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# BOUNDARY-CONTACT PROBLEMS FOR ELASTIC HEMITROPIC BODIES

*Dedicated to Mikheil Basheleishvili*
*on the occasion of his 80-th birthday*

**Abstract.** The contact problems of two elastic hemitropic bodies with different elastic properties under the condition of natural impenetrability of one medium into the other, is investigated. Using the theory of spatial variational inequalities, the existence and uniqueness of a weak solution is studied. The coercive case (when an elastic medium is fixed along a part of the boundary), as well as the non-coercive case (the boundaries of elastic media are not fixed) is considered. In the latter case, the necessary conditions for the existence of a solution are written out explicitly.

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[illegible]

## 1. Introduction

In the present work we consider the unilateral (one-sided) frictionless contact of two elastic hemitropic media with different physical properties under the condition of natural impenetrability, when under deformation one medium does not penetrate into the other. Such kind of problems with various modifications have been investigated in the classical theory of elasticity [16].

Here we consider the model of the theory of elasticity in which, unlike the classical theory, an elementary particle of a body along with displacements undergoes rotation, and hence the condition of mechanical equilibrium of the body is described by means of the three-component displacement vector and three-component micro-rotation vector.

The origin of the rational theories of polar continua goes back to brothers E. and F. Cosserat [3], [4], who gave a development of the mechanics of continuous media in which each material point has the six degrees of freedom defined by 3 displacement components and 3 microrotation components (for the history of the problem see [6], [19], [26], [34], and the references therein).

A micropolar solid which is not isotropic with respect to inversion is called *hemitropic, noncentrosymmetric,* or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules – DNA, as well as on a large scale, as in composites with helical or screw-shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For more details and applications see the references [1], [2], [3], [6], [8], [15], [20], [21], [27], [28], [34], [37], [39], [43].

Refined mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1], [2]. In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and moment stress tensor, which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor via the constitutive equations. All these quantities are expressed in terms of the components of the displacement and microrotation vectors. In turn the displacement and microrotation vectors satisfy a coupled complex system of second order partial differential equations. We note that the governing equations in this model become very involved and generate $6 \times 6$ matrix partial differential operator of second order. Evidently, the corresponding $6 \times 6$ matrix boundary differential operators describing the force stress and couple stress vectors have also an involved structure in comparison with the classical case.

In [29], [30], [31], [32] the fundamental matrices of the associated systems of partial differential equations of statics and steady state oscillations have been constructed explicitly in terms of elementary functions and the basic boundary value and transmission problems of hemitropic elasticity have been studied by the potential method for smooth and non-smooth Lipschitz domains. Particular problems of the elasticity theory of hemitropic continuum have been considered in [7], [20], [21], [22], [23], [34], [35], [36], [42].

The main goal of the present paper is the study of frictionless contact problems for hemitropic elastic solids, their mathematical modelling as transmission-boundary value problems with the natural impenetrability conditions and their analysis with the help of the spatial variational inequality technique.

In classical elasticity, similar problems have been considered in many monographs and papers (see, e.g., [5], [9], [10], [11], [14], [16], [17], [18], [38], and the references therein).

The work consists of Introduction and a number of sections. First we write out the basic equations of statics of the theory of elasticity for hemitropic media in vector and matrix forms, introduce the stress operator and the potential energy quadratic form. Then we formulate the contact problem for two elastic homogeneous hemitropic bodies with different elastic properties under the condition of natural impenetrability of one body into the other. We consider the coercive case when the bodies are fixed along some parts of their boundaries. The problem is reduced equivalently to the spatial variational inequality. We present an analysis of the existence and uniqueness of a weak solution of the variational inequality and investigate the question of continuous dependence of a solution on the data of the problem. Finally, we study a non-coercive case when the contacting bodies are not fixed. In this case, the necessary conditions for the existence of a solution are written out explicitly.

## 2. The Basic Equations and Green's Formula

Let $\Omega \in \mathbb{R}^3$ be a bounded simply connected domain with a piecewise smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$.

We assume that $\Omega$ is occupied by a homogeneous hemitropic elastic material. Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the *displacement vector* and the *micro-rotation vector*, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition.

In the hemitropic elasticity theory we have the following constitutive equations for the *force stress tensor* $\{\tau_{pq}\}$ and the *couple stress tensor* $\{\mu_{pq}\}$ :

$$\tau_{pq} = \tau_{pq}(U) := (\mu+\alpha)\partial_p u_q + (\mu-\alpha)\partial_q u_p + \lambda\delta_{pq}\operatorname{div} u + \delta\delta_{pq}\operatorname{div}\omega + \\ + (\varkappa+\nu)\partial_p\omega_q + (\varkappa-\nu)\partial_q\omega_p - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k, \tag{2.1}$$

$$\mu_{pq} = \mu_{pq}(U) := \delta\delta_{pq}\operatorname{div} u + (\varkappa+\nu)\Big[\partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\Big] + \beta\delta_{pq}\operatorname{div}\omega + \\ + (\varkappa-\nu)\Big[\partial_q u_p - \sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\Big] + (\gamma+\varepsilon)\partial_p\omega_q + (\gamma-\varepsilon)\partial_q\omega_p, \tag{2.2}$$

where $U=(u,\omega)^\top$, $\delta_{pq}$ is the Kronecker delta, $\partial=(\partial_1,\partial_2,\partial_3)$ with $\partial_j=\partial/\partial x_j$, $\varepsilon_{pqk}$ is the permutation (Levi-Civitá) symbol, and $\alpha$, $\beta$, $\gamma$, $\delta$, $\lambda$, $\mu$, $\nu$, $\varkappa$ and $\varepsilon$ are the material constants [1], [30].

The components of the force stress vector $\tau^{(n)}=(\tau_1^{(n)},\tau_2^{(n)},\tau_3^{(n)})^\top$ and the couple stress vector $\mu^{(n)}=(\mu_1^{(n)},\mu_2^{(n)},\mu_3^{(n)})^\top$, acting on a surface element with a normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau_q^{(n)}=\sum_{p=1}^{3}\tau_{pq}n_p,\quad \mu_q^{(n)}=\sum_{p=1}^{3}\mu_{pq}n_p,\quad q=1,2,3. \tag{2.3}$$

Denote by $T(\partial,n)$ the generalized $6\times 6$ matrix differential stress operator [30]

$$T(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6},\quad T^{(j)}=\big[T_{pq}^{(j)}\big]_{3\times 3},\quad j=\overline{1,4}, \tag{2.4}$$

where

$$\begin{aligned}
&T_{pq}^{(1)}(\partial,n)=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,\\
&T_{pq}^{(2)}(\partial,n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,\\
&T_{pq}^{(3)}(\partial,n)=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,\\
&T_{pq}^{(4)}(\partial,n)=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.
\end{aligned} \tag{2.5}$$

Here $\partial_n=\partial/\partial n$ denotes the directional derivative along the vector $n$ (normal derivative).

From formulas (2.1), (2.2) and (2.3) it can be easily checked that

$$(\tau^{(n)},\mu^{(n)})^\top=T(\partial,n)U.$$

The equilibrium equations of statics in the theory of hemitropic elasticity read as [30]

$$\begin{gathered}
\sum_{p=1}^{3}\partial_p\tau_{pq}(x)+\varrho F_q(x)=0,\\
\sum_{p=1}^{3}\partial_p\mu_{pq}(x)+\sum_{l,r=1}^{3}\varepsilon_{qlr}\tau_{lr}(x)+\varrho G_q(x)=0,\quad q=1,2,3,
\end{gathered}$$

where $\varrho$ is the mass density of the elastic material, and $F=(F_1,F_2,F_3)^\top$ and $G=(G_1,G_2,G_3)^\top$ are the body force and body couple vectors.

Using the constitutive equations (2.1) and (2.2) we can rewrite the equilibrium equations in terms of the displacement and micro-rotation vectors,

$$
\begin{gathered}
(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,u(x)+(\varkappa+\nu)\Delta\omega(x)+\\
+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,\omega(x)+2\alpha\,\mathrm{curl}\,\omega(x)+\varrho F(x)=0,\\
(\varkappa+\nu)\Delta u(x)+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,u(x)+2\alpha\,\mathrm{curl}\,u(x)+\\
+(\gamma+\varepsilon)\Delta\omega(x)+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\omega(x)+4\nu\,\mathrm{curl}\,\omega(x)-\\
-4\alpha\omega(x)+\varrho G(x)=0,
\end{gathered}
\tag{2.6}
$$

where $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator.

Let us introduce the matrix differential operator generated by the left hand side expressions of the system (2.6):

$$
L(\partial):=\begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial)\\ L^{(3)}(\partial) & L^{(4)}(\partial)\end{bmatrix}_{6\times 6},
\tag{2.7}
$$

where

$$
\begin{gathered}
L^{(1)}(\partial):=(\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),\\
L^{(2)}(\partial)=L^{(3)}(\partial):=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial)+2\alpha R(\partial),\\
L^{(4)}(\partial):=\big[(\gamma+\varepsilon)\Delta-4\alpha\big]I_3+(\beta+\gamma-\varepsilon)Q(\partial)+4\nu R(\partial).
\end{gathered}
\tag{2.8}
$$

Here and in the sequel $I_k$ stands for the $k\times k$ unit matrix and

$$
Q(\partial):=[\partial_k\partial_j]_{3\times 3},\quad R(\partial):=\begin{bmatrix} 0 & -\partial_3 & \partial_2\\ \partial_3 & 0 & -\partial_1\\ -\partial_2 & \partial_1 & 0\end{bmatrix}_{3\times 3}.
$$

It is easy to see that

$$
R(\partial)u=\begin{bmatrix}\partial_2u_3-\partial_3u_2\\ \partial_3u_1-\partial_1u_3\\ \partial_1u_2-\partial_2u_1\end{bmatrix}=\mathrm{curl}\,u,\quad Q(\partial)u=\mathrm{grad}\,\mathrm{div}\,u.
$$

Equations (2.6) can be written in matrix form as

$$
L(\partial)U(x)+\Psi(x)=0 \ \text{ with } \ U=(u,\omega)^\top,\quad \Psi=(\Psi^{(1)},\Psi^{(2)})^\top:=(\varrho F,\varrho G)^\top.
$$

Note that the operator $L(\partial)$ is formally self-adjoint, i.e., $L(\partial)=[L(-\partial)]^\top$.

2.1. **Green's formulas.** For real-valued vector functions $U=(u,\omega)^\top$ and $U'=(u',\omega')^\top$ from the class $[C^2(\overline{\Omega})]^6$ the following Green formula holds [30]

$$
\int\limits_{\Omega}\big[L(\partial)U\cdot U'+E(U,U')\big]\,dx=\int\limits_{S}\big\{T(\partial,n)U\big\}^+\cdot\{U'\}^+\,dS,
\tag{2.9}
$$

where $\{\cdot\}^+$ denotes the trace operator on $S$ from $\Omega$, while $E(\cdot,\cdot)$ is the bilinear form defined by the equality:

$$
E(U,U')=E(U',U)=\sum_{p,q=1}^{3}\Big\{(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp}+
$$

$$
+(\varkappa+\nu)(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq})+(\varkappa-\nu)(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp})+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq}+
$$

$$+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp})+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\}, \tag{2.10}$$

where $u_{pq}$ and $\omega_{pq}$ are the so called *strain* and *torsion* (*curvature*) tensors for hemitropic bodies,

$$u_{pq}=u_{pq}(U)=\partial_p u_q-\sum_{k=1}^{3}\epsilon_{pqk}\omega_k,\quad \omega_{pq}=\omega_{pq}(U)=\partial_p\omega_q,\quad p,q=1,2,3. \tag{2.11}$$

Here and in what follows $a\cdot b$ denotes the usual scalar product of two vectors $a,b\in\mathbb{R}^m$: $a\cdot b=\sum_{j=1}^m a_j b_j$.

From formulas (2.10) and (2.11) we get

$$\begin{aligned}
E(U,U')=&\ \frac{3\lambda+2\mu}{3}\Big(\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div}u'+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big)+\\
&+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega')+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'+\\
&+\frac{\mu}{2}\sum_{k,j=1,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big]\Big[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)\Big]+\\
&+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big]\Big[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big]+\\
&+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big(\frac{\partial\omega'_k}{\partial x_j}+\frac{\partial\omega'_j}{\partial x_k}\Big)+\\
&+\frac{1}{3}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big(\frac{\partial\omega'_k}{\partial x_k}-\frac{\partial\omega'_j}{\partial x_j}\Big)\Big]+\\
&+\alpha\Big(\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big)\cdot\Big(\operatorname{curl}u'+\frac{\nu}{\alpha}\operatorname{curl}\omega'-2\omega'\Big).
\end{aligned}\tag{2.12}$$

The potential energy density function $E(U,U)$ is a positive definite quadratic form with respect to the variables $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $c_0>0$ depending only on the material constants, such that

$$E(U,U)\geq c_0\sum_{p,q=1}^{3}\big[u_{pq}^2+\omega_{pq}^2\big]. \tag{2.13}$$

The necessary and sufficient conditions for the quadratic form $E(U,U)$ to be positive definite are the following inequalities (see [2], [6], [12])

$$\begin{gathered}
\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\
(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\
\mu\big[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2\big]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\
\mu\big[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2\big]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.
\end{gathered}$$

Let us note that, if the condition $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities

$$
\begin{gathered}
\mu > 0, \;\; \alpha > 0, \;\; \gamma > 0, \;\; \varepsilon > 0, \\
3\lambda + 2\mu > 0, \;\; \mu\gamma - \varkappa^2 > 0, \;\; \alpha\varepsilon - \nu^2 > 0, \\
(\mu+\alpha)(\gamma+\varepsilon) - (\varkappa+\nu)^2 > 0, \\
(3\lambda+2\mu)(3\beta+2\gamma) - (3\delta+2\varkappa)^2 > 0.
\end{gathered} \tag{2.14}
$$

For simplicity in what follows we assume that $3\lambda + 2\mu > 0$ and therefore conditions (2.14) imply positive definiteness of the energy quadratic form $E(U,U)$ defined by (2.12).

The following assertion describes the null space of the energy quadratic form $E(U,U)$ (see [30]).

**Lemma 2.1.** *Let $U = (u,\omega)^\top \in [C^1(\overline{\Omega})]^6$ and $E(U,U) = 0$ in $\Omega$. Then*

$$
u(x) = [a \times x] + b, \quad \omega(x) = a, \;\; x \in \Omega,
$$

*where $a$ and $b$ are arbitrary three-dimensional constant vectors and symbol $[\cdot \times \cdot]$ denotes the cross product of two vectors.*

Vectors of type $([a\times x]+b, a)$ are called *generalized rigid displacement vectors.* Note that a generalized rigid displacement vector vanishes identically if it vanishes at a single point.

Throughout the paper $L_p(\Omega)$ $(1 \le p < \infty)$ and $H^s(\Omega) = H^s_2(\Omega)$, $s \in \mathbb{R}$ denote Lebesgue and Bessel potential spaces (see, e.g., [24], [40], [41]). The corresponding norms we denote by symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. Denote by $\mathcal{D}(\Omega)$ the class of $C^\infty(\Omega)$ functions with support in the domain $\Omega$. If $M$ is an open proper part of the manifold $\partial\Omega$, i.e., $M \subset \partial\Omega$, $M \ne \partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ onto $M$,

$$
H^s(M) := \big\{ r_{_M}\varphi : \;\varphi \in H^s(\partial\Omega) \big\},
$$

where $r_{_M}$ denotes the restriction operator onto the set $M$. Further, let

$$
\widetilde{H}^s(M) := \big\{ \varphi \in H^s(\partial\Omega) : \; \operatorname{supp}\varphi \subset \overline{M} \big\}.
$$

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (2.11) it follows that

$$
B(U,U) := \int\limits_{\Omega} E(U,U)\, dx \ge 0. \tag{2.15}
$$

Moreover, there exist positive constants $c_1$ and $c_2$, depending only on the material parameters, such that the inequality

$$
\begin{aligned}
B(U,U) \ge c_1 \int\limits_{\Omega} \Big\{ \sum_{p,q=1}^{3} \big[(\partial_p u_q)^2 + (\partial_p \omega_q)^2\big] + \sum_{q=1}^{3} [u_q^2 + \omega_q^2] \Big\}\, dx \\
- c_2 \int\limits_{\Omega} \sum_{q=1}^{3} [u_q^2 + \omega_q^2]\, dx
\end{aligned} \tag{2.16}
$$

holds for an arbitrary real-valued vector function $U \in [C^1(\overline{\Omega})]^6$. By standard limiting arguments we easily conclude that for any $U \in [H^1(\Omega)]^6$ the following Korn's type inequality holds (cf. [10, Part I, § 12])

$$B(U,U) \geq c_1 \|U\|^2_{[H^1(\Omega)]^6} - c_2 \|U\|^2_{[H^0(\Omega)]^6}. \tag{2.17}$$

*Remark* 2.1. If $U \in [H^1(\Omega)]^6$ and on some open part $S^* \subset \partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $\{U\}^+_{S^*} = 0$, then we have the strict Korn's inequality

$$B(U,U) \geq c \|U\|^2_{[H^1(\Omega)]^6} \tag{2.18}$$

with some positive constant $c > 0$ which does not depend on the vector $U$. This follows from (2.17) and the fact that in this case $B(U,U) > 0$ for $U \neq 0$ (see, e.g., [33], [25], Ch.2, Exercise 2.17).

*Remark* 2.2. By standard limiting arguments Green's formula (2.9) can be extended to Lipschitz domains and to vector functions $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see, [33], [24]),

$$\int\limits_{\Omega} \big[ L(\partial)U \cdot U' + E(U,U') \big] \, dx = \big\langle \{T(\partial,n)U\}^+, \{U'\}^+ \big\rangle_{\partial\Omega}, \tag{2.19}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$, which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By this relation the generalized trace of the stress operator $\{T(\partial,n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is correctly determined. Note that for arbitrary real valued vector functions $V, V' \in [L_2(\partial\Omega)]^6$ we have

$$\langle V, V' \rangle_{\partial\Omega} = \int\limits_{\partial\Omega} V \cdot V' \, dS.$$

## 3. Statement of the Problem

Let $\Omega_q \in \mathbb{R}^3$, $q = 1,2$ be a simply connected bounded domain whose Lipschitz piecewise smooth boundary $S_q := \partial\Omega_q$ falls into three mutually disjoint portions $S_q^D$, $S_q^N$ and $S_c$, such that $\overline{S}_q^D \cup \overline{S}_q^N \cup \overline{S}_c = S_q$, $\overline{S}_q^D \cap \overline{S}_c = \varnothing$ and $S_c \subset C^{2,\alpha'}$, $\alpha' \in (0;1)$. Denote by $n^{(q)}(x)$ the unit, outward with respect to $\Omega_q$, normal at the point $x \in S_q$. Let $\Omega_1$ and $\Omega_2$ be filled with hemitropic materials of different elastic properties. We assume that the boundaries of the domains $\Omega_1$ and $\Omega_2$ have a part $S_c$ in common, such that $\overline{S}_c := S_1 \cap S_2$. The elastic constants corresponding to the elastic medium $\Omega_q$ are $\alpha^{(q)}$, $\beta^{(q)}$, $\gamma^{(q)}$, $\delta^{(q)}$, $\lambda^{(q)}$, $\mu^{(q)}$, $\nu^{(q)}$, $\varkappa^{(q)}$ and $\varepsilon^{(q)}$, $q = 1,2$. Analogously, $u^{(q)} = (u_1^{(q)}, u_2^{(q)}, u_3^{(q)})^\top$ and $\omega^{(q)} = (\omega_1^{(q)}, \omega_2^{(q)}, \omega_3^{(q)})^\top$ denote the displacement and microrotation vectors in the domain $\Omega_q$, $E^{(q)}(U^{(q)}, U^{(q)})$ designates the corresponding potential energy density, $L^{(q)}(\partial)$ and $T^{(q)}(\partial, n^{(q)})$ are the corresponding differential operators given by formulas (2.7), (2.8) and (2.4), (2.5).

3.1. **The statement of the problem and the corresponding variational inequality.** In the sequel, we will be concerned with weak solutions of the corresponding differential equations. By definition, the vector function $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$ is called a weak solution of the equation

$$L^{(q)}(\partial)U^{(q)} + \mathcal{G}^{(q)} = 0, \quad \mathcal{G}^{(q)} \in [L_2(\Omega_q)]^6 \tag{3.1}$$

in the domain $\Omega_q$, if for every $\Phi \in [\mathcal{D}(\Omega_q)]^6$

$$B^{(q)}(U^{(q)}, \Phi) := \int\limits_{\Omega_q} E^{(q)}(U^{(q)}, \Phi)\, dx, = \int\limits_{\Omega_q} \mathcal{G}^{(q)} \cdot \Phi\, dx, \quad q = 1, 2,$$

where $E^{(q)}(U^{(q)}, \Phi)$ is defined by formula (2.12).

Below, for the force and moment stress vectors we use the notation

$$\mathcal{T}^{(q)}U^{(q)} = T^{1(q)}u^{(q)} + T^{2(q)}\omega^{(q)}, \quad \mathcal{M}^{(q)}U^{(q)} = T^{3(q)}u^{(q)} + T^{4(q)}\omega^{(q)},$$

and for the normal and tangential components of the force stress vector we will use, respectively, the notation

$$(\mathcal{T}^{(q)}U^{(q)})_{n^{(q)}} := (\mathcal{T}^{(q)}U^{(q)}) \cdot n^{(q)},$$
$$(\mathcal{T}^{(q)}U^{(q)})_t := \mathcal{T}^{(q)}U^{(q)} - n^{(q)}(\mathcal{T}^{(q)}U^{(q)})_{n^{(q)}}.$$

Let

$$\mathcal{G}^{(q)} = (F^{(q)}, G^{(q)})^\top \in [L_2(\Omega_q)]^6, \quad \Psi^{(q)} = (\Psi^{1(q)}, \Psi^{2(q)})^\top \in \big[\widetilde{H}^{-1/2}(S_q^N)\big]^6$$

and consider the following boundary-contact problem.

**Problem (A).** Find vector functions $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top \in [H^1(\Omega_q)]^6$, $q = 1, 2$ which in the domains $\Omega_1$ and $\Omega_2$ are the solutions of equation (3.1) and satisfy the following conditions:

$$r_{S_q^D}\{U^{(q)}\}^+ = 0 \ \text{ on } \ S_q^D, \ \ q = 1, 2, \tag{3.2}$$

$$r_{S_q^N}\big\{T^{(q)}(\partial, n^{(q)})U^{(q)}\big\}^+ = \Psi^{(q)} \ \text{ on } \ S_q^N, \ \ q = 1, 2, \tag{3.3}$$

$$r_{S_c}\big\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\big\}^+ \le 0 \ \text{ on } \ S_c, \tag{3.4}$$

$$r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+ = r_{S_c}\big\{(\mathcal{T}^{(2)}U^{(2)})_{n^{(2)}}\big\}^+ \le 0 \ \text{ on } \ S_c, \tag{3.5}$$

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+, r_{S_c}\big\{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\big\}^+\Big\rangle_{S_c} = 0 \ \text{ on } \ S_c, \tag{3.6}$$

$$r_{S_c}\big\{(\mathcal{T}^{(q)}U^{(q)})_t\big\}^+ = 0 \ \text{ on } \ S_c, \ \ q = 1, 2, \tag{3.7}$$

$$r_{S_c}\big\{\mathcal{M}^{(q)}U^{(q)}\big\}^+ = 0 \ \text{ on } \ S_c, \ \ q = 1, 2. \tag{3.8}$$

We introduce here the notation

$$\mathbb{H}^1 = \Big\{U = (U^{(1)}, U^{(2)})^\top : \ U^{(q)} \in [H^1(\Omega_q)]^6, \ q = 1, 2\Big\},$$

$$\|U\|_{1,\Omega}^2 := \sum_{q=1}^2 \|U^{(q)}\|_{[H^1(\Omega_q)]^6}^2, \quad \mathcal{B}(U, V) = \sum_{q=1}^2 B^{(q)}(U^{(q)}, V^{(q)}),$$

$$\mathcal{K} = \Big\{ U \in \mathbb{H}^1 : \ r_{S_q^D} \{U^{(q)}\}^+ = 0, \ r_{S_c} \{u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)}\}^+ \leq 0 \Big\},$$

$$\mathcal{L}(V) = \sum_{q=1}^{2} \bigg[ \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)}\}^+ \big\rangle_{S_q^N} + \int\limits_{\Omega_q} \mathcal{G}^{(q)} \cdot V^{(q)} \, dx \bigg], \ \ \forall V \in \mathcal{K}.$$

On a convex closed set $\mathcal{K}$ we consider the following variational inequality: Find $U \in \mathcal{K}$, such that the inequality

$$\mathcal{B}(U, V - U) \geq \mathcal{L}(V - U) \tag{3.9}$$

is fulfilled for all $V \in \mathcal{K}$.

First of all, we investigate the question on the uniqueness of a solution of Problem (A).

3.2. **The uniqueness of a solution.** The following theorem is valid.

**Theorem 3.1.** *Problem* (A) *has no more than one solution.*

*Proof.* Let $U = (U^{(1)}, U^{(2)})^\top$, $U^{(q)} = (u^{(q)}, \omega^{(q)})^\top$ and $W = (W^{(1)}, W^{(2)})^\top$, $W^{(q)} = (v^{(q)}, w^{(q)})^\top$ be two distinct solutions of Problem (A). Then the difference $\widetilde{U} := U - W$ will satisfy the conditions (3.2), (3.7), (3.8), equation (3.1) with $\mathcal{G}^{(q)} = 0$ and the condition (3.3) with $\Psi^{(q)} = 0$. From condition (3.5) we have

$$r_{S_c} \big\{ (\mathcal{T}^{(1)} \widetilde{U}^{(1)})_{n^{(1)}} \big\}^+ = r_{S_c} \big\{ (\mathcal{T}^{(2)} \widetilde{U}^{(2)})_{n^{(2)}} \big\}^+.$$

Using Green's formula (see (2.19)) and taking into account the above conditions, we have

$$\sum_{q=1}^{2} \int\limits_{\Omega_q} E^{(q)}(\widetilde{U}^{(q)}, \widetilde{U}^{(q)}) \, dx = \sum_{q=1}^{2} \Big\langle \big\{ T^{(q)}(\partial, n^{(q)}) \widetilde{U}^{(q)} \big\}^+, \{\widetilde{U}^{(q)}\}^+ \Big\rangle_{S_q} =$$

$$= \sum_{q=1}^{2} \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(q)} \widetilde{U}^{(q)})_{n^{(q)}} \big\}^+, r_{S_c} \big\{ \widetilde{u}^{(q)} \cdot n^{(q)} \big\}^+ \Big\rangle_{S_c} =$$

$$= \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} \widetilde{U}^{(1)})_{n^{(1)}} \big\}^+, r_{S_c} \big\{ \widetilde{u}^{(1)} \cdot n^{(1)} + \widetilde{u}^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} =$$

$$= \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}} \big\}^+ - r_{S_c} \big\{ (\mathcal{T}^{(1)} W^{(1)})_{n^{(1)}} \big\}^+,$$
$$r_{S_c} \Big\{ u^{(1)} \cdot n^{(1)} - v^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)} - v^{(2)} \cdot n^{(2)} \Big\}^+ \Big\rangle_{S_c} =$$

$$= \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}} \big\}^+ - r_{S_c} \big\{ (\mathcal{T}^{(1)} W^{(1)})_{n^{(1)}} \big\}^+,$$
$$r_{S_c} \big\{ u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)} \big\}^+ - r_{S_c} \big\{ v^{(1)} \cdot n^{(1)} + v^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} =$$

$$= - \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}} \big\}^+, r_{S_c} \big\{ v^{(1)} \cdot n^{(1)} + v^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} -$$
$$- \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} W^{(1)})_{n^{(1)}} \big\}^+, r_{S_c} \big\{ u^{(1)} \cdot n^{(1)} + u^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} \leq 0.$$

Whence bearing in mind the fact that the quadratic form $E^{(q)}(\widetilde{U}^{(q)},\widetilde{U}^{(q)})$ is positive definite (see (2.13)), we have

$$E^{(q)}(\widetilde{U}^{(q)},\widetilde{U}^{(q)})=0,\quad q=1,2.$$

By Lemma 2.1

$$\widetilde{U}^{(q)}=\Big([a^{(q)}\times x]+b^{(q)},a^{(q)}\Big)^\top,\quad q=1,2.$$

Since $r_{S_q^D}\{\widetilde{U}^{(q)}\}^+=0$, we conclude $a^{(q)}=b^{(q)}=0$. Thus $\widetilde{U}^{(q)}=0$. $\square$

3.3. **The equivalence.** Now we prove the following equivalence theorem.

**Theorem 3.2.** *Problem* (A) *is equivalent to the variational inequality* (3.9)*, i.e., every weak solution of Problem* (A) *is a solution of inequality* (3.9)*, and vice versa.*

*Proof.* Let $U=(U^{(1)},U^{(2)})^\top\in\mathbb{H}^1$ be a solution of Problem (A). By virtue of the interior regularity theorems (see [10]), $U^{(q)}\in[H^2(\Omega_q')]^6$, $q=1,2$ for every domain $\overline{\Omega}_q'\subset\Omega_q$, and hence almost everywhere in the domain $\Omega_q$,

$$L^{(q)}(\partial)U^{(q)}+\mathcal{G}^{(q)}=0,\quad q=1,2. \tag{3.10}$$

For every vector $V=(V^{(1)},V^{(2)})^\top\in\mathcal{K}$, Green's formula (2.9) provides us with

$$0=\sum_{q=1}^{2}\int\limits_{\Omega_q}L^{(q)}(\partial)U^{(q)}\cdot(V^{(q)}-U^{(q)})\,dx=-\sum_{q=1}^{2}B^{(q)}(U^{(q)},V^{(q)}-U^{(q)})+$$
$$+\sum_{q=1}^{2}\Big\langle\big\{T^{(q)}(\partial,n^{(q)})U^{(q)}\big\}^+,\big\{V^{(q)}-U^{(q)}\big\}^+\Big\rangle_{S_q}. \tag{3.11}$$

Since $U=(U^{(1)},U^{(2)})^\top$ is a solution of Problem (A) and $U\in\mathcal{K}$, in view of conditions (3.2)–(3.8), we have

$$\sum_{q=1}^{2}\Big\langle\big\{T^{(q)}(\partial,n^{(q)})U^{(q)}\big\}^+,\big\{V^{(q)}-U^{(q)}\big\}^+\Big\rangle_{S_q}=$$
$$=\sum_{q=1}^{2}\Big[\big\langle\Psi^{(q)},r_{S_q^N}\big\{V^{(q)}-U^{(q)}\big\}^+\big\rangle_{S_q^N}+$$
$$+\Big\langle r_{S_c}\big\{T^{(q)}(\partial,n^{(q)})U^{(q)}\big\}^+,r_{S_c}\big\{V^{(q)}-U^{(q)}\big\}^+\Big\rangle_{S_c}\Big]=$$
$$=\sum_{q=1}^{2}\Big[\big\langle\Psi^{(q)},r_{S_q^N}\big\{V^{(q)}-U^{(q)}\big\}^+\big\rangle_{S_q^N}+$$
$$+\Big\langle r_{S_c}\big\{\mathcal{T}^{(q)}U^{(q)}\big\}^+,r_{S_c}\big\{v^{(q)}-u^{(q)}\big\}^+\Big\rangle_{S_c}\Big]=$$

$$= \sum_{q=1}^{2} \Big[ \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \big\rangle_{S_q^N} +$$

$$+ \Big\langle r_{S_c} \big\{ \big(\mathcal{T}^{(q)} U^{(q)}\big)_{n^{(q)}} \big\}^+, r_{S_c} \big\{ v^{(q)} \cdot n^{(q)} - u^{(q)} \cdot n^{(q)} \big\}^+ \Big\rangle_{S_c} \Big] =$$

$$= \sum_{q=1}^{2} \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \big\rangle_{S_q^N} + \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}} \big\}^+,$$

$$r_{S_c} \big\{ v^{(1)} \cdot n^{(1)} - u^{(1)} \cdot n^{(1)} + v^{(2)} \cdot n^{(2)} - u^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} =$$

$$= \sum_{q=1}^{2} \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \big\rangle_{S_q^N} +$$

$$+ \Big\langle r_{S_c} \big\{ (\mathcal{T}^{(1)} U^{(1)})_{n^{(1)}} \big\}^+, r_{S_c} \big\{ v^{(1)} \cdot n^{(1)} + v^{(2)} \cdot n^{(2)} \big\}^+ \Big\rangle_{S_c} \geq$$

$$\geq \sum_{q=1}^{2} \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \big\rangle_{S_q^N}.$$

Due to the obtained inequality and the relation (3.10), from (3.11) we find that

$$\sum_{q=1}^{2} B^{(q)}(U^{(q)}, V^{(q)} - U^{(q)}) \geq$$

$$\geq \sum_{q=1}^{2} \Big[ \int_{\Omega_q} \mathcal{G}^{(q)} \cdot (V^{(q)} - U^{(q)}) dx + \big\langle \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \big\rangle_{S_q^N} \Big],$$

i.e.,

$$\mathcal{B}(U, V - U) \geq \mathcal{L}(V - U), \;\; \forall\, V \in \mathcal{K}.$$

Let now $U = (U^{(1)}, U^{(2)})^\top \in \mathcal{K}$ be a solution of the variational inequality (3.9) and $\Phi^{(q)} \in [\mathcal{D}(\Omega_q)]^6$ be an arbitrary vector function. Then if we substitute into (3.9) first $U^{(1)} \pm \Phi^{(1)}$ instead of $V^{(1)}$ and null instead of $V^{(2)}$, and then $U^{(2)} \pm \Phi^{(2)}$ instead of $V^{(2)}$ and null instead of $V^{(1)}$, we will get

$$B^{(q)}(U^{(q)}, \Phi^{(q)}) = \int_{\Omega_q} \mathcal{G}^{(q)} \cdot \Phi^{(q)} dx, \quad q = 1, 2,$$

i.e., $U^{(q)} \in [H^1(\Omega_q)]^6$ is a weak solution of equation (3.10) in the domain $\Omega_q$. As above, using Green's formula for every $V = (V^{(1)}, V^{(2)})^\top \in \mathcal{K}$, from the inequality (3.9), in view of (3.10), we obtain

$$\sum_{q=1}^{2} \Big[ \Big\langle r_{S_c} \big\{ T^{(q)}(\partial, n^{(q)}) U^{(q)} \big\}^+, r_{S_c} \big\{ V^{(q)} - U^{(q)} \big\}^+ \Big\rangle_{S_c} +$$

$$+ \Big\langle r_{S_q^N} \big\{ T^{(q)}(\partial, n^{(q)}) U^{(q)} \big\}^+ - \Psi^{(q)}, r_{S_q^N} \{V^{(q)} - U^{(q)}\}^+ \Big\rangle_{S_q^N} \Big] \geq 0. \quad (3.12)$$

Substituting into (3.12) $U^{(q)} \pm \Phi^{(q)}$ instead of $V^{(q)}$, where $\Phi^{(q)} \in [H^1(\Omega_q)]^6$ and $\{\Phi^{(q)}\}^+ \in [\widetilde{H}^{1/2}(S_q^N)]^6$, we find that

$$r_{S_q^N}\big\{T^{(q)}(\partial, n^{(q)})U^{(q)}\big\}^+ = \Psi^{(q)}, \quad q = 1, 2, \tag{3.13}$$

i.e., the condition (3.3) is fulfilled. The conditions (3.2) and (3.4) are fulfilled automatically, since $U \in \mathcal{K}$. Taking into account (3.13), inequality (3.12) takes the form

$$\sum_{q=1}^{2}\Big\langle r_{S_c}\big\{T^{(q)}(\partial, n^{(q)})U^{(q)}\big\}^+, r_{S_c}\big\{V^{(q)} - U^{(q)}\big\}^+\Big\rangle_{S_c} \geq 0, \ \ \forall\, V \in \mathcal{K}. \tag{3.14}$$

Let $\Phi = (\Phi^{(1)}, \Phi^{(2)})^\top \in \mathbb{H}^1$ be such that $\{\Phi^{(q)}\}^+ \in [\widetilde{H}^{1/2}(S_c)]^6$, $q = 1, 2$, $\Phi^{(q)} = (\varphi^{(q)}, \psi^{(q)})^\top$, $r_{S_c}\{\psi^{(q)}\}^+ = 0$, $r_{S_c}\{\varphi_s^{(q)}\}^+ = 0$, $r_{S_c}\{\varphi^{(1)} \cdot n^{(1)}\}^+ = -r_{S_c}\{\varphi^{(2)} \cdot n^{(2)}\}^+ = \vartheta$, $\vartheta \in \widetilde{H}^{1/2}(S_c)$. Substituting in (3.14) $U^{(q)} \pm \Phi^{(q)}$ instead of $V^{(q)}$, we get

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+ - r_{S_c}\big\{(\mathcal{T}^{(2)}U^{(2)})_{n^{(2)}}\big\}^+, \vartheta\Big\rangle_{S_c} = 0, \ \ \forall\, \vartheta \in \widetilde{H}^{1/2}(S_c).$$

Thus we can conclude that the first condition in (3.5) is fulfilled.

Analogously, if $\Phi^{(q)} = (\varphi^{(q)}, \psi^{(q)})^\top \in [\widetilde{H}^{1/2}(S_c)]^6$ is such that the conditions

$$r_{S_c}\{\varphi^{(1)}\}^+ = r_{S_c}\{\varphi^{(2)}\}^+ = r_{S_c}\{\psi^{(2)}\}^+ = 0$$

and

$$r_{S_c}\{\psi^{(1)}\}^+ = \vartheta, \ \ \vartheta \in [\widetilde{H}^{1/2}(S_c)]^3$$

are satisfied, then it follows from (3.14) that

$$\Big\langle \big\{r_{S_c}\mathcal{M}^{(1)}U^{(1)}\big\}^+, \vartheta\Big\rangle_{S_c} = 0, \ \ \forall\, \vartheta \in [\widetilde{H}^{1/2}(S_c)]^3,$$

i.e.,

$$r_{S_c}\big\{\mathcal{M}^{(1)}U^{(1)}\big\}^+ = 0.$$

Just in the same way we find that

$$r_{S_c}\big\{\mathcal{M}^{(2)}U^{(2)}\big\}^+ = 0.$$

Consequently, the condition (3.8) is fulfilled.

We now choose $\Phi = (\Phi^{(1)}, \Phi^{(2)})^\top \in \mathbb{H}^1$ in such a way that $\{\Phi^{(1)}\}^+ \in [\widetilde{H}^{1/2}(S_c)]^6$, $\{\Phi^{(2)}\}^+ = 0$, $r_{S_c}\{\varphi^{(1)} \cdot n^{(1)}\}^+ = 0$, $r_{S_c}\{\psi^{(1)}\}^+ = 0$ and $r_{S_c}\{\varphi_t^{(1)}\}^+ = \vartheta$, $\vartheta \in [\widetilde{H}^{1/2}(S_c)]^3$. Then (3.14) yields

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_t\big\}^+, \vartheta\Big\rangle_{S_c} = 0, \ \ \forall\, \vartheta \in [\widetilde{H}^{1/2}(S_c)]^3,$$

i.e.,

$$r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_t\big\}^+ = 0.$$

Analogously, we obtain

$$r_{S_c}\big\{(\mathcal{T}^{(2)}U^{(2)})_t\big\}^+ = 0,$$

and thus the validity of equality (3.7) is proved.

Taking into account the obtained relations, (3.14) takes the form

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+, r_{S_c}\big\{(v^{(1)}-u^{(1)})\cdot n^{(1)}+(v^{(2)}-u^{(2)})\cdot n^{(2)}\big\}^+\Big\rangle_{S_c} \geq \\ \geq 0, \ \ \forall\, V=(V^{(1)},V^{(2)})^\top \in \mathcal{K}. \tag{3.15}$$

Let now $V=U+\Phi$, where

$$\Phi=(\Phi^{(1)},0)^\top \in \mathbb{H}^1, \ \ \Phi^{(1)}=(\varphi^{(1)},\psi^{(1)})^\top, \ \ \{\Phi^{(1)}\}^+ \in \big[\widetilde{H}^{1/2}(S_c)\big]^6, \\ r_{S_c}\{\psi^{(1)}\}^+=0, \ \ r_{S_c}\{\varphi^{(2)}_S\}^+=0$$

and

$$r_{S_c}\{\varphi^{(1)}\cdot n^{(1)}\}^+=\vartheta, \ \ \vartheta\in \widetilde{H}^{1/2}(S_c), \ \ \vartheta\leq 0.$$

It is not difficult to notice that $V\in\mathcal{K}$, and from (3.15) we conclude that

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+,\vartheta\Big\rangle_{S_c}\geq 0, \ \ \forall\,\vartheta\in\widetilde{H}^{1/2}(S_c), \ \ \vartheta\leq 0,$$

i.e.,

$$r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+\leq 0.$$

Thus the condition (3.5) is proved completely.

It remains to prove the condition (3.6).

We choose $V=(V^{(1)},V^{(2)})^\top\in\mathbb{H}^1$ in such a way that $\{V^{(q)}\}^+\in[\widetilde{H}^{1/2}(S_c)]^6$, $q=1,2$ and $r_{S_c}\{v^{(1)}\cdot n^{(1)}\}^+=r_{S_c}\{v^{(2)}\cdot n^{(2)}\}^+=0$. It is clear that $V\in\mathcal{K}$, and from (3.15) we find that

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+, r_{S_c}\big\{u^{(1)}\cdot n^{(1)}+u^{(2)}\cdot n^{(2)}\big\}^+\Big\rangle_{S_c}\leq 0. \tag{3.16}$$

Let now $V=(V^{(1)},V^{(2)})^\top\in\mathbb{H}^1$ be such that again $\{V^{(q)}\}^+\in[\widetilde{H}^{1/2}(S_c)]^6$, $q=1,2$, $r_{S_c}\{v^{(1)}\cdot n^{(1)}\}^+=2r_{S_c}\{u^{(1)}\cdot n^{(1)}\}^+$ and $r_{S_c}\{v^{(2)}\cdot n^{(2)}\}^+=2r_{S_c}\{u^{(2)}\cdot n^{(2)}\}^+$. Clearly, $V\in\mathcal{K}$, and the inequality

$$\Big\langle r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^+, r_{S_c}\big\{u^{(1)}\cdot n^{(1)}+u^{(2)}\cdot n^{(2)}\big\}^+\Big\rangle_{S_c}\geq 0 \tag{3.17}$$

is fulfilled. The inequalities (3.16) and (3.17) leads to the condition (3.6). The theorem is proved completely. □

## 4. The Existence of Solutions

4.1. **Basic existence results.** On a convex closed set $\mathcal{K}$ we consider the functional

$$J(V)=\frac{1}{2}\,\mathcal{B}(V,V)-\sum_{q=1}^{2}\bigg[\int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot V^{(q)}\,dx+\big\langle\Psi^{(q)},r_{S_q^N}\{V^{(q)}\}^+\big\rangle_{S_q^N}\bigg], \tag{4.1} \\ \forall\, V=(V^{(1)},V^{(2)})^\top\in\mathcal{K}.$$

It is not difficult to show that owing to the symmetric form $\mathcal{B}$, the existence of solutions in the variational inequality (3.9) is equivalent to the existence of a minimizing element on the set $\mathcal{K}$ of the functional (4.1), i.e., the problem of finding the vector $\widetilde{V}=(\widetilde{V}^{(1)},\widetilde{V}^{(2)})^\top\in\mathcal{K}$ for which

$$J(\widetilde{V})=\inf_{V\in\mathcal{K}}J(V) \tag{4.2}$$

and the variational inequality (3.9) are equivalent.

It can be shown that the functional (4.1) is continuous and strictly convex. Let us show that the functional $J$ is coercive on the set $\mathcal{K}$, i.e., let us show that

$$J(V)\to+\infty \ \text{ when } \ V\in\mathcal{K} \ \text{ and } \ \|V\|_{1,\Omega}\to\infty.$$

Since on the set $\mathcal{K}$ the form $\mathcal{B}$ is coercive (see (2.18)), by means of the trace operator properties, the coerciveness of $J$ follows directly from the following obvious estimate:

$$J(V)\geq\alpha_0\sum_{q=1}^2\|V^{(q)}\|^2_{[H^1(\Omega_q)]^6}-\alpha_1\sum_{q=1}^2\|V^{(q)}\|_{[H^1(\Omega_q)]^6}=\alpha_0\|V\|^2_{1,\Omega}-\alpha_1\|V\|_{1,\Omega},$$

where $\alpha_0>0$ and $\alpha_1>0$ are positive constants independent of $V$.

The general theory of variational inequalities (see [10], [13]) allows us now to conclude that the problem (4.2) is solvable uniquely, and hence the variational inequality (3.9) is likewise solvable uniquely owing to the equivalence.

Thus taking into account Theorem 3.2, we finally arrive at the following existence theorem.

**Theorem 4.1.** *If $\mathcal{G}^{(q)}\in[L_2(\Omega_q)]^6$ and $\Psi^{(q)}\in\big[\widetilde{H}^{-1/2}(S_q^N)\big]^6$, $q=1,2$, then Problem* (A) *has a unique solution in the space $\mathbb{H}^1$.*

4.2. **The continuous dependence of solutions on data of the problem.** Let $U=(U^{(1)},U^{(2)})^\top\in\mathbb{H}^1$ and $\widetilde{U}=(\widetilde{U}^{(1)},\widetilde{U}^{(2)})^\top\in\mathbb{H}^1$ be two solutions of Problem (A) (i.e., solutions of the variational inequality (3.9)) corresponding to the data $\mathcal{G}^{(q)}$, $\Psi^{(q)}$ and $\widetilde{\mathcal{G}}^{(q)}$, $\widetilde{\Psi}^{(q)}$, respectively. Since the convex closed set $\mathcal{K}$ does not depend on these data and $U,\widetilde{U}\in\mathcal{K}$, we have

$$\sum_{q=1}^2 B^{(q)}\big(U^{(q)},\widetilde{U}^{(q)}-U^{(q)}\big)\geq$$
$$\geq\sum_{q=1}^2\bigg[\int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot(\widetilde{U}^{(q)}-U^{(q)})\,dx+\big\langle\Psi^{(q)},r_{S_q^N}\{\widetilde{U}^{(q)}-U^{(q)}\}^+\big\rangle_{S_q^N}\bigg]$$

and

$$\sum_{q=1}^2 B^{(q)}\big(\widetilde{U}^{(q)},U^{(q)}-\widetilde{U}^{(q)}\big)\geq$$
$$\geq\sum_{q=1}^2\bigg[\int\limits_{\Omega_q}\widetilde{\mathcal{G}}^{(q)}\cdot(U^{(q)}-\widetilde{U}^{(q)})\,dx+\big\langle\widetilde{\Psi}^{(q)},r_{S_q^N}\{U^{(q)}-\widetilde{U}^{(q)}\}^+\big\rangle_{S_q^N}\bigg].$$

Adding the above inequalities, we obtain

$$-\sum_{q=1}^{2} B^{(q)}(U^{(q)}-\widetilde{U}^{(q)},U^{(q)}-\widetilde{U}^{(q)}) \geq$$
$$\geq -\sum_{q=1}^{2}\bigg[\int\limits_{\Omega_q}(\mathcal{G}^{(q)}-\widetilde{\mathcal{G}}^{(q)})\cdot(U^{(q)}-\widetilde{U}^{(q)})\,dx+$$
$$+\big\langle \Psi^{(q)}-\widetilde{\Psi}^{(q)}, r_{S_q^N}\{U^{(q)}-\widetilde{U}^{(q)}\}^{+}\big\rangle_{S_q^N}\bigg]. \quad (4.3)$$

Taking into account the coercivity of the form $\mathcal{B}$ on the set $\mathcal{K}$ (see (2.18)), as well as Hölder's inequality and the properties of the trace operator, we can conclude from (4.3) that

$$\|U-\widetilde{U}\|_{1,\Omega} \leq c\sum_{q=1}^{2}\Big(\big\|\mathcal{G}^{(q)}-\widetilde{\mathcal{G}}^{(q)}\big\|_{[L_2(\Omega_q)]^6}+\big\|\Psi^{(q)}-\widetilde{\Psi}^{(q)}\big\|_{[\widetilde{H}^{-1/2}(S_q^N)]^6}\Big),$$

where $c$ is a positive constant, independent of the data of the problem. This implies that solutions of Problem (A) depend continuously on the data.

## 5. The Non-Coercive Case

5.1. **The Statement of the Problem.** Let $S_q^D = \varnothing$, $S_q = \overline{S}_q^N \cup \overline{S}_c$, $\mathcal{G}^{(q)} \in [L_2(\Omega_q)]^6$ and $\Psi^{(q)} \in \big[\widetilde{H}^{-1/2}(S_q^N)\big]^6$, $q=1,2$. Consider the so-called non-coercive problem.

**Problem (B).** Find a vector-function $U=(U^{(1)},U^{(2)})^\top \in \mathbb{H}^1$ which is a weak solution of the equation

$$L^{(q)}(\partial)U^{(q)}+\mathcal{G}^{(q)}=0, \;\; q=1,2, \quad (5.1)$$

in the domain $\Omega_q$ and which satisfies the boundary conditions on $S_q^N$

$$r_{S_q^N}\big\{T^{(q)}(\partial,n^{(q)})U^{(q)}\big\}^{+}=r_{S_q^N}\Psi^{(q)}, \;\; q=1,2, \quad (5.2)$$

and the contact conditions on $S_c$

$$r_{S_c}\big\{u^{(1)}\cdot n^{(1)}+u^{(2)}\cdot n^{(2)}\big\}^{+}\leq 0, \quad (5.3)$$

$$r_{S_c}\big\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\big\}^{+}=r_{S_c}\big\{(\mathcal{T}^{(2)}U^{(2)})_{n^{(2)}}\big\}^{+}\leq 0, \quad (5.4)$$

$$\Big\langle r_{S_c}\{(\mathcal{T}^{(1)}U^{(1)})_{n^{(1)}}\}^{+}, r_{S_c}\big\{u^{(1)}\cdot n^{(1)}+u^{(2)}\cdot n^{(2)}\big\}^{+}\Big\rangle_{S_c}=0, \quad (5.5)$$

$$r_{S_c}\big\{(\mathcal{T}^{(q)}U^{(q)})_t\big\}^{+}=0, \;\; q=1,2, \quad (5.6)$$

$$r_{S_c}\big\{\mathcal{M}^{(q)}U^{(q)}\big\}^{+}=0, \;\; q=1,2. \quad (5.7)$$

To reduce the problem to the variational inequality, we introduce a closed convex set

$$\mathcal{K}_0=\Big\{U=(U^{(1)},U^{(2)})^\top \in \mathbb{H}^1 :\; r_{S_c}\big\{u^{(1)}\cdot n^{(1)}+u^{(2)}\cdot n^{(2)}\big\}^{+}\leq 0\Big\}$$

and consider the following variational inequality:

Find $U=(U^{(1)},U^{(2)})^\top\in\mathcal{K}_0$ such that

$$\sum_{q=1}^{2}B^{(q)}\big(U^{(q)},V^{(q)}-U^{(q)}\big)\geq$$

$$\geq\sum_{q=1}^{2}\bigg[\big\langle r_{S_q^N}\Psi^{(q)},r_{S_q^N}\{V^{(q)}-U^{(q)}\}^+\big\rangle_{S_q^N}+\int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot(V^{(q)}-U^{(q)})\,dx\bigg],\quad(5.8)$$

$$\forall\,V=(V^{(1)},V^{(2)})^\top\in\mathcal{K}_0.$$

Analogously, just as in the previous case (see Theorem 3.2), we can prove that the variational inequality (5.8) is equivalent to Problem (B), i.e., every solution of Problem (B) is a solution of the problem (5.8), and vice versa.

First of all, we derive the necessary conditions of solvability of inequality (5.8). Let $U=(U^{(1)},U^{(2)})^\top\in\mathcal{K}_0$ be a solution of inequality (5.8). First, we substitute $V=0$ in (5.8) and then $V=2U$. As a result, we obtain

$$\sum_{q=1}^{2}B^{(q)}(U^{(q)},U^{(q)})=$$

$$=\sum_{q=1}^{2}\bigg[\big\langle r_{S_q^N}\Psi^{(q)},r_{S_q^N}\{U^{(q)}\}^+\big\rangle_{S_q^N}+\int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot U^{(q)}\,dx\bigg].\qquad(5.9)$$

Taking into account this identity, from (5.8) we derive

$$\sum_{q=1}^{2}B^{(q)}(U^{(q)},V^{(q)})\geq$$

$$\geq\sum_{q=1}^{2}\bigg[\big\langle r_{S_q^N}\Psi^{(q)},r_{S_q^N}\{V^{(q)}\}^+\big\rangle_{S_q^N}+\int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot V^{(q)}\,dx\bigg]\qquad(5.10)$$

for all $V=(V^{(1)},V^{(2)})^\top\in\mathcal{K}_0$.

By $\Lambda^{(q)}(S_q)$, $q=1,2$ we denote the set of traces on $S_q$ of vectors of rigid displacements, i.e.,

$$\Lambda^{(q)}(S_q)=\Big\{\chi^{(q)}=\big([a^{(q)}\times x]+b^{(q)},a^{(q)}\big)^\top,\ a^{(q)},b^{(q)}\in\mathbb{R}^3,\ x\in S_q\Big\}$$

and let

$$\Lambda:=\Big\{\chi=(\chi^{(1)},\chi^{(2)})^\top:\ \chi^{(q)}\in\Lambda^{(q)}(S_q),\ q=1,2\Big\}.$$

Consider the set

$$\mathcal{R}:=\mathcal{K}_0\cap\Lambda=$$

$$=\Big\{\chi\in\Lambda:\ \big([a^{(1)}\times x]+b^{(1)}\big)\cdot n^{(1)}+\big(\big[a^{(2)}\times x\big]+b^{(2)}\big)\cdot n^{(2)}\leq0,\ x\in S_c\Big\}.$$

Since at the points of $S_c$, $n^{(1)}=-n^{(2)}$, we have

$$\mathcal{R} = \Big\{ \chi = (\chi^{(1)}, \chi^{(2)})^\top \in \Lambda : \\ \big( \big[ (a^{(1)} - a^{(2)}) \times x \big] + b^{(1)} - b^{(2)} \big) \cdot n^{(1)} \le 0, \ x \in S_c \Big\}.$$

Let now $\chi \in \mathcal{R}$, and substitute $\chi$ into (5.10) instead of $V$. Then taking into account the fact that $B^{(q)}(U^{(q)}, \chi^{(q)}) = 0$, $q = 1, 2$, we find

$$\sum_{q=1}^{2} \bigg[ \big\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \chi^{(q)} \big\rangle_{S_q^N} + \int\limits_{\Omega_q} \mathcal{G}^{(q)} \cdot \chi^{(q)} dx \bigg] \le 0, \ \ \forall \chi \in \mathcal{R}. \qquad (5.11)$$

Thus we finally obtain that (5.11) is the necessary condition for the existence of a solution in the variational inequality (5.8).

Let

$$\mathcal{R}^* := \Big\{ \chi = (\chi^{(1)}, \chi^{(2)})^\top \in \mathcal{R} : \\ \big( \big[ (a^{(1)} - a^{(2)}) \times x \big] + b^{(1)} - b^{(2)} \big) \cdot n^{(1)} = 0, \ x \in S_c \Big\}$$

and inequality (5.11) be fulfilled in a strong sense, i.e., equality (5.11) holds if and only if $\chi \in \mathcal{R}^*$.

Relying on the general theory of variational inequalities (see [10], [13]), the condition (5.11) in this case becomes also sufficient for the solvability of inequality (5.8).

Investigate now the uniqueness of a solution of inequality (5.8).

Suppose that $U = (U^{(1)}, U^{(2)})^\top \in \mathcal{K}_0$ and $\widetilde{U} = (\widetilde{U}^{(1)}, \widetilde{U}^{(2)})^\top \in \mathcal{K}_0$ are two arbitrary solutions of the variational inequality (5.8). Then

$$\sum_{q=1}^{2} B^{(q)}(U^{(q)}, \widetilde{U}^{(q)} - U^{(q)}) \ge \\ \ge \sum_{q=1}^{2} \bigg[ \big\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \{ \widetilde{U}^{(q)} - U^{(q)} \}^+ \big\rangle_{S_q^N} + \int\limits_{\Omega_q} \mathcal{G}^{(q)} \cdot (\widetilde{U}^{(q)} - U^{(q)}) \, dx \bigg] \quad (5.12)$$

and

$$\sum_{q=1}^{2} B^{(q)}(\widetilde{U}^{(q)}, U^{(q)} - \widetilde{U}^{(q)}) \ge \\ \ge \sum_{q=1}^{2} \bigg[ \big\langle r_{S_q^N} \Psi^{(q)}, r_{S_q^N} \{ U^{(q)} - \widetilde{U}^{(q)} \}^+ \big\rangle_{S_q^N} + \int\limits_{\Omega_q} \mathcal{G}^{(q)} \cdot (U^{(q)} - \widetilde{U}^{(q)}) \, dx \bigg]. \quad (5.13)$$

Adding the above inequalities and taking into account the fact that the quadratic form $B^{(q)}$, $q = 1, 2$ is positive definite, we obtain

$$B^{(q)} \big( U^{(q)} - \widetilde{U}^{(q)}, U^{(q)} - \widetilde{U}^{(q)} \big) = 0, \ \ q = 1, 2,$$

from which we can conclude that

$$U^{(q)} - \widetilde{U}^{(q)} = \chi^{(q)}, \ \ q = 1, 2.$$

Since $B^{(q)}(U^{(q)},\chi^{(q)}) = 0$ and $B^{(q)}(\widetilde{U}^{(q)},\chi^{(q)}) = 0$, it follows from inequalities (5.12) and (5.13) that

$$\sum_{q=1}^{2}\Big[\big\langle r_{S_q^N}\Psi^{(q)}, r_{S_q^N}\{\chi^{(q)}\}\big\rangle_{S_q^N} + \int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot\chi^{(q)}\,dx\Big] \geq 0$$

and

$$\sum_{q=1}^{2}\Big[\big\langle r_{S_q^N}\Psi^{(q)}, r_{S_q^N}\{\chi^{(q)}\}\big\rangle_{S_q^N} + \int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot\chi^{(q)}\,dx\Big] \leq 0,$$

i.e.,

$$\sum_{q=1}^{2}\Big[\big\langle r_{S_q^N}\Psi^{(q)}, r_{S_q^N}\{\chi^{(q)}\}\big\rangle_{S_q^N} + \int\limits_{\Omega_q}\mathcal{G}^{(q)}\cdot\chi^{(q)}\,dx\Big] = 0. \tag{5.14}$$

Thus we obtain the following

**Theorem 5.1.** *Let $\mathcal{G}^{(q)} \in [L_2(\Omega_q)]^6$ and $\Psi^{(q)} \in [\widetilde{H}^{-1/2}(S_q^N)]^6$, $q = 1,2$. Then the condition* (5.11) *is necessary for Problem* (B) *to be solvable. If the condition* (5.11) *is fulfilled in a strong sense, i.e., the equality in* (5.11) *holds if and only if $\chi = (\chi^{(1)},\chi^{(2)})^\top \in \mathcal{R}^*$, then a solution of Problem* (B) *exists and it is defined modulo a vector of rigid displacement for which the condition* (5.14) *is fulfilled.*

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## References

1. E. L. Aero and E. V. Kuvshinskii, Continuum theory of asymmetric elasticity. The problem of "internal" rotation. (Russian) *Fiz. Tverd. Tela* **5** (1963), 2591–2598; English transl.: *Soviet Physics Solid State* **5** (1963), 1892–1897.
2. E. L. Aero and E. V. Kuvshinskii, Continuum theory of asymmetric elasticity. Equilibrium of an isotropic body. (Russian) *Fiz. Tverd. Tela* **6** (1965), 2689–2699; English transl.: *Soviet Physics Solid State* **6** (1965), 2141–2148.
3. E. Cosserat and F. Cosserat, Sur les fonctions potentielles de la théorie de l'élasticité. (French) *C. R.* **126** (1898), 1129–1132, *Reviewer: Kotter, F.; Prof.* (*Berlin*).
4. E. Cosserat and F. Cosserat, Théorie des corps déformables. (French) *A. Hermann et Fils., Paris*, 1909.
5. G. Duvaut and J.-L. Lions, Les inéquations en mécanique et en physique. (French) Travaux et Recherches Mathématiques, No. 21. *Dunod, Paris*, 1972.
6. J. Dyszlewicz, Micropolar theory of elasticity. Lecture Notes in Applied and Computational Mechanics, 15. *Springer-Verlag, Berlin*, 2004.
7. M. J. Elphinstone and A. Lakhtakia, Plane-wave representation of an elastic chiral solid slab sandwiched between achiral solid half-spaces. *J. Acoust. Soc. Am.* **95** (1994), 617–627.
8. A. C. Eringen, Microcontinuum field theories. I. Foundations and solids. *Springer-Verlag, New York*, 1999.

9. G. Fichera, Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno. (Italian) *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8)* **7** (1963/1964), 91–140.
10. G. Fichera, Existence theorems in elasticity. Handb. der Physik, Bd. 6/2, *Springer Verlag, Heidelberg*, 1973.
11. A. Gachechiladze and D. Natroshvili, Boundary variational inequality approach in the anisotropic elasticity for the Signorini problem. *Georgian Math. J.* **8** (2001), No. 3, 469–492.
12. R. Gachechiladze, J. Gwinner, and D. Natroshvili, A boundary variational inequality approach to unilateral contact with hemitropic materials. *Mem. Differential Equations Math. Phys.* **39** (2006), 69–103.
13. R. Glowinski, J.-L. Lions, and R. Tremolieres, Numerical analysis of variational inequalities. Translated from the French. Studies in Mathematics and its Applications, 8. *North-Holland Publishing Co., Amsterdam–New York*, 1981.
14. J. Gwinner and E. P. Stephan, A boundary element procedure for contact problems in plane linear elastostatics. *RAIRO Model. Math. Anal. Numer.* **27** (1993), No. 4, 457–480.
15. Z. Haijun and O. Zhong-can, Bending and twisting elasticity: a revised Marko–Sigga model on DNA chirality. *Phys. Rev. E* **58(4)** (1998), 4816–4821.
16. I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek, Solution of variational inequalities in mechanics. Translated from the Slovak by J. Jarnik. Applied Mathematical Sciences, 66. *Springer-Verlag, New York*, 1988.
17. N. Kikuchi and J. T. Oden, Contact problems in elasticity: a study of variational inequalities and finite element methods. SIAM Studies in Applied Mathematics, 8. *Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA*, 1988.
18. D. Kinderlehrer, Remarks about Signorini's problem in linear elasticity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8** (1981), No. 4, 605–645.
19. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Russian) *Izdat. "Nauka", Moscow*, 1976; English transl.: North-Holland Series in Applied Mathematics and Mechanics, 25. *North–Holland Publishing Co., Amsterdam–New York*, 1979.
20. R. Lakes, Elastic and viscoelastic behavior of chiral materials. *Int. J. Mech. Sci.* **43** (2001), No. 7, 1579–1589.
21. R. S. Lakes and R. L. Benedict, Noncentrosymmetry in micropolar elasticity. *Int. J. Eng. Sci.* **20** (1982), 1161–1167.
22. A. Lakhtakia, V. K. Varadan, and V. V. Varadan, Elastic wave propagation in noncentrosymmetric isotropic media: dispersion and field equations. *J. Appl. Phys.* **64** (1988), 5246–5250.
23. A. Lakhtakia, V. V. Varadan, and V. K. Varadan, Elastic wave scattering by an isotropic noncentrosymmetric sphere. *J. Acoust. Soc. Am.* **91** (1992), 680–684.
24. J.-L. Lions and E. Magenes, Problemes aux limites non homogenes et applications. Vol. 2. (French) Travaux et Recherches Mathematiques, No. 18 *Dunod, Paris*, 1968.
25. W. McLean, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.
26. R. D. Mindlin, Micro-structure in linear elasticity. *Arch. Rational Mech. Anal.* **16** (1964), 51–78.
27. T. Mura, Mircomechanics of defects in solids. 2nd, rev. ed. (English) Mechanics of Elastic and Inelastic Solids, 3. *Martinus Nijhoff Publishers, a member of the Kluwer Academic Publishers Group. XIII, Dordrecht–Boston–Lancaster*, 1987.
28. T. Mura, Some new problems in the micromechanics. *Materials Science and Engineering A (Structural Materials: Properties, Microstructure and Processing)* **A285** (2000), No. 1-2, 224–228.

29. D. Natroshvili, R. Gachechiladze, A. Gachechiladze, and I. G. Stratis, Transmission problems in the theory of elastic hemitropic materials. *Appl. Anal.* **86** (2007), No. 12, 1463–1508.
30. D. Natroshvili, L. Giorgashvili, and I. G. Stratis, Mathematical problems of the theory of elasticity of chiral materials. *Appl. Math. Inform. Mech.* **8** (2003), No. 1, 47–103, 127.
31. D. Natroshvili, L. Giorgashvili, and Sh. Zazashvili, Steady state oscillation problems in the theory of elasticity for chiral materials. *J. Integral Equations Appl.* **17** (2005), No. 1, 19–69.
32. D. Natroshvili and I. G. Stratis, Mathematical problems of the theory of elasticity of chiral materials for Lipschitz domains. *Math. Methods Appl. Sci.* **29** (2006), No. 4, 445–478.
33. J. Nečas, Les méthodes directes en théorie des équations elliptiques. (French) *Masson et Cie, Editeurs, Paris; Academia, Éditeurs, Prague*, 1967.
34. W. Nowacki, Theory of asymmetric elasticity. Translated from the Polish by H. Zorski. *Pergamon Press, Oxford; PWN—Polish Scientific Publishers, Warsaw*, 1986.
35. J. P. Nowacki, Green functions for a hemitropic micropolar continuum. *Bull. Acad. Polon. Sci. Ser. Sci. Tech.* **25** (1977), No. 8, 619–625.
36. J. P. Nowacki and W. Nowacki, Some problems of hemitropic micropolar continuum. *Bull. Acad. Polon. Sci. Ser. Sci. Tech.* **25** (1977), No. 4, 297–305.
37. R. Ro, Elastic activity of the chiral medium. *J. Appl. Phys.* **85** (1999), No. 5, 2508–2513.
38. J.-F. Rodrigues, Obstacle problems in mathematical physics. North-Holland Mathematics Studies, 134. Notas de Matematica [Mathematical Notes], 114. *North-Holland Publishing Co., Amsterdam*, 1987.
39. P. Sharma, Size-dependent elastic fields of embedded inclusions in isotropic chiral solids. *Int. J. Solids Struct.* **41** (2004), No. 22-23, 6317–6333.
40. H. Triebel, Interpolation theory, function spaces, differential operators. North-Holland Mathematical Library, 18. *North–Holland Publishing Co., Amsterdam–New York*, 1978.
41. H. Triebel, Theory of function spaces. Monographs in Mathematics, 78. *Birkhauser Verlag, Basel*, 1983.
42. Y. Weitsman, Initial stress and skin effects in a hemitropic Cosserat continuum. *J. Appl. Math.* **349** (1967), 160–164.
43. J. F. C. Yang and R. S. Lakes, Experimental study of micropolar and couple stress elasticity in compact bone bending. *J. Biomechanics* **15** (1982), No. 2, 91–98.


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