h - h_0\rangle + j(\xi) - j(\xi_0) \geq \int\limits_{\Gamma_2} \big[F_N(\xi_N - \xi_{0N}) + \psi \cdot (\eta - \eta_0)\big]\, ds \tag{9}$$

for all $h=(\xi,\eta)\in\mathcal{K}$.

Let us prove that Problem $(F)^-$ and the variational inequality (9) are equivalent.

The following theorem is valid.

Theorem 1. *The boundary variational inequality* (9) *and Problem* $(F)^-$ *are equivalent.*

Proof. It should be noted that the equivalence is understood in the sense that if $U\in(H^1_{\rm loc}(\Omega^-))^6$ is a solution of Problem $(F)^-$, then $U\big|_\Gamma=h_0$ is a solution of inequality (9), and vice versa, if $h_0\in\mathcal{K}$ is a solution of inequality (9), then $G^-h_0\in(H^1_{\rm loc}(\Omega^-))^6$ is a solution of Problem $(F)^-$.

Let $U\in(H^1_{\rm loc}(\Omega^-))^6$ be a solution of Problem $(F)^-$ and $U\big|_\Gamma=h_0$ (by the definition of Green's operator, it is clear that $U=G^-h_0$).

It can be easily verified that if the conditions of Problem $(F)^-$ are fulfilled, then the inequality

$$\sigma_T(G^-h_0)\cdot(\xi_T-\xi_{0T})+g\big(|\xi_T-\varphi_T|-|\xi_{0T}-\varphi_T|\big)\geq 0. \tag{10}$$

is valid on Γ_2.

Integrating (10) on Γ_2, we obtain

$$\int\limits_{\Gamma_2}\sigma_T(G^-h_0)\cdot(\xi_T-\xi_{0T})\,ds+\int\limits_{\Gamma_2}\mu(G^-h_0)\cdot(\eta-\eta_0)\,ds+$$
$$+\int\limits_{\Gamma_2}\sigma_N(G^-h_0)(\xi_N-\xi_{0N})\,ds+j(\xi)-j(\xi_0)\geq$$
$$\geq\int\limits_{\Gamma_2}\big[\sigma_N(G^-h_0)(\xi_N-\xi_{0N})+\mu(G^-h_0)\cdot(\eta-\eta_0)\big]\,ds,$$

i.e., inequality (9) is fulfilled.

Conversely, let $h_0\in\mathcal{K}$ be a solution of inequality (9). By the definition of Green's operator $U=G^-h_0$ is the weak solution of equation (5) and $U\big|_{\Gamma_1}=G^{-1}h_0\big|_{\Gamma_1}=h_0\big|_{\Gamma_1}=0$, since $h_0\in\mathcal{K}$.

Let $h=(\xi,\eta)\in\mathcal{K}$ such that $\xi_T=\xi_{0T}$, $\eta=\eta_0$, $\xi_N=\xi_{0N}\pm\theta$, where $\theta\in H^{1/2}(\Gamma)$, $\operatorname{supp}\theta\subset\Gamma_2$. Then $j(\xi)=j(\xi_0)$, and from (9) we find that

$$\big\langle\sigma_N(G^-h_0),\theta\big\rangle=\int\limits_{\Gamma_2}F_N\theta\,ds,\ \ \forall\,\theta\in H^{1/2}(\Gamma),\ \ \operatorname{supp}\theta\subset\Gamma_2.$$

Therefore

$$\sigma_N(G^-h_0)\big|_{\Gamma_2}=F_N. \tag{11}$$

Similarly, choosing $h\in\mathcal{K}$ appropriately, we have

$$\mu(G^-h_0)\big|_{\Gamma_2}=\psi. \tag{12}$$

If we take now into account (11) and (12), inequality (9) will take the form

$$\int\limits_{\Gamma_2} \big[\sigma_T(G^- h_0)\cdot \chi_T + g|\chi_T|\big]\,ds - \int\limits_{\Gamma_2} \big[\sigma_T(G^- h_0)\cdot \chi_{0T} + g|\chi_{0T}|\big]\,ds \geq 0, \quad (13)$$

where $\chi_T = \xi_T - \varphi_T$ and $\chi_{0T} = \xi_{0T} - \varphi_T$.

Let

$$\Theta = \Big\{\zeta \in \big(H^{1/2}(\Gamma)\big)^3 : \ \zeta\big|_{\Gamma_1} = 0\Big\}.$$

Substituting in (13) $\chi_{0T} \pm \zeta_T$ instead of χ_T, where $\zeta \in \Theta$, and taking into account $|\zeta_T| \leq |\zeta|$, after certain reasoning we obtain that

$$\bigg|\int\limits_{\Gamma_2} \sigma_T(G^- h_0)\cdot \zeta\,ds\bigg| \leq \int\limits_{\Gamma_2} g|\zeta|\,ds, \ \ \forall\,\zeta \in \Theta \quad (14)$$

and

$$\int\limits_{\Gamma_2} \big[\sigma_T(G^- h_0)\cdot \chi_{0T} + g|\chi_{0T}|\big]\,ds \leq 0. \quad (15)$$

Consider on the set Θ the functional

$$\Phi(\zeta) = \int\limits_{\Gamma_2} \sigma_T(G^- h_0)\cdot \zeta\,ds, \quad \forall\,\zeta \in \Theta.$$

By virtue of (14), the functional Φ in the space $\Theta \subset (L^1(\Gamma))^3$ is linear and continuous in the induced topology, and its norm does not exceed unity.

Since $\Theta\big|_{\Gamma_2}$ is dense in $(L^1(\Gamma_2))^3$, by the Hahn–Banach theorem we have $\Phi \in (L^\infty(\Gamma_2))^3$ and $\|\Phi\| \leq 1$, i.e.,

$$\sigma_T(G^- h_0) \in (L^\infty(\Gamma_2))^3.$$

We represent the functional Φ in somewhat different form:

$$\Phi(\zeta) = \int\limits_{\Gamma_2} g^{-1}\sigma_T(G^- h_0)\cdot g\zeta\,ds \ \ \text{(we mean that } g \geq g_0 > 0). \quad (16)$$

Reasoning analogously for the functional (16), we find that

$$|\sigma_T(G^- h_0)| \leq g. \quad (17)$$

If we take into account (17), from (15) we obtain

$$\sigma_T(G^- h_0)\cdot \chi_{0T} + g|\chi_{0T}| = 0,$$

which first of all implies that the friction conditions of Problem $(F)^-$ are fulfilled.

Thus the theorem is proved. □

Let us now consider the question on the existence and uniqueness of the solution of the variational inequality (9). To this end, we show that the operator S^- satisfies the following conditions:

(i) $\langle S^- h, g\rangle = \langle S^- g, h\rangle$ for all h, $g \in (H^{1/2}(\Gamma))^6$;

(ii) $S^- : (H^{1/2}(\Gamma))^6 \longrightarrow (H^{-1/2}(\Gamma))^6$ is a continuous mapping;

(iii) there exists $c > 0 :\ \langle S^- h, h\rangle \geq c\|h\|^2_{1/2,\Gamma}$ for all $h \in (H^{1/2}(\Gamma))^6$.

Properties (i) and (ii) follow directly from the fact that the bilinear form $\mathcal{B}$ is symmetric and the singular integral operator in the space $(H^{-1/2}(\Gamma))^6$ is continuous. To prove (iii), we consider an auxiliary problem.

Let $r = \|r_{ij}\|_{6\times 6}$, $r_{ii} = \rho$, $i = 1, 2, 3$, $r_{ii} = \mathcal{I}$, $i = 4, 5, 6$ and $r_{ij} = 0$, $i \neq j$ where ρ and $\mathcal{I}$ are physical characteristics of an elastic medium.

Problem $(I_k)^-$. Find $U \in (H^1(\Omega^-))^6$ which is a weak solution of the equation

$$\mathcal{M}(\partial)U(x) - k^2 rU(x) = 0,\ \ x \in \Omega^-\ \ \big(k \in \mathbb{R}\setminus\{0\}\big)$$

and

$$U\big|_\Gamma = g,\ \ g \in \big(H^{1/2}(\Gamma)\big)^6,\quad \lim_{|x|\to\infty} U(x) = 0.$$

A unique solution of Problem $(I_k)^-$ which we denote by $G_k^- g$ is given as a simple layer potential

$$(G_k^- g)(x) = \int_\Gamma \Psi(x-y,k)(\mathcal{H}_k^{-1} g)(y)\, d_yS,\ \ x \in \Omega^-,$$

where

$$(\mathcal{H}_k g)(y) = \lim_{\Omega^- \ni z \to y \in \Gamma} \int_\Gamma \Psi(z-t,k) g(t)\, d_tS,$$

and $\Psi(x,k)$ is the fundamental solution of the differential operator $\mathcal{M}(\partial) - k^2 rI$ which is given by the formula

$$\Psi(x,k) = \left\| \begin{matrix} \Psi^{(1)}(x,k) & \vdots & \Psi^{(2)}(x,k) \\ \cdots\cdots & \cdots & \cdots\cdots \\ \Psi^{(3)}(x,k) & \vdots & \Psi^{(4)}(x,k) \end{matrix} \right\|_{6\times 6},$$

$$\Psi^{(l)}(x,k) = \|\Psi^{(l)}_{ij}(x,k)\|_{3\times 3},\ \ l = 1, 2, 3, 4,$$

$$\Psi^{(1)}_{ij}(x,k) = \sum_{l=1}^{4} \Big\{\delta_{ij}\alpha_l + \beta_l \frac{\partial^2}{\partial x_i \partial x_j}\Big\} \frac{e^{-\sigma_l|x|}}{|x|},$$

$$\Psi^{(2)}_{ij}(x,k) = \Psi^{(3)}_{ij}(x,k) = \frac{2\alpha}{\mu+\alpha} \sum_{l=1}^{4} \varepsilon_l \varepsilon ijp \frac{\partial}{\partial x_p} \frac{e^{-\sigma_l|x|}}{|x|},$$

$$\Psi_{ij}^{(4)}(x,k)=\sum_{l=1}^{4}\Big\{\delta_{ij}\gamma_l+\delta_l\,\frac{\partial^2}{\partial x_i\partial x_j}\Big\}\,\frac{e^{-\sigma_l|x|}}{|x|}\,,$$

$$\alpha_l=\frac{(-1)^l(\delta_{3l}+\delta_{4l})(k_2^2-\sigma_l^2)}{2\pi(\mu+\alpha)(\sigma_3^2-\sigma_4^2)}\,,\quad \beta_l=\frac{\delta_{1l}}{2\pi\rho k^2}-\frac{\alpha_l}{\delta_l^2}\,,$$

$$\gamma_l=\frac{(-1)^l(k_1^2-\sigma_l^2)(\delta_{3l}+\delta_{4l})}{2\pi(\nu+\beta)(\sigma_3^2-\sigma_4^2)}\,,\quad \delta_l=\frac{\delta_{2l}}{2\pi(\mathcal{I}k^2+4\alpha)}-\frac{\gamma_l}{\sigma_l^2}\,,$$

$$\varepsilon_l=\frac{(-1)^l(\delta_{3l}+\delta_{4l})}{2\pi(\nu+\beta)(\sigma_3^2-\sigma_4^2)}\,,\quad k_1^2=\frac{\rho k^2}{\mu+\alpha}\,,\quad k_2^2=\frac{\mathcal{I}k^2+4\alpha}{\nu+\beta}\,,$$

$$\sigma_1^2=\frac{\rho k^2}{\lambda+2\mu}\,,\quad \sigma_2^2=\frac{\mathcal{I}k^2+4\alpha}{2\nu+\varepsilon}\,,$$

$$\sigma_3^2+\sigma_4^2=k_1^2+k_2^2-\frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}\,,\quad \sigma_3^2\sigma_4^2=k_1^2k_2^2.$$

Note that relations (7) are valid for the operators $\mathcal{H}_k$ and $\mathcal{H}_k^{-1}$.

Introduce the operator $S_k^-:(H^{1/2}(\Gamma))^6\longrightarrow(H^{-1/2}(\Gamma))^6$ by the formula

$$S_k^-h=\big\{\mathcal{N}(G_k^-h)\big\}_\Gamma^-,\quad \forall\,h\in(H^{1/2}(\Gamma))^6.$$

Properties (i) and (ii) are satisfied for the operator S_k^- as well. Let us prove property (iii).

From Green's formula and the coerciveness of the bilinear form $\mathcal{B}_k$ ($\mathcal{B}_k(U,V)=\mathcal{B}(U,V)+k^2\int\limits_{\Omega^-}rU\cdot V\,dx$) it follows that

$$\langle S_k^-h,h\rangle=\mathcal{B}_k(G_k^-h,G_k^-h)\geq c\|G_k^-h\|_{1,\Omega^-}^2. \tag{18}$$

Since the operator S_k^- is continuous, we have

$$\big|\langle S_k^-h,h\rangle\big|\leq c_1\|h\|_{1/2,\Gamma}^2.$$

Thus the operator

$$G_k^-:\big(H^{1/2}(\Gamma)\big)^6\longrightarrow\big(\widetilde{H}^1(\Omega^-)\big)^6=$$
$$=\Big\{V\in\big(H^1(\Omega^-)\big)^6:\ \mathcal{M}V-k^2rV=0,\ \lim_{|x|\to\infty}V(x)=0\Big\}$$

satisfies the condition

$$\|G_k^-h\|_{1,\Omega^-}\leq\text{const}\,\|h\|_{1/2,\Gamma}. \tag{19}$$

Taking into account inequality (19) and the fact that the space $(\widetilde{H}^1(\Omega^-))^6$ is complete, we find that the operator G_k^- is continuous, and since it is surjective, the inverse operator $(G_k^-)^{-1}$ is, by the Banach theorem, also continuous, i.e.,

$$\|G_k^-h\|_{1,\Omega^-}\geq c\|h\|_{1/2,\Gamma}.$$

Thus taking into account (18), we can conclude that property (iii) is fulfilled for the operator S_k^-.

Consider the operator $S_k^- - S^-$. We have

$$(S_k^- - S^-)h = \big\{\mathcal{N}(G_k^- - G^-)h\big\}_\Gamma^-$$

for all $h \in (H^{1/2}(\Gamma))^6$;

$$(G_k^- - G^-)h(x) = \int_\Gamma \big[\Psi(x-y,k) - \Psi(x,y)\big](\mathcal{H}_k^{-1}(h))(y)\,d_yS + \\ + \int_\Gamma \Psi(x-y)\big[(\mathcal{H}_k^{-1} - \mathcal{H}^{-1})h\big](y)\,d_yS = I_1 + I_2.$$

Denoting by $\sigma_{\mathcal{H}}(\xi')$ and $\sigma_{\mathcal{H}_k}(\xi')$ the principal symbols respectively of the operators $\mathcal{H}$ and $\mathcal{H}_k$, after simple, but cumbersome calculations we obtain their representations explicitly:

$$\sigma_{\mathcal{H}}(\xi') = \\ = \left\| \begin{array}{ccc:ccc} A\xi_1^2+B & A\xi_1\xi_2 & 0 & 0 & 0 & E\xi_2 \\ A\xi_1\xi_2 & A\xi_2^2+B & 0 & 0 & 0 & -E\xi_1 \\ 0 & 0 & \widetilde{A}+B & -E\xi_2 & E\xi_1 & 0 \\ \hdashline 0 & 0 & E\xi_2 & C\xi_1^2+D & C\xi_1\xi_2 & 0 \\ 0 & 0 & -E\xi_1 & C\xi_1\xi_2 & C\xi_2^2+D & 0 \\ -E\xi_2 & E\xi_1 & 0 & 0 & 0 & \widetilde{C}+D \end{array} \right\|$$

and

$$\sigma_{\mathcal{H}_k}(\xi') = \\ = \left\| \begin{array}{ccc:ccc} A_1\xi_1^2+B_1 & A_1\xi_1\xi_2 & 0 & 0 & 0 & E_1\xi_2 \\ A_1\xi_1\xi_2 & A_1\xi_2^2+B_1 & 0 & 0 & 0 & -E_1\xi_1 \\ 0 & 0 & \widetilde{A}_1+B_1 & -E_1\xi_2 & E_1\xi_1 & 0 \\ \hdashline 0 & 0 & E_1\xi_2 & C_1\xi_1^2+D_1 & C_1\xi_1\xi_2 & 0 \\ 0 & 0 & -E_1\xi_1 & C_1\xi_1\xi_2 & C_1\xi_2^2+D_1 & 0 \\ -E_1\xi_2 & E_1\xi_1 & 0 & 0 & 0 & \widetilde{C}_1+D_1 \end{array} \right\|,$$

where

$$
\begin{aligned}
A &= -\frac{(\mu+\alpha)(\nu+\beta)}{8\mu^2(\lambda+2\mu)}\Big(\frac{a}{2|\xi'|^3}-\frac{1}{|\xi'|}+\frac{1}{\sqrt{|\xi'|^2+a}}\Big)-\\
&\quad-\frac{(\lambda+\mu-\alpha)(\nu+\beta)}{8\mu^2(\lambda+2\mu)}\Big(\frac{ab}{2(b-a)|\xi'|^3}-\frac{1}{|\xi'|}+\frac{1}{\sqrt{|\xi'|^2+a}}\Big),\\
B &= \frac{\nu+\beta}{8\mu\alpha}\Big(\frac{b}{|\xi'|}-\frac{b-a}{\sqrt{|\xi'|^2+a}}\Big),\\
C &= \frac{1}{2\mu(a-c)}\Big\{\frac{c-a}{|\xi'|}-\frac{c}{\sqrt{|\xi'|^2+a}}+\frac{a}{\sqrt{|\xi'|^2+c}}\Big\}+\\
&\quad+\frac{\varepsilon+\nu-\beta}{2(a-c)(\nu+\beta)(\varepsilon+2\nu)}\Big\{\frac{1}{\sqrt{|\xi'|^2+a}}-\frac{1}{\sqrt{|\xi'|^2+c}}\Big\},\\
D &= \frac{1}{2(\nu+\beta)}\,\frac{1}{\sqrt{|\xi'|^2+a}}\,,\\
\widetilde{A} &= -\frac{(\mu+\alpha)(\nu+\beta)}{8\mu^2(\lambda+2\mu)}\Big(\frac{a}{2|\xi'|}+|\xi'|-\sqrt{|\xi'|^2+a}\Big)-\\
&\quad-\frac{(\lambda+\mu-\alpha)(\nu+\beta)}{8\mu^2(\lambda+2\mu)}\Big(\frac{ab}{2(b-a)|\xi'|}+|\xi'|-\sqrt{|\xi'|^2+a}\Big),\\
\widetilde{C} &= \frac{1}{2\mu(c-a)}\Big\{(c-a)|\xi'|-c\sqrt{|\xi'|^2+a}+a\sqrt{|\xi'|^2+c}\Big\}+\\
&\quad+\frac{\varepsilon+\nu-\beta}{2(a-c)(\nu+\beta)(\varepsilon+2\nu)}\Big\{\sqrt{|\xi'|^2+c}-\sqrt{|\xi'|^2+a}\Big\},\\
E &= \frac{i}{4\mu}\Big(\frac{1}{|\xi'|}-\frac{1}{\sqrt{|\xi'|^2+a}}\Big),\\
a &= \frac{4\alpha\mu}{(\mu+\alpha)(\nu+\beta)}\,,\quad b=\frac{4\alpha}{\nu+\beta}\,,\quad c=\frac{4\alpha}{\varepsilon+2\nu}\,,\\
A_1 &= \frac{[(\mu+\alpha)(\lambda+2\mu)]^{-1}}{(\sigma_3^2-\sigma_1^2)(\sigma_4^2-\sigma_1^2)(\sigma_3^2-\sigma_4^2)}\times\\
&\quad\times\Big\{\frac{2\alpha^2}{\nu+\beta}\Big(\frac{\sigma_4^2-\sigma_3^2}{\sqrt{|\xi'|^2+\sigma_1^2}}+\frac{\sigma_1^2-\sigma_4^2}{\sqrt{|\xi'|^2+\sigma_3^2}}+\frac{\sigma_3^2-\sigma_1^2}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big)+\\
&\quad+\frac{\lambda+\mu-\alpha}{2}\Big(\frac{(k_2^2-\sigma_1^2)(\sigma_4^2-\sigma_3^2)}{\sqrt{|\xi'|^2+\sigma_1^2}}+\\
&\quad+\frac{(k_2^2-\sigma_3^2)(\sigma_1^2-\sigma_4^2)}{\sqrt{|\xi'|^2+\sigma_3^2}}+\frac{(k_2^2-\sigma_4^2)(\sigma_3^2-\sigma_1^2)}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big)\Big\},\\
B_1 &= \frac{1}{2(\mu+\alpha)(\sigma_4^2-\sigma_3^2)}\Big(\frac{k_2^2-\sigma_3^2}{\sqrt{|\xi'|^2+\sigma_3^2}}-\frac{k_2^2-\sigma_4^2}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big),
\end{aligned}
$$

$$C_1=\frac{[(\nu+\beta)(\varepsilon+2\nu)]^{-1}}{(\sigma_3^2-\sigma_2^2)(\sigma_4^2-\sigma_2^2)(\sigma_3^2-\sigma_4^2)}\times$$
$$\times\Big\{\Big(\frac{\sigma_4^2-\sigma_3^2}{\sqrt{|\xi'|^2+\sigma_2^2}}+\frac{\sigma_2^2-\sigma_4^2}{\sqrt{|\xi'|^2+\sigma_3^2}}+\frac{\sigma_3^2-\sigma_2^2}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big)\frac{2\alpha^2}{\mu+\alpha}+$$
$$+\frac{\varepsilon+\nu-\beta}{2}\Big(\frac{(k_1^2-\sigma_2^2)(\sigma_4^2-\sigma_3^2)}{\sqrt{|\xi'|^2+\sigma_2^2}}+$$
$$+\frac{(k_1^2-\sigma_3^2)(\sigma_2^2-\sigma_4^2)}{\sqrt{|\xi'|^2+\sigma_3^2}}+\frac{(k_1^2-\sigma_4^2)(\sigma_3^2-\sigma_2^2)}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big)\Big\},$$
$$D_1=\frac{1}{2(\nu+\beta)(\sigma_4^2-\sigma_3^2)}\Big(\frac{k_1^2-\sigma_3^2}{\sqrt{|\xi'|^2+\sigma_3^2}}-\frac{k_1^2-\sigma_4^2}{\sqrt{|\xi'|^2+\sigma_4^2}}\Big),$$
$$\widetilde{A}_1=\frac{[(\mu+\alpha)(\lambda+2\mu)]^{-1}}{(\sigma_3^2-\sigma_1^2)(\sigma_4^2-\sigma_1^2)(\sigma_3^2-\sigma_4^2)}\Big\{\frac{2\alpha^2}{\nu+\beta}\Big(\sqrt{|\xi'|^2+\sigma_1^2}(\sigma_3^2-\sigma_4^2)+$$
$$+\sqrt{|\xi'|^2+\sigma_3^2}(\sigma_4^2-\sigma_1^2)+\sqrt{|\xi'|^2+\sigma_4^2}(\sigma_1^2-\sigma_3^2)\Big)+$$
$$+\frac{\lambda+\mu-\alpha}{2}\Big(\sqrt{|\xi'|^2+\sigma_1^2}(k_2^2-\sigma_1^2)(\sigma_3^2-\sigma_4^2)+$$
$$+\sqrt{|\xi'|^2+\sigma_3^2}(k_2^2-\sigma_3^2)(\sigma_4^2-\sigma_1^2)+\sqrt{|\xi'|^2+\sigma_4^2}(k_2^2-\sigma_4^2)(\sigma_1^2-\sigma_3^2)\Big)\Big\},$$
$$\widetilde{C}_1=\frac{[(\nu+\beta)(\varepsilon+2\nu)]^{-1}}{(\sigma_3^2-\sigma_2^2)(\sigma_3^2-\sigma_4^2)(\sigma_4^2-\sigma_2^2)}\Big\{\frac{2\alpha^2}{\mu+\alpha}\Big(\sqrt{|\xi'|^2+\sigma_2^2}(\sigma_3^2-\sigma_4^2)+$$
$$+\sqrt{|\xi'|^2+\sigma_3^2}(\sigma_4^2-\sigma_2^2)+\sqrt{|\xi'|^2+\sigma_4^2}(\sigma_2^2-\sigma_3^2)\Big)+$$
$$+\frac{\varepsilon+\nu-\beta}{2}\Big(\sqrt{|\xi'|^2+\sigma_2^2}(k_1^2-\sigma_2^2)(\sigma_3^2-\sigma_4^2)+$$
$$+\sqrt{|\xi'|^2+\sigma_3^2}(k_1^2-\sigma_3^2)(\sigma_4^2-\sigma_2^2)+\sqrt{|\xi'|^2+\sigma_4^2}(k_1^2-\sigma_4^2)(\sigma_2^2-\sigma_3^2)\Big)\Big\},$$
$$E_1=\frac{\alpha i}{(\mu+\alpha)(\nu+\beta)(\sigma_3^2-\sigma_4^2)}\Big(\frac{1}{\sqrt{|\xi'|^2+\sigma_4^2}}-\frac{1}{\sqrt{|\xi'|^2+\sigma_3^2}}\Big),$$

$\xi'=(\xi_1,\xi_2)$ and k_1^2, k_2^2, σ_1^2, σ_2^2, σ_3^2 and σ_4^2 have been defined above.

It can be easily verified that

$$\sigma_{\mathcal{H}}(\xi')-\sigma_{\mathcal{H}_k}(\xi')=O(|\xi'|^{-3}),$$

and hence

$$I_1:(H^s(\Gamma))^6\longrightarrow(H^{s+7/2}_{\rm loc}(\Omega^-))^6,\ \ \forall\, s\in\mathbb{R}.$$

Let

$$\sigma^{-1}_{\mathcal{H}}(\xi')-\sigma^{-1}_{\mathcal{H}_k}(\xi')=L(\xi'),$$

then

$$\sigma_{\mathcal{H}_k}(\xi')-\sigma_{\mathcal{H}}(\xi')=\sigma_{\mathcal{H}}(\xi')L(\xi')\sigma_{\mathcal{H}_k}(\xi').$$

Clearly, the operator with the principal symbol $L(\xi')$ is of order -1, i.e.,

$$(\mathcal{H}_k^{-1} - \mathcal{H}^{-1}) : (H^s(\Gamma))^6 \longrightarrow \big(H^{s+1}(\Gamma)\big)^6, \ \ \forall\, s \in \mathbb{R}.$$

Consequently,

$$I_2 : \big(H^s(\Gamma)\big)^6 \longrightarrow \big(H^{s+3/2}_{\rm loc}(\Omega^-)\big)^6.$$

Thus we finally find that

$$(G_k^- - G^-) : \big(H^{1/2}(\Gamma)\big)^6 \longrightarrow \big(H^3_{\rm loc}(\Omega^-)\big)^6$$

and for the operator $S_k^- - S^-$ we have

$$(S_k^- - S^-) : \big(H^{1/2}(\Gamma)\big)^6 \longrightarrow \big(H^{3/2}(\Gamma)\big)^6. \tag{20}$$

Taking into account (20), property (iii) for the operator S_k^-, and the fact that the operator of embedding of the space $(H^{1/2-\gamma}(\Gamma))^6$ $(0 < \gamma < 1/2)$ in the space $(H^{-3/2}(\Gamma))^6$ is compact, we obtain

$$\begin{aligned}\langle S^- h, h\rangle &= \langle S_k^- h, h\rangle - \langle (S_k^- - S^-)h, h\rangle \geq \\ &\geq \langle S_k^- h, h\rangle - \|(S_k^- - S^-)h\|_{-1/2,\Gamma}\|h\|_{1/2,\Gamma} \geq \\ &\geq c_1\|h\|^2_{1/2,\Gamma} - c_0\|h\|_{1/2-\gamma,\Gamma}\|h\|_{1/2,\Gamma}.\end{aligned}$$

Whence for every positive number N we have

$$\langle S^- h, h\rangle \geq \Big(c_1 - \frac{c_0^2}{2N^2}\Big)\|h\|^2_{1/2,\Gamma} - \frac{N^2}{2}\|h\|^2_{1/2-\gamma,\Gamma} \tag{21}$$

for all $h \in (H^{1/2}(\Gamma))^6$.

By Erling's lemma (see [11]), for all $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\|h\|_{1/2-\gamma,\Gamma} \leq \delta\|h\|_{1/2,\Gamma} + c(\delta)\|h\|_{0,\Gamma}. \tag{22}$$

If we take into account (22) and choose appropriately the positive numbers δ and N, from (21) we get

$$\langle S^- h, h\rangle \geq c\|h\|^2_{1/2,\Gamma} - \|h\|^2_{0,\Gamma} \tag{23}$$

for all $h \in (H^{1/2}(\Gamma))^6$.

Here we shall use the lemma whose proof can be found in [5] and which is formulated as follows.

Let H and Y be the real Hilbert spaces, $H \subset Y$, H be dense in Y, and the embedding operator $I : H \longrightarrow Y$ be compact. Moreover, let $a : H\times H \longrightarrow \mathbb{R}$ be a nonnegative, symmetric, continuous bilinear form for which there exist positive numbers α_1 and α_2 such that

$$a(u,u) \geq \alpha_1\|u\|^2_H - \alpha_2\|u\|^2_Y$$

for all $u \in H$.

By $I - P$ we denote the operator of the orthogonal projection (in the sense of H) of the space H onto $\operatorname{Ker} a$. Then the following lemma is valid.

Lemma. $\exists\, c > 0$ *such that*

$$a(u,u) \geq c\|Pu\|^2_H$$

for all $u \in H$.

Taking now into account estimate (23) and the fact that the equation $\langle S^- h, h\rangle = 0$ has only trivial solution, from that lemma ($H = (H^{1/2}(\Gamma))^6$, $Y = (L^2(\Gamma))^6$, $a(h,g) = \langle S^- h, g\rangle$) we finally conclude that condition (iii) is fulfilled for the operator S^-:

$$\langle S^- h, h\rangle \geq c\|h\|^2_{1/2,\Gamma}$$

for all $h \in (H^{1/2}(\Gamma))^6$.

Finally, for investigating the variational inequality (9), we consider on a convex closed set $\mathcal{K}$ the functional

$$I(h) = -\frac{1}{2}\,\langle S^- h, h\rangle + j(\xi) - \int\limits_{\Gamma_2} (F_N \xi_N + \psi \cdot \eta)\, ds, \quad \forall\, h \in (\xi, \eta) \in \mathcal{K}.$$

It can be easily verified that by virtue of property (i) of the operator S^- the solution of inequality (9) is equivalent to the minimization of the functional $I(h)$ on the set $\mathcal{K}$. Taking into account property (iii) of the operator S^- and the fact that $j(\xi) \geq 0$, we obtain the coerciveness of the functional $I(h)$ (i.e. $I(h) \to +\infty$ as $\|h\|_{1/2,\Gamma} \to \infty$):

$$I(h) \geq c\|h\|^2_{1/2,\Gamma} - c_1\|h\|_{1/2,\Gamma}$$

for all $h \in \mathcal{K}$.

On the basis of the well-known results concerning the variational inequalities (see [12], [13), we conclude that Problem $(F)^-$ has the unique solution despite the fact that Γ_1 is of positive measure or empty (in this case $\Gamma_2 = \Gamma$ and the corresponding changes taking place in the statement of Problem $(I)^-$ are clear).

Thus we obtain the following theorem

Theorem 2. *If* $F_N \in L^\infty(\Gamma_2)$, $\varphi \in (H^{1/2}(\Gamma))^3$, $\psi \in (L^\infty(\Gamma_2)^3$ *and* $\mathcal{F} \in L^\infty(\Gamma_2)$ $(\mathcal{F} \geq 0)$, *then Problem* $(F)^-$ *has the unique solution of the class* $(H^1_{\rm loc}(\Omega^-))^6$.

In conclusion, it should be noted that the problem formulated below is investigated analogously to Problem $(I)^-$.

Problem $(II)^-$**.** Let $\mathcal{G} \in (L^2_{loc}(\Omega^-))^6$, $\chi \in (H^1(\Gamma_1))^6$ and $f, \varphi \in L^\infty(\Gamma_2)$. Find the vector function $U \in (H^1_{\rm loc}(\Omega^-))^6$ which is a weak solution of equation (4), tangential components of force and moment stresses on Γ_2 are the functions of the class $(L^\infty(\Gamma_2))^3$, $U = \chi$ on Γ_1, and on Γ_2 the following conditions are fulfilled:

$$\sigma_n(U) = f, \quad \mu_n(U) = \varphi, \quad |\sigma_{\scriptscriptstyle T}(U)| < g_1 \Longrightarrow u_{\scriptscriptstyle T} = 0, \quad |\sigma_{\scriptscriptstyle T}(U)| = g_1 \Longrightarrow$$

$$\Longrightarrow \exists\, \gamma_1 > 0 : \; u_{_T} = -\gamma_1 \sigma_{_T}(U);$$

$$|\mu_{_T}(U)| < g_2 \Longrightarrow \omega_{_T} = 0, \quad |\mu_{_T}(U)| = g_2 \Longrightarrow \exists\, \gamma_2 > 0 : \; \omega_{_T} = -\gamma_2 \mu_{_T}(U),$$

where $g_1 = \mathcal{F}|(\sigma_n(U)|$ and $g_2 = |\mathcal{F}|(\mu_n(U)|$.

References

1. G. Duvaut and J.-L. Lions, Les inéquations en mécanique et en physique. *Dunod, Paris,* 1972; Russian transl.: *Nauka, Moscow,* 1980.
2. I. Hlavaček, J. Haslinger, J. Nečas, and J. Lovišek, Solution of variational inequalities in mechanics. (Slovak) *ALFA, Bratislava,* 1982; Russian transl.: *Mir, Moscow,* 1986.
3. J. Jarušek, Contact problems with bounded friction. Coercive case. *Czechoslovak Math. J.* **33**(1983), No. 2, 237–261.
4. R. Gachechiladze, Signorinis problem with friction for a layer in the couple-stress elasticity. *Proc. A. Razmadze Math. Inst.* **122**(2000), 45–57.
5. R. Gachechiladze and O. Maisaia, Some boundary value problems of the couple-stress elasticity with friction. *Proc. A. Razmadze Math. Inst.* **119**(1999), 25–41.
6. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. *North-Holland Series in Applied Mathematics and Mechanics,* **25**, *North-Holland Publ. Co., Amsterdam-New York-Oxford* 1979; (Russian) *Nauka, Moscow,* 1976.
7. J.-L. Lions and E. Magenes, Problemes aux limites nonhomogenes et applications. **1, 2,** *Dunod, Paris,* 1968; Russian transl.: *Mir, Moscow,* 1971.
8. H. Triebel, Interpolation theory, functional spaces, differential operators. *North-Holland, Amsterdam,* 1968; Russian transl.: *Mir, Moscow*, 1980.
9. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks. *Proc. I. Vekua Inst. Appl. Math.* **39**(1990), 68–84.
10. A. Gachechiladze and D. Natroshvili, Boundary variational inequality approach in the anisotropic elasticity for the Signorini problem. *Georgian Math. J.* **8**(2001), No. 3, 469–492.
11. G. Fichera, Existence theorems in elastisity. *Handb. Phys.* **VI/2**, No. 3, *Springer-Verlag, Heidelberg,* 1972; Russian transl.: *Mir, Moscow,* 1974.
12. J.-L. Lions, Some methods of solution of non-linear boundary value problems. *Dunod, Paris,* 1969; Russian transl.: *Mir, Moscow,* 1972.
13. A. Fridman, Variational principles and free boundary problems. *New York, Chichester, Brisbone, Toronto, Singapore,* 1982; Russian transl.: *Nauka, Moscow,* 1990.

(Received 19.05.2003)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia