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BOUNDARY CONTACT PROBLEMS WITH FRICTION OF DYNAMICS FOR HEMITROPIC ELASTIC SOLIDS

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with C^∞ smooth boundary $S := \partial\Omega$. Throughout the paper, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at the point $x \in S$.

We assume that Ω is occupied by a homogeneous hemitropic elastic material. Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the *displacement vector* and the *micro-rotation vector*, respectively. The symbol $(\cdot)^\top$ denotes transposition.

The equilibrium equations in terms of the displacement and micro-rotation vectors read as [1]

$$
\begin{aligned}
&(\mu+\alpha)\,\Delta\,u(x,t) + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,u(x,t) + (\varkappa+\nu)\,\Delta\,\omega(x,t) + \\
&\quad +(\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,\omega(x,t) + 2\alpha\,\mathrm{curl}\,\omega(x,t) + \varrho\,F(x,t) = \varrho\,\frac{\partial^2 u(x,t)}{\partial t^2}, \\
&(\varkappa+\nu)\,\Delta\,u(x,t) + (\delta+\varkappa-\nu)\,\mathrm{grad\,div}\,u(x,t) + 2\alpha\,\mathrm{curl}\,u(x,t) + \\
&\quad +(\gamma+\varepsilon)\,\Delta\,\omega(x,t) + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\omega(x,t) + 4\nu\,\mathrm{curl}\,\omega(x,t) - \\
&\quad -4\alpha\,\omega(x,t) + \varrho\,G(x,t) = \mathcal{J}\,\frac{\partial^2 \omega(x,t)}{\partial t^2},
\end{aligned}
$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \varkappa$, and ε are the material constants, t is the time variable, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors per unit mass, ϱ is the mass density of the elastic material, and $\mathcal{J}$ is a constant characterizing the so-called spin torque corresponding to the interior micro-rotations (i.e., the moment of inertia per unit volume). Using the matrix differential operator $L(\partial)$ of dimension 6×6, corresponding to the left side of the previous system, we can write the equilibrium equations in the matrix form

$$
L(\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}, \quad x\in\Omega, \ 0<t<T_0, \tag{1}
$$

where T_0 is an arbitrary positive number, $U = (u, \omega)^\top$, $\mathcal{G} = (\varrho F,\, \varrho G)^\top$, $P = [p_{ij}]_{6\times 6}$, $p_{ii} = \varrho$, for $i = 1, 2, 3$, $p_{ii} = \mathcal{J}$, for $i = 4, 5, 6$ and $p_{ij} = 0$, when $i \neq j$.

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For real-valued vector functions $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ from the class $[H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$, the following Green formula holds [2]:

$$\int_\Omega [L(\partial)U \cdot U' + E(U, U')]\, dx = \int_S \{T(\partial, n)U\}^+ \cdot \{U'\}^+ \, dS,$$

where $T(\partial, n)$ is the matrix differential stress operator, $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω, $E(\cdot\,,\,\cdot)$ is the bilinear form and $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$ ($L_2(\Omega)$, and $H^s(\Omega)$, $s \in R$, denote the Lebesque and Bessel potential spaces).
Below we shall deal with the weak solution of the equation (1).

The vector-function $U : (0; T_0) \to [H^1(\Omega)]^6$ is said to be a weak solution of equation (1) for $\mathcal{G} : (0; T_0) \to [L_2(\Omega)]^6$ if $U(t), U'(t) \in L_\infty(0, T_0; [H^1(\Omega)]^6), U''(t) \in L_\infty(0, T_0; [L_2(\Omega)]^6)$, and for every $\Phi \in [C_0^\infty(\Omega)]^6$

$$(PU''(t), \Phi) + B(U(t), \Phi) = (\mathcal{G}(t), \Phi)$$

for almost all t from the interval $(0; T_0)$, where the symbol $(\cdot, \cdot)$ denotes the inner product in the space $L_2(\Omega)$ and

$$B(U, U) := \int_\Omega E(U, U) dx \geq 0.$$

Let the boundary S of the domain Ω be divided into two open, connected and non-overlapping parts S_1 and S_2 of positive measure, $S = \overline{S_1} \cup \overline{S_2}$, $S_1 \cap S_2 = \varnothing$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2.

Let $\mathcal{G} : (0; T_0) \to [L_2(\Omega)]^6$, $\varphi : (0; T_0) \to [H^{-1/2}(S_2)]^3$, $f : (0; T_0) \to L_\infty(S_2)$, $\mathcal{F} : S_2 \times (0; T_0) \to [0; +\infty)$ be a bounded measurable function and

$$g := \mathcal{F}|f| \geq 0.$$

with $\mathcal{F}(x, t)$ being the friction coefficient at the point (x, t). It is a non-negative scalar function which depends on the geometry of the contacting surfaces and also on the physical properties of interacting materials.
Consider the following contact problem of dynamics with a friction.

Problem (A_0). Find a weak solution $U : (0; T_0) \to [H^1(\Omega)]^6$ of the equation

$$L(\partial)U(x, t) + \mathcal{G}(x, t) = P\frac{\partial^2 U(x, t)}{\partial t^2}, \quad x \in \Omega, \;\; t \in (0; T_0),$$

satisfying the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2 \times (0; T_0))]^3$ and the boundary and the initial conditions (for almost all t from the interval $(0; T_0)$):

$$
\begin{array}{lll}
r_{S_1}\{U\}^+ = 0 & \text{on} & S_1 \times (0;T_0); \\
r_{S_2}\{(\mathcal{T}U)_n\}^+ = f & \text{on} & S_2 \times (0;T_0); \\
r_{S_2}\{\mathcal{M}U\}^+ = \varphi & \text{on} & S_2 \times (0;T_0); \\
\text{if } |r_{S_2}\{(\mathcal{T}U)_s\}^+| < g, & \text{then} & r_{S_2}\Big\{\dfrac{\partial\, u_s}{\partial\, t}\Big\}^+ = 0;
\end{array}
$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions $\lambda_1$and λ_2, not vanishing simultaneously, and

$$
\lambda_1 r_{S_2}\Big\{\frac{\partial\, u_s}{\partial\, t}\Big\}^+ = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+;
$$

moreover,

$$
U(0) = U'(0) = 0,
$$

where $\mathcal{T}U$ and $\mathcal{M}U$ are the force stress and couple stress vectors respectively, F_n and F_s stand for the normal and tangential components of the vector F: $F_n = F \cdot n$ and $F_s = F - (F \cdot n)n$.

This problem can be reformulated in terms of a variational inequality. Find a vector-function $U = (u, \omega)^\top \in \mathcal{K}$ such that the variational inequality

$$
\begin{aligned}
&(PU''(t), V - U'(t)) + B(U(t), V - U'(t)) + j(V) - j(U'(t)) \geq \\
&\geq (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ ds + \\
&+ \langle \varphi(t), r_{S_2}\{w - \omega'(t)\}^+\rangle_{S_2}
\end{aligned} \tag{2}
$$

holds for all $V = (v, w)^\top \in \mathcal{K}_0$ (and for almost all $t \in (0;T_0)$), where

$$
\begin{aligned}
&\mathcal{K} = \{V | V(t), V'(t) \in L_\infty(0, T_0; [H^1(\Omega)]^6), \\
&V''(t) \in L_\infty(0, T_0; [L_2(\Omega)]^6), \;\; r_{S_1}\{V\}^+ = 0, \;\; V(0) = V'(0) = 0\}, \\
&\mathcal{K}_0 = \{V | V \in [H^1(\Omega)]^6, \;\; r_{S_1}\{V\}^+ = 0\},
\end{aligned}
$$

and

$$
j(V) = \int\limits_{S_2} g|\{v_s\}^+| ds, \qquad V = (v, w)^\top : (0;T_0) \to [H^1(\Omega)]^6.
$$

We prove that the variational inequality (2) and the Problem (A_0) are equivalent, i.e., any solution of the Problem (A_0) is a solution of the inequality (2), and vice versa. So, the investigation of the Problem (A_0) can be reduced to the study of the inequality (2).

The uniqueness of solution to the variational inequality (2) can be proved with the standard arguments.

The investigation of the existence of a solution to the variational inequality (2) is carried out by the following scheme. First of all, we replace the variational inequality (2) by an equivalent regularized equation (depending on a parameter) whose solvability is studied by the Faedo-Galerkin approximation method. Then we establish some a priori estimates which allow us to pass to the limit with respect to the dimension and to the parameter. The limiting function turns out to be a solution of the variational inequality (2), and consequently it is a solution of the problem (A_0), as well.
Finally, we have obtained the following of the existence and uniqueness theorem for the Problem (A_0).

Theorem 1. *Let* $\mathcal{G},\mathcal{G}',\mathcal{G}'' \in L_2(0,T_0;[L_2(S_2)]^6)$, $f,f',f'' \in L_2(0,T_0; L_2(S_2))$, $\varphi,\varphi',\varphi'' \in L_2(0,T_0;[H^{-1/2}(S_2)]^3)$, g *be independent of* t *and there exist the vector-function* $U_0 \in [L_2(\Omega)]^6$ *such that*

$$(U_0,V)=(\mathcal{G}(0),V)+\int\limits_{S_2} f(0)\{v_n\}^+ds+\langle\varphi(0),r_{S_2}\{w\}^+\rangle_{S_2} \quad \forall V=(v,w)^\top \in \mathcal{K}_0.$$

Then there exists a unique function $U \in \mathcal{K}$ *which is a solution of the variational inequality* (2)*, i.e.,* U *is a unique solution of the problem* A_0*, as well.*

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