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Topological completeness of the provability logic GLP

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ABSTRACT

Provability logic **GLP** is well-known to be incomplete w.r.t. Kripke semantics. A natural topological semantics of **GLP** interprets modalities as derivative operators of a polytopological space. Such spaces are called GLP-spaces whenever they satisfy all the axioms of **GLP**. We develop some constructions to build nontrivial GLP-spaces and show that **GLP** is complete w.r.t. the class of all GLP-spaces.

1. Introduction

This paper continues the study of topological semantics of an important polymodal provability logic **GLP** initiated in [12,9]. This system was introduced by Giorgi Japaridze in [26,25]. The language of **GLP** extends that of classical propositional logic by unary modalities $[n]$, for each $n \in \omega$. A natural provability interpretation of **GLP** is to translate $[n]\varphi$ into the language of Peano arithmetic PA as the statement *“φ is provable from the axioms of* PA *together with all true arithmetical Π^0_n-sentences.”* The dual modalities $\langle n\rangle\varphi := \neg[n]\neg\varphi$ then correspond to the standard uniform Σ^0_n-reflection principles for the theory $\mathsf{PA} + \varphi$. Thus, Japaridze and Ignatiev [26,24] have shown that **GLP** is complete with respect to a very natural proof-theoretic semantics.

The logic **GLP** has been extensively studied in the early 1990s by Ignatiev and Boolos who simplified and extended Japaridze’s work (see [15]). More recently, interesting applications of **GLP** have been found

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in the proof theory and ordinal analysis of arithmetic. In particular, **GLP** gives rise to a natural system of ordinal notations for the ordinal ε_0. Based on the use of **GLP**, the first author of this paper gave a proof-theoretic analysis of Peano arithmetic, which stimulated further interest towards **GLP** (see [4,5] for a detailed survey).

The main obstacle in the study of **GLP** is that it is incomplete w.r.t. any class of Kripke frames. However, a more general topological semantics for the Gödel–Löb provability logic **GL** has been known since the work of Simmons [27] and Esakia [16]. In the sense of this semantics, the diamond modality is interpreted as the topological derivative operator acting on a scattered topological space. Esakia established that the Gödel–Löb logic is, in fact, complete w.r.t. the class of all scattered spaces. Moreover, Abashidze [1] and Blass [14] independently improved this result by showing that **GL** is complete w.r.t. a natural scattered topological space: any ordinal $\alpha \geqslant \omega^\omega$ equipped with its standard order topology.

The idea to extend this approach to the polymodal logic **GLP** comes quite naturally.[1] Each modality of **GLP** individually behaves like the one of **GL** and can therefore be interpreted as a derivative operator of a polytopological space $(X, \tau_0, \tau_1, \ldots)$. The additional axioms of **GLP** imply certain dependencies between the scattered topologies τ_i, which lead the authors of [12] to the concept of *GLP-space*. Thus, GLP-spaces provide an adequate topological semantics for **GLP**.

The question of completeness of **GLP** w.r.t. this semantics turned out to be rather difficult. Icard [22,23] showed that the variable-free fragment of **GLP** is complete w.r.t. a sequence of natural topologies on the ordinal ε_0. The main contribution of [12] was to show that the fragment of **GLP** with only two modalities and no restriction on variables was topologically complete. However, already for the fragment with three modalities the question remained open. The present paper answers this question positively for the language with infinitely many modalities and shows that **GLP** is complete w.r.t. the semantics of GLP-spaces.

2. Preliminaries

GLP is a propositional modal logic formulated in a language with infinitely many modalities $[0], [1], [2], \ldots$. As usual, $\langle n\rangle\varphi$ stands for $\neg[n]\neg\varphi$, and $\bot$ is the logical constant 'false'. **GLP** is given by the following axiom schemata and inference rules.

Axioms:

(i) Boolean tautologies;
(ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;
(iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$ (Löb's axiom);
(iv) $[m]\varphi \to [n]\varphi$, for $m < n$;
(v) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$.

Rules:

(i) $\vdash \varphi,\ \vdash \varphi \to \psi \ \Rightarrow \vdash \psi$ (modus ponens);
(ii) $\vdash \varphi \ \Rightarrow \vdash [n]\varphi$, for each $n \in \omega$ (necessitation).

In other words, for each modality, **GLP** contains the axioms and inference rules of the Gödel–Löb logic **GL**. Axioms (iv) and (v) relate different modalities to one another.

Neighborhood semantics for modal logic can be seen both as a generalization of Kripke semantics and as a particular kind of algebraic semantics. Let X be a nonempty set and let $\delta_n : \mathcal{P}(X) \to \mathcal{P}(X)$, for each

[1] Leo Esakia raised this question several times in conversations with the first author.

$n \in \omega$, be some unary operators acting on the Boolean algebra of all subsets of X. Such a structure X will be called a *neighborhood frame*.

A *valuation* on X is a map $v : \mathrm{Var} \to \mathcal{P}(X)$ from the set of propositional variables to the powerset of X, which is extended to all formulas in the language of **GLP** as follows:

- $v(\varphi \vee \psi) = v(\varphi) \cup v(\psi)$, $v(\neg\varphi) = X \setminus v(\varphi)$, $v(\bot) = \varnothing$,
- $v(\langle n\rangle\varphi) = \delta_n(v(\varphi))$, $v([n]\varphi) = \tilde{\delta}_n(v(\varphi))$, where $\tilde{\delta}_n(A) := X \setminus \delta_n(X \setminus A)$, for any $A \subseteq X$.

A formula φ is *valid in* X, denoted $X \vDash \varphi$, if $v(\varphi) = X$ for all v. The *logic of* X is the set $\mathrm{Log}(X)$ of all formulas valid in X. This set obviously contains the set of all Boolean tautologies and is closed under the modus ponens and substitution rules and, in this sense, is a propositional polymodal logic.

Next we observe that any neighborhood frame of **GLP** is, essentially, a polytopological space, in which all operators δ_n can be interpreted as the derived set operators.

Suppose (X, τ) is a topological space. The *derived set operator* on X is the map $d_\tau : \mathcal{P}(X) \to \mathcal{P}(X)$ associating with each $A \subseteq X$ its set of limit points, denoted $d_\tau(A)$. In other words, $x \in d_\tau(A)$ iff every open neighborhood of x contains a point $y \neq x$ such that $y \in A$. We shall write dA for $d_\tau(A)$ whenever the topology τ is given from the context.

A topological space (X, τ) is called *scattered* if every nonempty subspace $A \subseteq X$ has an isolated point. A polytopological space $(X, \tau_0, \tau_1, \dots)$ is called a *GLP-space* (cf. [12]) if the following conditions hold, for each $n < \omega$:

- τ_n is scattered;
- $\tau_n \subseteq \tau_{n+1}$;
- $d_{\tau_n}(A)$ is τ_{n+1}-open, for each $A \subseteq X$.

This concept is justified by the basic observation that GLP-spaces are equivalent to the neighborhood frames validating all the axioms of **GLP**. Thus, to each GLP-space we associate a neighborhood frame $(X, d_0, d_1, \dots)$ where d_n abbreviates d_{τ_n}, for each $n < \omega$. Then the following proposition holds.

Proposition 2.1.

(i) *If* $(X, \tau_0, \tau_1, \dots)$ *is a GLP-space, then in the associated neighborhood frame all the theorems of* **GLP** *are valid*: $(X, d_0, d_1, \dots) \vDash$ **GLP**.

(ii) *Suppose* $(X, \delta_0, \delta_1, \dots)$ *is a neighborhood frame such that* $X \vDash$ **GLP**. *Then there are naturally defined topologies* $\tau_0, \tau_1, \dots$ *on* X *such that* $\delta_n = d_{\tau_n}$, *for each* $n < \omega$. *Moreover,* $(X, \tau_0, \tau_1, \dots)$ *is a GLP-space.*

A proof of this proposition builds upon the ideas of H. Simmons [27] and L. Esakia [16,17], which by now have become almost folklore, but it is somewhat lengthy. For the reader's convenience we give this proof in Appendix A.

By Proposition 2.1, the study of neighborhood semantics for **GLP** becomes the study of GLP-spaces. Since **GLP** is well-known to be incomplete w.r.t. any class of Kripke frames the following question naturally arises:

- Is **GLP** complete w.r.t. neighborhood semantics?

In other words, we ask whether there is a suitable class of neighborhood frames $\mathcal{C}$ such that any formula is valid in all frames in $\mathcal{C}$ iff it is provable in **GLP**. Equivalently, this problem was stated in [12] as the question whether **GLP** is the logic of the class of all GLP-spaces.

This question was positively answered for the language with only two modalities in [12]. However, for the case of three or more modalities even a more basic problem was open:

- Is there a GLP-space in which all the topologies are non-discrete?

Some difficulties surrounding these problems are exposed in the papers [12,9,7]. Given a scattered space (X, τ) we can define a new topology τ^+ on X as the coarsest topology containing $\tau \cup \{d_\tau(A): A \subseteq X\}$. Then $(X, \tau, \tau^+, \tau^{++}, \ldots)$ becomes a GLP-space which we call a GLP-space *naturally generated from* (X, τ).

As a fundamental example, one can consider the class of GLP-spaces naturally generated from the standard order topology $\tau_<$ on the ordinals. We call them *ordinal GLP-spaces*. Quite unexpectedly, these spaces turned out to have some deep relations with set theory, in particular, with stationary reflection. For example, it can be shown that the first limit point of $\tau_<^+$ is the cardinal $\aleph_1$, whereas the first limit point of $\tau_<^{++}$ is the so-called *doubly reflecting cardinal*. The existence of this (relatively weak) large cardinal is, however, independent from the axioms of ZFC. Thus, it is independent from ZFC whether τ^{++} is discrete on any ordinal GLP-space.

Andreas Blass [14] was the first to consider the question of topological completeness of **GL** w.r.t. the topology $\tau_<^+$ on an ordinal (using the language of filters rather than the topological one). He showed that the question of completeness of **GL** in this semantics cannot be settled within ZFC. More precisely, the statement '**GL** *is incomplete*' is equiconsistent with the existence of Mahlo cardinals. This result has been generalized to the bimodal logic corresponding to the pair of topologies $(\tau_<, \tau_<^+)$ in [9]. However, to the best of our knowledge, the question whether **GLP** is complete w.r.t. some ordinal GLP-space (under some natural set-theoretic assumptions) remains open so far.

In spite of the above, the present paper gives positive answers to both questions formulated above for general GLP-spaces while firmly standing on the grounds of ZFC.[2] This is achieved by developing new topological techniques related to the study of maximal rank-preserving extensions of scattered topologies. In particular, we introduce a certain class of topologies we call ℓ*-maximal* and show that they are sufficiently well-behaved w.r.t. the operation $\tau \mapsto \tau^+$.

As another ingredient of the topological completeness proof, we introduce an operation on scattered spaces called *d-product*. It can be seen as a generalization of the usual multiplication operation on the ordinals (considered as linear orderings) to arbitrary scattered spaces. We think that this operation could be of some interest in its own right.

The paper is organized as follows. In Section 3 we introduce some useful standard notions related to scattered spaces and prove a few facts about the Cantor–Bendixson rank function. Maximal rank preserving and ℓ-maximal spaces are introduced in Section 4. In Section 5 we show how this techniques allows one to build a non-discrete GLP-space. Section 6 essentially deals with logic and contains a reduction of the topological completeness theorem to some statement of purely topological and combinatorial nature (main lemma). The rest of the paper is devoted to a proof of this lemma. In Section 7 the d-product operation is introduced and a few basic properties of this operation are established. Using d-products, as well as the techniques of Sections 4 and 5, two basic constructions on GLP-spaces are presented in Section 6. Finally, Section 8 contains a proof of the main lemma.

3. Scattered spaces, ranks and *d*-maps

Given a scattered space $X = (X, \tau)$ one can define a transfinite *Cantor–Bendixson sequence* of closed subsets $d^\alpha X$ of X, for any ordinal α, as follows:

- $d^0 X = X$; $d^{\alpha+1} X = d(d^\alpha X)$ and
- $d^\alpha X = \bigcap_{\beta<\alpha} d^\beta X$ if α is a limit ordinal.

[2] It is also natural to ask whether (it is provable in ZFC that) **GLP** is complete w.r.t. any naturally generated GLP-space in the sense given above. This question remains open.

Since X is a scattered space, $d^{\alpha+1}X \subset d^\alpha X$ is a strict inclusion unless $d^\alpha X = \varnothing$. Therefore, from cardinality considerations, for some ordinal α we must have $d^\alpha X = \varnothing$. Call the least such α the *Cantor–Bendixson rank* of X and denote it by $\rho(X)$. The *rank function* $\rho_X : X \to \mathrm{On}$ is defined by

$$\rho_X(x) := \min\{\alpha\colon\ x \notin d^{\alpha+1}(X)\}.$$

Notice that ρ_X maps X onto $\rho_X(X) = \{\alpha\colon\ \alpha < \rho(X)\}$. Also, $\rho_X(x) \geqslant \alpha$ iff $x \in d^\alpha X$. We omit the subscript X whenever there is no danger of confusion.

Example 3.1. Let Ω be an ordinal equipped with its *left topology* $\tau_\leftarrow$, that is, a subset $U \subseteq \Omega$ is open iff $\forall\alpha \in U\ \forall\beta < \alpha\ \beta \in U$. Then $\rho(\alpha) = \alpha$, for all α.

Example 3.2. Let Ω be an ordinal equipped with its *order topology* (also called *interval topology*) $\tau_<$ generated by $\{0\}$ and the intervals $(\alpha, \beta]$, for all $\alpha < \beta \leqslant \Omega$. Then ρ is the function r defined by

$$r(0) = 0; \quad r(\alpha) = \beta \quad \text{if } \alpha = \gamma + \omega^\beta, \text{ for some } \gamma, \beta.$$

By the Cantor normal form theorem, for any $\alpha > 0$, such a β is uniquely defined.

A map $f : X \to Y$ between topological spaces is called a *d-map* if f is continuous, open and *pointwise discrete*, that is, $f^{-1}(\{y\})$ is a discrete subspace of X for each $y \in Y$. It is well-known that d-maps satisfy the properties expressed in the following lemma (see [13]).

Lemma 3.1. *Let $f : X \to Y$ be a d-map, d_X and d_Y denote the derivative operators of X and Y, respectively.*

(i) *$f^{-1}(d_Y(A)) = d_X(f^{-1}(A))$, for any $A \subseteq Y$;*
(ii) *$f^{-1} : (\mathcal{P}(Y), d_Y) \to (\mathcal{P}(X), d_X)$ is a homomorphism of neighborhood frames;*
(iii) *If f is onto, then $\mathrm{Log}(X) \subseteq \mathrm{Log}(Y)$.*

In fact, (i) is easy to check directly; (ii) follows from (i) and (iii) from (ii). From (i) we easily obtain the following corollary by transfinite induction.

Corollary 3.2. *Suppose $f : X \to Y$ is a d-map. Then, for each ordinal α, $d_X^\alpha X = f^{-1}(d_Y^\alpha Y)$.*

The following lemma states that the rank function, when the ordinals are equipped with their left topology, becomes a d-map. It is also uniquely characterized by this property.

Lemma 3.3. *Let Ω be the ordinal $\rho(X)$ taken with its left topology. Then*

(i) *$\rho_X : X \twoheadrightarrow \Omega$ is an onto d-map;*
(ii) *If $f : X \to \lambda$ is a d-map, where λ is an ordinal with its left topology, then $f(X) = \Omega$ and $f = \rho_X$.*

Proof. Let ρ denote ρ_X.

(i) ρ is continuous, because the set $\rho^{-1}[0, \alpha) = X \setminus d^\alpha X$ is open.

The map ρ being open means that, for each open $U \subseteq X$, whenever $\alpha \in \rho(U)$ and $\beta < \alpha$ one has $\beta \in \rho(U)$. Fix an $x \in U$ such that $\rho(x) = \alpha$. Consider the set $X_\beta := \rho^{-1}(\{\beta\}) = d^\beta X \setminus d(d^\beta X)$. For any subset A of a scattered space we have $d(A) = d(A \setminus dA)$, hence $dX_\beta = d(d^\beta X) \supseteq d^\alpha X$. Since $\rho(x) = \alpha$ it follows that $x \in dX_\beta$. Hence $U \cap X_\beta \neq \varnothing$, that is, $\beta \in \rho(U)$.

The map ρ being pointwise discrete means $X_\alpha = \rho^{-1}(\{\alpha\})$ is discrete, for each $\alpha < \Omega$. In fact, $X_\alpha = d^\alpha X \setminus d(d^\alpha X)$ is the set of isolated points of $d^\alpha X$. Thus, it cannot help being discrete.

(ii) Since f is a d-map, by Corollary 3.2 we obtain that $f^{-1}[\alpha,\lambda) = d^\alpha X$, for each $\alpha < \lambda$. Hence, $f^{-1}(\{\alpha\}) = \rho^{-1}(\{\alpha\})$, for each $\alpha < \lambda$, that is, $f = \rho$ and $f(X) = \rho(X) = \Omega$. □

Corollary 3.4. *If $f : X \to Y$ is a d-map, then $\rho_X = \rho_Y \circ f$.*

Proof. Clearly, $\rho_Y \circ f : X \to \Omega$ is a d-map. Statement (ii) of the previous lemma yields the result. □

Note that if $U \in \tau$ is open, then the image of U under the map ρ is always a leftwards closed interval of ordinals and thus is itself an ordinal, which we denote $\rho(U)$. We denote the complement of a set $d^\alpha X$ by $O_\alpha(X)$ or simply O_α when there is no danger of confusion.

4. Maximal and ℓ-maximal topologies

First we introduce two notions: that of a *rank-preserving extension* of a scattered topology, and a more restrictive notion of an *ℓ-extension*. The first one is quite natural and it will help us to build a non-discrete GLP-space. The second is the one we actually need for the proof of the topological completeness theorem.

Definition 4.1. Let (X,τ) be a scattered space.

- A topology σ on X is called a rank-preserving *extension* of τ, if $\sigma \supseteq \tau$ and $\rho_\sigma(x) = \rho_\tau(x)$, for all $x \in X$.
- σ is an *ℓ-extension* of τ, if it is a rank-preserving extension of τ and the identity function $id : (X,\tau) \to (X,\sigma)$ is continuous at all points of successor rank, that is,

  (ℓ) For any $U \in \sigma$ and any $x \in U$ with $\rho(x) \notin \mathrm{Lim}$ there exists $V \in \tau$ such that $x \in V \subseteq U$.

Since the rank function is preserved, ρ stands for both ρ_τ and ρ_σ. Also notice that if $\rho(x) = 0$ then condition (ℓ) is obviously satisfied at x (one can take $V = \{x\}$).

We also note that the notions of rank-preserving extension and of ℓ-extension are transitive and, in fact, define partial orders on the set of all scattered topologies on X. The following observation will be repeatedly used below.

Lemma 4.2. *The following conditions are equivalent*:

(i) *σ is a rank-preserving extension of τ;*
(ii) *ρ_τ is an open map in the topology σ on X;*
(iii) *$\rho_\tau(U)$ is leftwards closed, for each $U \in \sigma$.*

This statement follows from Lemma 3.3.

We are interested in the maximal rank-preserving and the maximal ℓ-extensions. These are naturally defined as follows.

Definition 4.3.

(i) (X,τ) is *maximal*[3] if (X,τ) does not have any proper rank-preserving extensions, in other words, if

$$\forall\sigma\ \big(\sigma \supsetneqq \tau \ \Rightarrow\ \exists x\ \rho_\sigma(x) \neq \rho_\tau(x)\big).$$

(ii) (X,τ) is *ℓ-maximal* if (X,τ) does not have any proper ℓ-extensions.

[3] In the standard terminology used in general topology, *maximal* or *maximal scattered* would mean something different than defined here. Throughout this paper we use the term *maximal* as a shorthand for *maximal scattered with the given rank function.*

It is worth noting that any maximal topology is ℓ-maximal, but not necessarily conversely.

Lemma 4.4.

(i) *Any* (X, τ) *has a maximal extension*;
(ii) *Any* (X, τ) *has an* ℓ*-maximal* ℓ*-extension.*

Proof. Consider the set of all (ℓ-)extensions of a given topology τ ordered by inclusion. We verify, for each of the two orderings, that every chain in it has an upper bound. The result then follows by Zorn's lemma.

Suppose $(\tau_i)_{i\in I}$ is a chain of extensions. Then the topology σ generated by the union $\upsilon = \bigcup_{i\in I} \tau_i$ is apparently a scattered topology containing τ. Note that υ is closed under finite intersections and thus serves as a base for σ. Let $\rho : X \twoheadrightarrow \Omega$ be the common rank function of each of the τ_i. In order to apply Lemma 4.2 we check that ρ is open w.r.t. σ. In fact, any basic $U \in \upsilon$ is open in the sense of some τ_i, and hence $\rho(U)$ must be open in Ω. Lemma 4.2 shows that ρ is the rank function of σ. Hence (i) holds.

Suppose now that $(\tau_i)_{i\in I}$ is a chain of ℓ-extensions. Since any ℓ-extension is an extension, σ (defined as above) is an extension of τ. To check the condition (ℓ) suppose $U \in \sigma$ is given and $x \in U$ is such that $\rho(x) \notin \mathrm{Lim}$. Since σ is generated by the base υ, there exists $U' \in \upsilon$ with $x \in U' \subseteq U$. It follows that $U' \in \tau_i$ for some i. As τ_i is an ℓ-extension of τ, there exists $V \in \tau$ such that $x \in V \subseteq U'$. Since $U' \subseteq U$, we are done. $\square$

Next we prove a workable characterization of ℓ-maximal topologies.

Lemma 4.5. *Let* (X, τ) *be a scattered space and* ρ *its rank function. Then* X *is* ℓ*-maximal iff the following condition holds.*

(*lm*) *For any* $x \in X$ *with rank* $\lambda = \rho(x) \in \mathrm{Lim}$ *and any open* $V \subseteq O_\lambda$*, either* $V \cup \{x\} \in \tau$ *or there is a neighborhood* U *of* x *such that* $\rho(V \cap U) < \lambda$.

Intuitively, condition (*lm*) means that in the neighborhood of a point x of limit rank any open set V is either very large (contains a punctured neighborhood of x), or relatively small (there is a punctured neighborhood whose intersection with V has bounded rank).

Proof. (only if) Suppose the condition (*lm*) is not met. Thus, there exists an $x \in X$ with $\rho(x) = \lambda \in \mathrm{Lim}$ and an open $V_0 \subseteq O_\lambda$ such that $V := V_0 \cup \{x\}$ is not open and $\rho(U \cap V_0) = \lambda$, for any neighborhood U of x.

Let us generate a new topology σ by adding V to τ. We claim that σ is an ℓ-extension of τ. First, we observe that the neighborhood filter at any point $z \in X$, $z \neq x$, in σ is the same as in τ. In fact, any σ-neighborhood W of z either contains a τ-neighborhood of z or contains a subset of the form $V \cap U$ where $U \in \tau$ and $z \in V \cap U = (V_0 \cap U) \cup \{x\}$. In the former case we are done. In the latter case, if $z \neq x$, we have $z \in V_0 \cap U \in \tau$ and $V_0 \cap U \subseteq W$.

From this observation we conclude that $id : (X, \tau) \to (X, \sigma)$ is continuous at all the points $z \neq x$, in particular, condition (ℓ) holds. We show that $\rho_\sigma = \rho$ by applying Lemma 4.2. To check that $\rho : (X, \sigma) \to \Omega$ is open it is sufficient to show that $\rho(W)$ is a neighborhood of $\lambda = \rho(x)$ (in the left topology) for any σ-neighborhood W of x. For all the other points the statement is obvious by the previous observation.

We know that W contains a set of the form $V \cap U$ with $x \in U \in \tau$. Clearly, $V \cap U = (V_0 \cap U) \cup \{x\}$. By the choice of V_0, we have $\rho(V_0 \cap U) = \lambda$ and hence $\rho(W) \supseteq \rho(V \cap U) = [0, \lambda]$ is a neighborhood of λ, as required.

Thus, σ is a proper ℓ-extension of τ, hence X is not ℓ-maximal.

(if) Suppose (X, τ) is not ℓ-maximal and let σ be some proper ℓ-extension of τ. Then the map id : $(X, \tau) \to (X, \sigma)$ is not continuous at certain points. Let $x \in X$ be such a point with the least rank $\rho(x) = \lambda$.

It follows from condition (ℓ) that $\lambda \in \mathrm{Lim}$. Since the map id is not continuous at x, there exists a σ-open neighborhood V of x which contains no τ-open neighborhood of x. Denote $V_0 := V \cap O_\lambda$. It is clear that $V_0 \in \sigma$. It follows from the minimality of λ that $V_0 \in \tau$. From the discontinuity of id at x we may conclude that $V_0 \cup \{x\} \notin \tau$. However, $\{x\} \cup V_0 = V \cap (\{x\} \cup O_\lambda) \in \sigma$, hence, for any τ-neighborhood U of x, we have $(U \cap V_0) \cup \{x\} = U \cap (V_0 \cup \{x\})$ is a σ-neighborhood of x. It follows that $\rho(U \cap V_0) = \lambda$. Thus x and V_0 witness that the condition (lm) is violated for τ. □

Our next objective is to show that whenever $f : X \to Y$ is an onto d-map and Y' is any ℓ-maximal ℓ-extension of Y, one can always find a suitable ℓ-maximal ℓ-extension X' of X so that $f : X' \to Y'$ is still a d-map. We need an auxiliary lemma.

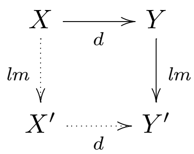
Lemma 4.6. *Let $f : X \to Y$ be a d-map between a scattered space $X = (X, \tau)$ and an ℓ-maximal space $Y = (Y, \sigma)$. Let $X' = (X, \tau')$ be any ℓ-extension of X. Then $f : X' \to Y$ is also a d-map.*

Proof. That $f : X' \to Y$ is continuous and pointwise discrete follows from the fact that $\tau' \supseteq \tau$. We only have to show that $f : X' \to Y$ is open. For the sake of contradiction suppose f is not. Then there exists a point $x \in X'$ and a neighborhood $U \in \tau'$ of x such that $f(U)$ does not contain a neighborhood of $y = f(x)$. We can take such an x of the minimal possible rank λ. This ensures that the restriction of f to the subspace $O_\lambda(X')$ is open, hence a d-map. (Since X' is a rank-preserving extension of X, the set $O_\lambda = O_\lambda(X)$ is the same as $O_\lambda(X')$.)

Since $id : X \to X'$ is continuous at the points of non-limit ranks and $f : X \to Y$ is a d-map, we observe that $\lambda \in \mathrm{Lim}$. Otherwise, for a sufficiently small τ-neighborhood V of x we would have $V \subseteq U$, and then $f(V) \subseteq f(U)$ would be a σ-neighborhood of $f(x)$.

Since $O_\lambda \in \tau$, we may assume that the selected neighborhood U has the form $U = U_0 \cup \{x\}$ where $U_0 \subseteq O_\lambda$ and $U_0 \in \tau$. Thus, $\rho(x) = \lambda \in \mathrm{Lim}$, $V_0 := f(U_0)$ is open, and $V := f(U) = V_0 \cup \{y\}$ is not open in Y. Since Y is ℓ-maximal, by Lemma 4.5 we obtain an open neighborhood W of y such that $\beta := \rho(V_0 \cap W) < \lambda$. We notice that $f(U_0 \cap f^{-1}(W)) = V_0 \cap W$. Hence, $\rho(U_0 \cap f^{-1}(W)) = \beta$. Since $f^{-1}(W) \in \tau$ and $U \in \tau'$ we obtain that $U_1 := U \cap f^{-1}(W) = (U_0 \cap f^{-1}(W)) \cup \{x\}$ is a τ'-open neighborhood of x. Therefore, on the one hand, $\rho(U_1) = \rho_{\tau'}(U_1) = [0, \lambda]$, as τ' is a rank-preserving extension of τ. However, on the other hand, $\rho(U_1) = \rho((U_0 \cap f^{-1}(W)) \cup \{x\}) = \beta \cup \{\lambda\}$, a contradiction. □

Lemma 4.7. *Let $X = (X, \tau)$ and $Y = (Y, \sigma)$ be scattered spaces, let $Y' = (Y, \sigma')$ be an ℓ-maximal ℓ-extension of Y and let $f : X \to Y$ be a d-map. Then there exists an ℓ-maximal ℓ-extension $X' = (X, \tau')$ of X such that $f : X' \to Y'$ is a d-map.*



Proof. Let θ be the topology on X generated by τ and all sets $\{f^{-1}(U)\colon\ U \in \sigma'\}$. A base of θ consists of sets of the form $V \cap f^{-1}(U)$ where $V \in \tau$ and $U \in \sigma'$. It is readily seen that $f : (X, \theta) \to (Y, \sigma')$ is continuous and pointwise discrete. Moreover, it is also an open map, since $f(V \cap f^{-1}(U)) = f(V) \cap U$ is open in σ', for any $V \in \tau$ and $U \in \sigma'$, since $f : X \to Y$ is a d-map. Hence, $f : (X, \theta) \to (Y, \sigma')$ is a d-map. In particular, by Corollary 3.4, θ is a rank-preserving extension of τ.

To see that the condition (ℓ) for θ is met, take any $x \in X$ of successor rank and any basic open neighborhood $W := V \cap f^{-1}(U) \ni x$ such that $V \in \tau$ and $U \in \sigma'$. We have $f(x) \in f(W) = f(V) \cap U \in \sigma'$. Since $f(x)$ is of the same rank as x, by condition (ℓ) applied to σ', there exists $W_1 \in \sigma$ with $f(x) \in W_1 \subseteq$

$f(W)$. Take $U_1 := V \cap f^{-1}(W_1)$. Since $W_1 \in \sigma$ we have $f^{-1}(W_1) \in \tau$, so $U_1 \in \tau$. Also, $W_1 \subseteq f(V) \cap U \subseteq U$ whence $f(U_1) = f(V) \cap W_1 \subseteq U$. It follows that $U_1 \subseteq f^{-1}(U)$, whence $x \in U_1$ and $U_1 \subseteq V \cap f^{-1}(U) = W$, as required.

Therefore, θ is an ℓ-extension of τ. Take any ℓ-maximal ℓ-extension τ' of θ. By Lemma 4.6 we obtain that $f : (X', \tau') \to Y'$ is an onto d-map. Since τ' is also an ℓ-maximal ℓ-extension of τ, the proof is finished. □

5. Building a non-discrete GLP-space

Recall that the *derivative topology* τ^+ on X is generated by τ and $\{d(A)\colon A \subseteq X\}$. Let X^+ denote the space (X, τ^+). The following lemma gives a useful characterization of the derivative topology for ℓ-maximal spaces.

Lemma 5.1. *Suppose (X, τ) is ℓ-maximal. Then τ^+ is generated by τ and the sets $\{d^{\beta+1}(X)\colon \beta < \rho(X)\}$.*

Proof. Let (X, τ) be ℓ-maximal and let τ' denote the topology generated by τ and the sets $\{d^{\beta+1}(X)\colon \beta < \rho(X)\}$. It is clear that each set $d^{\beta+1}(X) = d(d^\beta X)$ is open in τ^+. We show the converse.

Let $A \subseteq X$, we show that $d(A)$ is open in τ'. Consider any $x \in d(A)$ and let $\alpha = \rho(x)$. If α is not a limit ordinal, $\{x\}$ is open in τ'. In fact, since ρ is a d-map, $\rho^{-1}(\alpha)$ is discrete as a subspace of (X, τ). Moreover, $\rho^{-1}(\alpha) = d^\alpha(X) \setminus d^{\alpha+1}(X)$, hence it is clopen in τ'. It follows that x is isolated in τ'.

Suppose $\alpha \in \mathrm{Lim}$ and let C denote the interior of $O_\alpha \setminus A$. Since $x \in dA$ we have $\{x\} \cup C \notin \tau$. Hence, by condition (lm), there is an open $U \in \tau$ with $x \in U$ and a $\beta < \alpha$ such that $U \cap C \subseteq O_\beta$. Consider $V := U \cap d^{\beta+1}X$. Since U is open in τ, V is open in τ'. Moreover, $x \in V$. Thus, we only have to show that $V \subseteq dA$.

Suppose the contrary, that $z \in V \setminus dA$ for some z. Then there exists an open set $U_z \cup \{z\}$ such that $U_z \cap A = \varnothing$ and $U_z \subseteq O_\alpha$. It follows that $U_z \subseteq C$ and hence $U_z \cap U \subseteq O_\beta$. Since $z \in V \subseteq U$, we have that $U' := (U_z \cap U) \cup \{z\} = (U_z \cup \{z\}) \cap U$ is an open neighborhood of z. As ρ is an open map, $\rho(U')$ must be leftwards closed. We have $\rho(z) \geqslant \beta$, since $z \in d^{\beta+1}X$, however $\rho(U_z \cap U) \subseteq \rho(O_\beta) \subseteq \beta$, a contradiction. □

Lemma 5.2. *Suppose (X, τ) is ℓ-maximal and $f : X \to Y$ a d-map. Then f is a d-map between X^+ and Y^+.*

Proof. We only have to show that $f : X^+ \to Y^+$ is open. From the previous lemma we know that τ^+ is generated by τ and $d_X^{\beta+1}X$ for $\beta < \alpha$. Consider a τ^+-open set of the form $A \cap d_X^{\beta+1}X$. Since $f^{-1}(d_Y^{\beta+1}Y) = d_X^{\beta+1}X$ (f is rank preserving), we have $f(A \cap d_X^{\beta+1}X) = f(A) \cap d_Y^{\beta+1}Y$, which is open in Y^+. □

Let Ω denote an ordinal with its left topology. It is easy to check (see [12]) that Ω^+ coincides with the usual order topology on Ω. Let r denote its rank function (see above). In general, for an arbitrary scattered space X let ρ_X^+ denote the rank function of X^+.

Corollary 5.3. *If X is ℓ-maximal, then $\rho_X^+ = r \circ \rho_X$.*

Proof. Let $\Omega := \rho(X)$ be the rank of X. Consider the d-map $\rho : X \twoheadrightarrow \Omega$. By Lemma 5.2, $\rho : X^+ \twoheadrightarrow \Omega^+$ is a d-map. Since r is the rank function of Ω^+, $r : \Omega^+ \to \Omega$ is also a d-map. Hence, $r \circ \rho : X^+ \to \Omega$ is a d-map and coincides with the rank function of X^+. □

Remark 5.1. For an arbitrary scattered space X we only have $\rho_X^+ \leqslant r \circ \rho_X$.

Remark 5.2. In general, the 'derivative topology' operation is non-monotonic: There is a space X such that X^+ is discrete while $(X')^+$ is not, where X' is some maximal extension of X. As such an X one can take the ordinal $\omega^\omega + 1$ with the usual order topology. Since every limit point in X is a unique limit of a countable sequence, every point in X^+ is open, that is, X^+ is discrete. However, Corollary 5.3 shows that $(X')^+$ is non-discrete, for any ℓ-maximal (hence, for any maximal) extension of X. In fact, its rank function ρ is just $r \circ r$ and thus $\rho(\omega^\omega) = 1 \neq 0$.

Now we are ready to specify a suitable class of GLP-spaces which will be used for the topological completeness proof.

Definition 5.4. Let (X, τ) be a scattered space. A polytopological space $(X, \tau_0, \tau_1, \ldots)$ is called an *lme-space based on* τ if[4] τ_0 is an ℓ-maximal ℓ-extension of τ and, for each n, τ_{n+1} is an ℓ-maximal ℓ-extension of τ_n^+.

Clearly, any lme-space is a GLP-space. $(X, \tau_0, \tau_1, \ldots)$ is called an *ordinal lme-space* if X is an ordinal (or an interval of the ordinals) and τ is the order topology on X. Given an lme-space X, let ρ_n denote the rank function of τ_n.

Lemma 5.5. $\rho_{n+1} = r \circ \rho_n$.

Proof. τ_{n+1} has the same rank function as τ_n^+, being its ℓ-extension, hence $\rho_{n+1} = \rho_n^+$. By Corollary 5.3, $\rho_n^+ = r \circ \rho_n$. □

Now we can give an example of a GLP-space in which all topologies are non-discrete. Take any scattered space (X, τ) whose rank Ω satisfies $\omega^\Omega = \Omega$, for example, $X = \varepsilon_0$ with the order topology. Generate some lme-space $(X, \tau_0, \tau_1, \ldots)$ based on τ. Then clearly $\rho_n(X) = r^n(\rho_0(X)) = r^n(\Omega) = \Omega$, for each n. In particular, any topology τ_n is non-discrete. Thus, we have proved

Theorem 5.6. *There is a countable GLP-space* $(X, \tau_0, \tau_1, \ldots)$ *such that each* τ_n *is non-discrete.*

6. Topological completeness of GLP

In this section we reduce the construction of a polytopological space whose logic is **GLP** to a technical lemma. The rest of the paper is devoted to a proof of this lemma.

Our proof of topological completeness will make use of a subsystem of **GLP** introduced in [8] and denoted **J**. This logic is defined by weakening axiom (iv) of **GLP** to the following axioms (vi) and (vii) both of which are theorems of **GLP**:

(vi) $[m]\varphi \to [n][m]\varphi$, for $n \geqslant m$;
(vii) $[m]\varphi \to [m][n]\varphi$, for $n > m$.

J is the logic of a simple class of frames, which is established by standard methods [8, Theorem 1].

Lemma 6.1. **J** *is sound and complete with respect to the class of (finite) frames* $(W, R_0, R_1, \ldots)$ *such that, for all* $x, y, z \in W$,

1. R_k *are transitive and dually well-founded binary relations*;
2. *If* xR_ny, *then* xR_mz *iff* yR_mz, *for* $m < n$;
3. xR_my *and* yR_nz *imply* xR_mz, *for* $m < n$.

[4] The abbreviation *lme* stands for *limit maximal extension.*

Notice that Conditions 2. and 3. imply the following closure property:

$$\text{If } xR_ny \text{ and } yR_mz, \text{ then } xR_kz, \text{ where } k = \min(m,n). \tag{+}$$

Let R_n^* denote the relation $R_n \cup R_{n+1} \cup \cdots$. By (+) we obtain that R_n^* is transitive. Let E_n denote the reflexive, symmetric, transitive closure of R_n^*. Obviously, each E_{n+1} refines E_n. We call each E_n equivalence class an *n-sheet*. By 2. and the dual well-foundedness, all points in an n-sheet are R_m incomparable, for $m < n$. But R_n defines a natural ordering on $(n+1)$-sheets in the following sense: if α and β are $(n+1)$-sheets, then $\alpha R_n \beta$, iff $\exists x \in \alpha\ \exists y \in \beta\ xR_ny$. By the standard techniques, one can improve on Lemma 6.1 to show that **J** is complete for such frames, in which the set of $(n+1)$-sheets contained in each n-sheet is a tree under R_n, and if $\alpha R_n \beta$ then xR_ny for all $x \in \alpha$, $y \in \beta$ (see [8, Theorem 2 and Corollary 3.3]). Every such structure is automatically a J-frame and we call such frames *tree-like J-frames*.

As shown in [8], **GLP** is reducible to **J** in the following sense. Let

$$M(\varphi) := \bigwedge_{i<s} \bigwedge_{k=m_i+1}^{n} \big([m_i]\varphi_i \to [k]\varphi_i\big),$$

where $[m_i]\varphi_i$, $i < s$, are all subformulas of φ of the form $[m]\psi$ and $n := \max_{i<s} m_i$. Also, let $M^+(\varphi) := M(\varphi) \wedge \bigwedge_{m \leqslant n}[m]M(\varphi)$.[5]

Proposition 6.2. *(See [8].)* **GLP** $\vdash \varphi$ *iff* **J** $\vdash M^+(\varphi) \to \varphi$.

For the proof below we will only need the trivial implication from the right to the left. We obtain another proof of this proposition as a byproduct of the topological completeness proof below.

Let $\mathcal{L}_n$ denote the modal language with modalities $[0], [1], \dots, [n]$. Denote by $\mathbf{J}_n$ the logic **J** restricted to $\mathcal{L}_n$. Analogously for $\mathbf{GLP}_n$.[6]

Let $T = (T, R_0, \dots, R_n)$ be a tree-like J_n-frame (or J_n-tree for short). Recall that $w \in T$ is called a *hereditary k-root* if for no $j \geqslant k$ and no $v \in T$ is it true that vR_jw. Note that since T is a J_n-tree, for each $w \in T$ and each $k \leqslant n$ there exists a hereditary k-root $v \in T$ such that $v = w$ or vR_kw.

Definition 6.3. We view T as a polytopological space $T = (T, \sigma_0, \dots, \sigma_n)$ by considering all R_i-upsets to be σ_i-open. Given a GLP$_n$ space $X = (X, \tau_0, \dots, \tau_n)$ and a map $f : X \to T$ we will say that f is a *J_n-morphism* iff:

- (j_1) $f : (X, \tau_n) \to (T, \sigma_n)$ is a d-map;
- (j_2) $f : (X, \tau_k) \to (T, \sigma_k)$ is an open map for all $k \leqslant n$;
- (j_3) For each $k < n$ and each hereditary $(k+1)$-root $w \in T$, the sets $f^{-1}(R_k^*(w))$ and $f^{-1}(R_k^*(w) \cup \{w\})$ are open in τ_k;
- (j_4) For each $k < n$ and each hereditary $(k+1)$-root $w \in T$, the set $f^{-1}(\{w\})$ is a τ_k-discrete subspace of X.

Here, for any binary relation R, we denote by $R(x)$ the set $\{y \in W\colon\ xRy\}$; hence $R_k^*(w)$ denotes the set $\bigcup_{i=k}^n R_i(w)$. Also notice that (j_1) would follow from (j_2)–(j_4) if one also stated them for $k = n$ assuming that $R_{n+1} = \varnothing$. In this case each element of T would be an $(n+1)$-root. The same definition also applies to general J_n-models.

[5] The formula $M(\varphi)$ was defined in [8] incorrectly, however with the present modification everything in [8] works.
[6] Here, to simplify some formulas below, we deviate slightly from the more common notation $\mathbf{GLP}_{n+1}$ for this logic.

A J_n-morphism $f : X \to T$ can be thought of as a map which is a weak kind of d-map from (X, τ_k) to (T, σ_k), for each $k \leqslant n$. As a consequence, we obtain the following simple but useful observation.

Lemma 6.4. *Suppose X, Y are GLP_n-spaces, $g : (Y, \theta_k) \to (X, \tau_k)$ is a d-map, for each $k \leqslant n$, and $f : X \to T$ is a J_n-morphism. Then $f \circ g$ is a J_n-morphism from Y to T.*

Recall that $\tilde{d}(A)$ abbreviates $X \setminus d(X \setminus A)$. Obviously, $x \in \tilde{d}(A)$ iff A contains some punctured neighborhood of x.

Lemma 6.5. *Conditions (j_3) and (j_4) together are equivalent to the following one: for any hereditary $(k+1)$-root w,*

$$f^{-1}\big(R_k^*(w) \cup \{w\}\big) \subseteq \tilde{d}_k\big(f^{-1}\big(R_k^*(w)\big)\big). \tag{$*$}$$

Proof. Suppose $(*)$ holds. Then $f^{-1}(R_k^*(w))$ contains a punctured neighborhood of every point $a \in f^{-1}(R_k^*(w) \cup \{w\})$, hence a neighborhood of every $a \in f^{-1}(R_k^*(w))$. So, $f^{-1}(R_k^*(w))$ is open. It also follows that $f^{-1}(R_k^*(w) \cup \{w\})$ contains a neighborhood of every point $a \in f^{-1}(R_k^*(w) \cup \{w\})$, hence $f^{-1}(R_k^*(w) \cup \{w\})$ is also open.

To show that $f^{-1}(\{w\})$ is τ_k-discrete assume $a \in f^{-1}(\{w\})$. Select a punctured neighborhood V_a of a such that $V_a \subseteq f^{-1}(R_k^*(w))$. Since $w \notin R_k^*(w)$ we have $V_a \cap f^{-1}(\{w\}) = \varnothing$, as required.

Suppose (j_3) and (j_4) hold, we show $(*)$. Assume $a \in f^{-1}(R_k^*(w) \cup \{w\})$. We have to construct a punctured neighborhood of a contained in $f^{-1}(R_k^*(w))$. Consider

$$U := f^{-1}\big(R_k^*(w) \cup \{w\}\big) = f^{-1}\big(R_k^*(w)\big) \cup f^{-1}\big(\{w\}\big).$$

By the second part of (j_3), U is a neighborhood of a. If $a \in f^{-1}(R_k^*(w))$ then $V := f^{-1}(R_k^*(w))$ is a neighborhood of a by the first part of (j_3), so $V - \{a\}$ is as required.

If $a \in f^{-1}(\{w\})$ then by (j_4) there is a neighborhood V_a such that $V_a \cap f^{-1}(\{w\}) = \{a\}$. Then,

$$V_a \cap U = \big(V_a \cap f^{-1}\big(R_k^*(w)\big)\big) \cup \{a\}$$

is a neighborhood of a. Then, $(V_a \cap U) \setminus \{a\}$ is a punctured neighborhood of a contained in $f^{-1}(R_k^*(w))$. □

The following theorem is crucial.

Theorem 6.6. *Let X be a GLP_n-space, T a J_n-tree, $f : X \to T$ a J_n-morphism and φ an $\mathcal{L}_n$-formula. Then $X \vDash \varphi$ iff $T \vDash M^+(\varphi) \to \varphi$.*

Proof. Suppose $T \nvDash M^+(\varphi) \to \varphi$. Then for some valuation ν on T and some point $w \in T$ (assume without loss of generality that w is the hereditary 0-root of T) we have that $w \in \nu(M^+(\varphi))$ but $w \notin \nu(\varphi)$. Consider a valuation ν' on X by taking $\nu'(p) = f^{-1}(\nu(p))$.

Lemma 6.7. *For all subformulas θ of φ, we have $\nu'(\theta) = f^{-1}(\nu(\theta))$.*

Proof. We argue by induction on the complexity of θ. If θ is a propositional letter, the claim is provided by the definition of ν'. The case of Boolean connectives is trivial.

If $\theta = [n]\psi$, then the claim follows by condition (j_1) of f being a J_n-morphism.

Suppose $\theta = [k]\psi$ for some $k < n$. To show that $\nu'(\theta) \subseteq f^{-1}(\nu(\theta))$ assume $x \in \nu'(\theta)$. Then there exists a $U \subseteq X$ such that $\{x\} \cup U \in \tau_k$ and $U \subseteq \nu'(\psi)$. By the IH we obtain $U \subseteq f^{-1}(\nu(\psi))$. Hence $f(U) \subseteq$

$f(f^{-1}(\nu(\psi))) = \nu(\psi)$. By (j_2), the set $f(\{x\} \cup U) = \{f(x)\} \cup f(U)$ is an R_k-upset and so $R_k(f(x)) \subseteq f(U) \subseteq \nu(\psi)$. It follows that $f(x) \in \nu([k]\psi)$. In other words, $x \in f^{-1}(\nu(\theta))$.

For the converse inclusion suppose $x \in f^{-1}(\nu(\theta))$, that is, $f(x) \vDash [k]\psi$. We must show $x \in \nu'(\theta)$. By the induction hypothesis,

$$\nu'(\theta) = \tilde{d}_k(\nu'(\psi)) = \tilde{d}_k(f^{-1}(\nu(\psi))).$$

Let $v \in T$ be a hereditary $(k+1)$-root such that $v = f(x)$ or $vR_{k+1}f(x)$. Since v and $f(x)$ are in the same $(k+1)$-sheet, $R_k(v) = R_k(f(x))$. Thus $v \vDash [k]\psi$. We also have $v \vDash M^+(\varphi)$. In particular, $v \vDash [k]\psi \to [k']\psi$ for any k' with $k \leqslant k' \leqslant n$ and hence $v \vDash [k']\psi$. It follows that for each k' between k and n we have $R_{k'}(v) \subseteq \nu(\psi)$. Therefore $R^*_k(v) \subseteq \nu(\psi)$ and hence $f^{-1}(R^*_k(v)) \subseteq f^{-1}(\nu(\psi))$. By the construction of v, $x \in f^{-1}(R^*_k(v) \cup \{v\})$. Hence, by Lemma 6.5,

$$x \in \tilde{d}_k(f^{-1}(R^*_k(v))) \subseteq \tilde{d}_k(f^{-1}(\nu(\psi))),$$

as required. □

From this lemma we obtain $y \notin \nu'(\varphi) = f^{-1}(\nu(\varphi))$, for any y with $f(y) = w$. Consequently, $X \nvDash \varphi$. □

The proof of the following lemma will be provided later on.

Lemma 6.8 *(Main). For each finite J_n-tree T there exist an ordinal lme-space $X = ([1, \lambda], \tau_0, \dots, \tau_n)$ and an onto J_n-morphism $f : X \twoheadrightarrow T$, where $\lambda < \epsilon_0$.*

Using this lemma we can prove that the logic **GLP** is topologically complete. Let $\mathcal{L}_\omega$ denote the modal language with modalities $[k]$, $k < \omega$.

Theorem 6.9. *Let φ be a formula of $\mathcal{L}_\omega$. If* **GLP** $\nvdash \varphi$ *then φ can be refuted on a GLP-space.*

Proof. Suppose **GLP** $\nvdash \varphi$ and let n be the maximal such that $[n]$ occurs in φ. Obviously, $\mathbf{J}_n \nvdash M^+(\varphi) \to \varphi$. Then there exists a finite J_n-tree T such that $T \nvDash M^+(\varphi) \to \varphi$. By Lemma 6.8 there exists a GLP$_n$-space $X = ([1, \lambda], \tau_0, \dots, \tau_n)$ and a J_n-morphism $f : X \twoheadrightarrow T$. By Theorem 6.6 we have $X \nvDash \varphi$. Let X_ω denote the GLP-space $X_\omega = (X, \tau_0, \dots, \tau_n, \tau_{n+1}, \dots)$ where each topology τ_i is discrete for $i > n$. It is obvious that $X_\omega \nvDash \varphi$. □

The topological completeness theorem can also be stated in a stronger *uniform* way. Recall that ε_0 is the supremum of the countable ordinals ω_k recursively defined by $\omega_0 = 1$ and $\omega_{k+1} = \omega^{\omega_k}$.

Theorem 6.10. *There is an ordinal lme-space $X = (\varepsilon_0, \tau_0, \tau_1, \dots)$ such that* $\mathrm{Log}(X) =$ **GLP**.

Proof. Let $\varphi_0, \varphi_1, \dots$ be an enumeration of all the formulas of $\mathcal{L}_\omega$ unprovable in **GLP**. Using Theorem 6.9 select ordinal lme-spaces $X_i = ([1, \lambda_i], \tau^i_0, \tau^i_1, \dots)$ in such a way that $X_i \nvDash \varphi_i$, for each $i < \omega$. We can assume that $\lambda_i < \varepsilon_0$, for each $i < \omega$. Consider the ordinal $\lambda := \sum_{i<\omega} \lambda_i$. The interval $[1, \lambda)$ is naturally identified (as a set) with the disjoint union $\bigsqcup_{i<\omega} [1, \lambda_i]$. Hence, we can define the topologies τ_i on $[1, \lambda)$ in such a way that $X = ([1, \lambda), \tau_0, \tau_1, \dots)$ is isomorphic to the topological sum $\bigsqcup_{i<\omega} X_i$. Then clearly $\lambda \leqslant \varepsilon_0$ and each formula φ such that **GLP** $\nvdash \varphi$ is refutable on X. Hence, $\mathrm{Log}(X) =$ **GLP**.

In fact, λ must coincide with ε_0. Assume $\lambda < \omega_n$. Then for the topology τ_n we have $\rho_n(X) \leqslant r^{n+1}(\omega_n) = 0$ by Theorem 5.6. However, this contradicts the fact that the unprovable formula $[n]\bot$ is refutable in X. Therefore, $\lambda = \varepsilon_0$ and X is isomorphic to an ordinal lme-space based on ε_0. □

In order to prove the main lemma we introduce the notion of *d-product* of scattered spaces.

7. *d*-Product

Definition 7.1. Let (X, τ_X) and (Y, τ_Y) be any topological spaces. We define their *d-product space* (Z, τ_Z), denoted $X \otimes_d Y$, as follows.

Notice that Y is a union of its isolated points and limit points, $Y = iso(Y) \cup d(Y)$. For all $y \in iso(Y)$, let X_y denote pairwise disjoint copies of X, and let $i_y : X \to X_y$ be the associated homeomorphism maps.

Let Z_0 be the topological sum of $\{X_y\colon\ y \in iso(Y)\}$, that is, $Z_0 := \bigsqcup_{y \in iso(Y)} X_y$. The space Z_0 can also be defined as the cartesian product $X \times iso(Y)$ of X and the discrete space $iso(Y)$. The projection $\pi_0 : Z_0 \twoheadrightarrow X$ is defined in a natural way, that is, $\pi_0(i_y(x)) = x$, for each $y \in iso(Y)$.

Let Z_1 be a copy of the set dY disjoint from Z_0, and $\pi : Z_1 \to dY$ the associated bijection. Put $Z := Z_0 \cup Z_1$. We set $\pi_1(x) := y$, if $x \in X_y$ and $y \in iso(Y)$, and $\pi_1(x) := \pi(x)$, if $x \in Z_1$. It is also convenient to let $X_y := \{y\}$, if $y \in dY$, thus, $X_y = \pi_1^{-1}(y)$, for each $y \in Y$.

Let a topology τ_Z on Z be generated by the one inherited from Z_0 (with the basic open sets $\{i_y(V)\colon\ V \in \tau_X,\ y \in iso(Y)\}$) and by all sets $\{\pi_1^{-1}(U)\colon\ U \in \tau_Y\}$.

We note that, for each $y \in iso(Y)$ and $U \subseteq Y$, the set $\pi_1^{-1}(U) \cap X_y$ is either empty or coincides with X_y. Hence, the above basic open sets form a base of topology τ_Z. It follows that any open set of τ_Z has the form $V \cup \pi_1^{-1}(U)$, where V is open in Z_0 and $U \in \tau_Y$. (Pay attention that this union need not be disjoint.) It also follows that the topologies induced from Z on Z_0 and Z_1 are homeomorphic to those of the product $X \times iso(Y)$ and dY, respectively.

As a typical example, consider the d-product of two compact ordinal spaces $[1, \lambda]$ and $[1, \mu]$ taken with their interval topologies. We claim that $[1, \lambda] \otimes_d [1, \mu]$ is homeomorphic to $[1, \lambda\mu]$ (with the interval topology). Indeed, every $\alpha \in [1, \lambda\mu]$ either has the form $\lambda\beta$ with $\beta \in \mathrm{Lim}$, or belongs to a (clopen) interval $I_{\beta+1} := [\lambda\beta + 1, \lambda(\beta + 1)]$ isomorphic to $[1, \lambda]$. In the former case, $\alpha = \lambda\beta$ corresponds to a limit point $\beta \in [1, \mu]$. In the latter case, α belongs to a copy of $[1, \lambda]$ corresponding to an isolated point $\beta + 1$ of $[1, \mu]$.

The described bijection is, in fact, a homeomorphism: an interval of the form $(\delta, \alpha]$, where $\delta < \alpha \leqslant \lambda\mu$ is a neighborhood of α in the d-product topology. This is clear if $\alpha \in I_{\beta+1}$. If $\alpha = \lambda\gamma$ with $\gamma \in \mathrm{Lim}$, then for all sufficiently large $\beta < \gamma$, $I_\beta \subseteq (\delta, \alpha]$, if $\beta \in \mathrm{Suc}$, and $\lambda\beta \in (\delta, \alpha]$, if $\beta \in \mathrm{Lim}$; hence, the claim. The converse is also clear: a neighborhood of α in the d-product topology contains a suitable interval of the form $(\delta, \alpha]$.

Lemma 7.2.

(i) $\pi_0 : Z_0 \twoheadrightarrow X$ *is a d-map*;
(ii) *The map* $\pi_1 : Z \twoheadrightarrow Y$ *is continuous and open.*

Proof. (i) This follows from the fact that Z_0 is homeomorphic to the product $X \times iso(Y)$ with $iso(Y)$ discrete.

(ii) The continuity of π_1 is clear. To show that it is open, we check that $\pi_1(U)$ is open in Y, for each basic open set U of Z. If U is $\pi_1^{-1}(V)$ for a set $V \in \tau_Y$, we are done. If $U = i_y(V)$, for some nonempty $V \in \tau_X$ and $y \in iso(Y)$, then $\pi_1(U) = \{y\} \in \tau_Y$ as well. □

The following observations will also be helpful.

Lemma 7.3.

(i) *Suppose* $x \in Z_1$. *Then* U *is a punctured neighborhood of* x *in* τ_Z *iff* $\{y \in Y\colon\ X_y \subseteq U\}$ *is a punctured neighborhood of* $\pi_1(x)$ *in* τ_Y.

(ii) *Let* $A \subseteq Z$, $x \in Z_1$. *Then*, $x \in d_Z(A)$ *iff*

$$\pi_1(x) \in d_Y\{y \in Y\colon\ A \cap X_y \neq \varnothing\}.$$

Clearly, $X \otimes_d Y$ is scattered if so are X and Y. Let us compute the rank function of $X \otimes_d Y$.

Lemma 7.4.

(i) *If* $x \in Z_0$ *then* $\rho_Z(x) = \rho_X(\pi_0(x))$.
(ii) *If* $x \in Z_1$ *then* $\rho_Z(x) = \rho(X) + \rho_{dY}(\pi_1(x))$. (*Obviously*, $1 + \rho_{dY}(y) = \rho_Y(y)$.)

Proof. For (i), we just notice that $\rho_Z(x) = \rho_{Z_0}(x)$, since Z_0 is open in Z. Since $\pi_0 : Z_0 \to X$ is a d-map, we have $\rho_{Z_0}(x) = \rho_X(\pi_0(x))$.

For (ii) we first prove that $Z_1 \subseteq d_Z^\beta(Z)$, for each $\beta < \rho(X)$. This goes by transfinite induction on β. The cases when $\beta = 0$ or $\beta \in \mathrm{Lim}$ are easy. Suppose the claim is true for all $\alpha \leqslant \beta$. We prove that $Z_1 \subseteq d_Z^{\beta+1}(Z) = d_Z(d_Z^\beta(Z))$. By (i), if $\beta < \rho(X)$ then $d_Z^\beta(X_y) = i_y(d_X^\beta(X)) \neq \varnothing$, for all $y \in iso(Y)$. Hence, any $y \in Z_1$ is a limit point of $d^\beta(Z_0)$, hence of $d_Z^\beta(Z)$, as required.

As a consequence we obtain that $d_Z^{\rho(X)}(Z) = Z_1$. Hence, $d_Z^{\rho(X)+\alpha}(Z) = d_Z^\alpha(Z_1) = \pi_1^{-1}(d_Y^{1+\alpha}(Y))$, for each α. □

Next we would like to show that d-product is well-behaved w.r.t. ℓ-extensions.

Lemma 7.5. *Suppose* X', Y' *are* ℓ-*extensions of* X, Y, *respectively. Then* $X' \otimes_d Y'$ *is an* ℓ-*extension of* $X \otimes_d Y$.

Proof. The rank function is preserved by the previous lemma. We only have to check that the identity function $id : X \otimes_d Y \to X' \otimes_d Y'$ is continuous at the points x of successor rank. Let $Z = X \otimes_d Y$. If $x \in Z_0$, the claim follows from the hypothesis about X'.

Suppose $x \in Z_1$. By Lemma 7.4, $\rho_Y(\pi_1(x))$ is not a limit. Consider a basic open neighborhood V' of x in $Z' = X' \otimes_d Y'$. The set V' has the form $\pi_1^{-1}(U')$, where U' is a Y'-neighborhood of $\pi_1(x)$. Since Y' is an ℓ-extension of Y, there is a Y-neighborhood $U \subseteq U'$ such that $\pi_1(x) \in U$. Then $x \in \pi_1^{-1}(U) \subseteq V'$, as required. □

Lemma 7.6. *Suppose* X *and* Y *are* ℓ-*maximal and* $\rho(X) \in \mathrm{Suc}$. *Then* $X \otimes_d Y$ *is* ℓ-*maximal.*

Proof. We use Lemma 4.5. Let $Z = X \otimes_d Y$ and suppose $x \in Z$ and $\rho_Z(x) = \lambda \in \mathrm{Lim}$. Consider any open $V \subseteq O_\lambda(Z) = \{z \in Z\colon\ \rho_Z(z) < \lambda\}$. We show that either $V \cup \{x\}$ is open, or there is an open neighborhood U_x of x such that $\rho_Z(V \cap U_x) < \lambda$.

Case 1: $x \in Z_0$. In this case, $V \subseteq \mathcal{O}_\lambda(Z) \subseteq Z_0$ by Lemma 7.4. Also, Z_0 is ℓ-maximal as a topological sum of ℓ-maximal spaces. Hence, the claim follows from ℓ-maximality of Z_0.

Case 2: $x \in Z_1$. In this case we represent V as a union $W \cup \pi_1^{-1}(U)$, where W is open in Z_0 and U in Y. Let $y := \pi_1(x)$ and let $\mu := \rho_Y(y)$. By Lemma 7.4(ii) we have $\rho(X) + \mu' = \lambda$ where $\mu = 1 + \mu'$. Since λ is a limit ordinal, so is μ (unless $\mu' = 0$ and $\lambda = \rho(X)$, but then $\rho(X)$ would be a limit). Hence, we can use the ℓ-maximality of Y for y, μ, and $U \subseteq O_\mu(Y)$.

Suppose $\rho_Y(U \cap U_y) = \beta < \mu$, for some open neighborhood U_y of y in Y. Let $U_x := \pi_1^{-1}(U_y)$. Then $U_x \cap \pi_1^{-1}(U) = \pi_1^{-1}(U \cap U_y)$ is a neighborhood of x (by the continuity of π_1). We also have $\rho_Z(V \cap U_x) \leqslant \rho(X) + \rho_{dY}(U \cap U_y) \leqslant \rho(X) + \beta < \lambda$.

If, on the other hand, $U \cup \{y\}$ is open in Y, then $\pi_1^{-1}(U) \cup \{x\}$ is open in Z, by the continuity of π_1. Hence, so is $V \cup \{x\} = W \cup \pi_1^{-1}(U) \cup \{x\}$. □

Consider now two spaces $X = [1, \lambda]$ and $Y = [1, \mu]$ equipped with the interval topologies. Notice that since X is compact there is an ordinal $\alpha \in X$ whose rank is maximal. Then $\rho(X) = r(\alpha) + 1 \in \mathrm{Suc}$. Let X' and Y' be any ℓ-maximal ℓ-extensions of X and Y, respectively. Combining the previous two lemmas we obtain the following corollary.

Corollary 7.7. *$X' \otimes_d Y'$ is an ℓ-maximal ℓ-extension of $[1, \lambda\mu]$ taken with the interval topology.*

Next, we investigate how the d-product topology behaves w.r.t. the plus operation, for the case of ℓ-maximal spaces.

Lemma 7.8. *Suppose X and Y are ℓ-maximal and $\rho(X) \in \mathrm{Suc}$. Then $(X \otimes_d Y)^+ \simeq (X^+ \times iso(Y)) \sqcup (dY)^+$.*

Here $\sqcup$ denotes the topological sum and $iso(Y)$ comes with the discrete topology. Also notice that, if $Z = X \otimes_d Y$ then $X^+ \times iso(Y)$ is homeomorphic to Z_0^+, and that $(dY)^+$ is homeomorphic to the restriction of Y^+ to the set dY. (Any set dA on Y is contained in dY.)

Proof. Let $Z = X \otimes_d Y$ and let W denote $(X^+ \times iso(Y)) \sqcup (dY)^+$. We can assume that Z and W have the same underlying set. By Lemma 5.1 the topology of W is generated by sets of the form

1. $i_y(V)$, where $y \in iso(Y)$, $V \in \tau_X$ or $V = d^{\alpha+1}X$ with $\alpha < \rho(X)$;
2. $\pi_1^{-1}(U \cap dY)$ for $U \in \tau_Y$ and $\pi_1^{-1}(d^{\beta+1}Y)$ with $\beta < \rho(Y)$.

To prove the inclusion of τ_W into τ_Z^+ we check that all these basic open sets are open in Z^+.

If $V \in \tau_X$ then $i_y(V) \in \tau_Z$, hence it is open in Z^+. If $V = d^{\alpha+1}X$ then $i_y(V) = X_y \cap d^{\alpha+1}Z$, which is open in Z^+ as the intersection of two open sets. If $U \in \tau_Y$, then $\pi_1^{-1}(U \cap dY) = \pi_1^{-1}(U) \cap Z_1$ is open in Z^+. In fact, $Z_1 = d^{\rho(X)}Z$ is open in Z^+, since $\rho(X) \in \mathrm{Suc}$. If $U = \pi_1^{-1}(d^{\beta+1}Y)$, then $U = d_Z^{\beta+1}Z_1 = d_Z^{\rho(X)+\beta+1}Z$ which is open in Z^+.

Now we check that τ_Z^+ is included in τ_W. Since $X \oplus_d Y$ is ℓ-maximal, τ_Z^+ is generated by τ_Z and sets of the form $d^{\alpha+1}Z$ for $\alpha < \rho(Z)$. By Lemma 7.4

$$d^{\alpha+1}Z = \begin{cases} d^{\alpha+1}Z_0 \cup Z_1, & \text{if } \alpha < \rho(X), \\ \pi_1^{-1}(d^{\beta+1}Y), & \text{if } \alpha = \rho(X) + \beta. \end{cases}$$

In both cases it is clearly open in W. On the other hand, open sets in Z are generated by $i_y(V)$ with $V \in \tau_X$, in which case we are done, and $\pi_1^{-1}(U)$ with $U \in \tau_Y$. Let $U_0 := U \cap iso(Y)$ and $U_1 := U \cap dY$. Notice that $\pi_1^{-1}(U_0) = \bigcup_{y \in U_0} X_y$ is open in Z_0, and hence in W, whereas $\pi_1^{-1}(U_1)$ is open in dY, hence in $(dY)^+$ and W. Hence, τ_Z is included in τ_W and we are done. □

8. Some operations on lme-spaces

Recall that $(X, \tau_0, \ldots, \tau_n)$ is an *lme-space* based on a scattered topology τ if $\tau_0 = \tau'$ and $\tau_{i+1} = (\tau_i^+)'$, for each $i < n$, where σ' denotes any ℓ-maximal ℓ-extension of σ. Obviously, any such space is a GLP_n-space. We call $(X, \tau_0, \ldots, \tau_n)$ an *ordinal lme-space* if X is an ordinal and τ is the interval topology on X. We specify two constructions on lme-spaces.

First, we extend the operation of d-product to GLP-spaces.

Definition 8.1. Suppose $(X, \tau_0, \ldots, \tau_n)$ and $(Y, \sigma_0, \ldots, \sigma_n)$ are two GLP_n-spaces. Let (Z, θ_0) be the d-product $(X, \tau_0) \otimes_d (Y, \sigma_0)$. For each $i = 1, \ldots, n$ we specify a topology θ_i on Z as the sum of the topologies τ_i on

X_y, for each $y \in iso(Y)$, and of σ_i on dY, where $iso(Y)$ and dY refer to the space (Y, σ_0). In other words, θ_i consists of the sets of the form

$$\bigcup_{y \in iso(Y)} i_y(U_y) \cup \pi_1^{-1}(V \cap dY)$$

where $U_y \subseteq X$, $U_y \in \tau_i$ and $V \in \sigma_i$. We note that the functions $\pi_0 : (Z_0, \theta_i \upharpoonright Z_0) \twoheadrightarrow (X, \tau_i)$ and $\pi_1 : (Z_1, \theta_i \upharpoonright Z_1) \twoheadrightarrow (dY, \sigma_i \upharpoonright dY)$ are d-maps, for $i = 1, \ldots, n$.

Lemma 8.2. *$(Z, \theta_0, \theta_1, \ldots, \theta_n)$ is a GLP_n-space.*

Proof. We make use of the fact that the plus operation on topologies distributes over topological sums. Hence, $\theta_i^+ \subseteq \theta_{i+1}$ on Z, for all $i = 1, \ldots, n-1$. Thus, we only have to show that $\theta_0^+ \subseteq \theta_1$.

Consider any $A \subseteq Z$. By Lemma 7.3

$$d_Z(A) = \pi_1^{-1}\big(d_Y\{y\colon\ A \cap X_y \neq \varnothing\}\big) \cup d_{Z_0}(A \cap Z_0).$$

In fact, any $x \in d_Z(A) \cap Z_0$ must belong to $d_{Z_0}(A \cap Z_0)$, since Z_0 is open in Z, hence the claim. However, both $\pi_1^{-1}(d_Y\{y\colon\ A \cap X_y \neq \varnothing\})$ and $d_{Z_0}(A \cap Z_0)$ are open in θ_1. This is because $d_Y\{y\colon\ A \cap X_y \neq \varnothing\}$ is open in (dY, σ_1) and $d_{Z_0}(A \cap Z_0)$ is open in (Z_0, θ_1). □

Lemma 8.3. *Suppose $(X, \tau_0, \ldots, \tau_n)$ and $(Y, \sigma_0, \ldots, \sigma_n)$ are lme-spaces based on τ and σ, respectively, such that both $\rho(X, \tau)$ and $\rho(Y, \sigma)$ are successor ordinals. Then $X \otimes_d Y$ is an lme-space based on $(X, \tau) \otimes_d (Y, \sigma)$. Moreover, $\rho((X, \tau) \otimes_d (Y, \sigma))$ is a successor ordinal.*

Proof. Let $Z = (Z, \theta_0, \ldots, \theta_n)$ denote $X \otimes_d Y$. The fact that (Z, θ_0) is an ℓ-maximal ℓ-extension of $(X, \tau) \otimes_d (Y, \sigma)$ follows from Lemmas 7.5 and 7.6.

We show that (Z, θ_1) is an ℓ-maximal ℓ-extension of (Z, θ_0^+). By Lemma 7.8

$$\big(Z, \theta_0^+\big) \simeq \big(\big(X, \tau_0^+\big) \times iso(Y)\big) \sqcup \big(dY, \sigma_0^+\big).$$

On the other hand, by definition,

$$(Z, \theta_1) \simeq \big((X, \tau_1) \times iso(Y)\big) \sqcup (dY, \sigma_1).$$

We have that (X, τ_1) is an ℓ-maximal ℓ-extension of (X, τ_0^+) and that (dY, σ_1) is one of (dY, σ_0^+). This relation then holds for the respective topological sums.

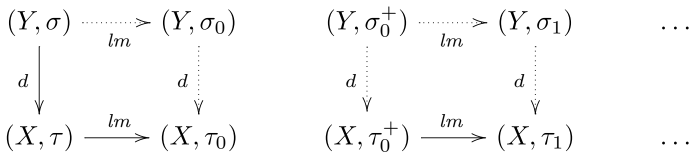
Finally, we remark that (Z, θ_{i+1}) is an ℓ-maximal ℓ-extension of (Z, θ_i^+), for $i = 1, \ldots, n-1$, because (X, τ_{i+1}) is an ℓ-maximal ℓ-extension of (X, τ_i^+) and (dY, σ_{i+1}) is an ℓ-maximal ℓ-extension of (dY, σ_i^+). These relations then must also hold for the respective topological sums. □

Corollary 8.4. *Let X and Y be ordinal lme-spaces on $[1, \lambda]$ and $[1, \mu]$, respectively. Then $X \otimes_d Y$ is an ordinal lme-space on $[1, \lambda\mu]$.*

We are going to introduce another key operation on lme-spaces called *lifting*. Before doing it we state a simple 'pullback' lemma.

Lemma 8.5. *Let $(X, \tau_0, \ldots, \tau_n)$ be an lme-space based on τ, and let $h : (Y, \sigma) \to (X, \tau)$ be a d-map. Then there is an lme-space $(Y, \sigma_0, \ldots, \sigma_n)$ based on σ such that $h : (Y, \sigma_i) \to (X, \tau_i)$ is a d-map, for each $i \leqslant n$.*

Proof. This statement is proved by a repeated application of Lemmas 4.7 and 5.2 as indicated in the following diagram.



Here, the arrows labeled by 'd' indicate d-maps; the arrows labeled by 'lm' indicate ℓ-maximal ℓ-extensions. Dotted arrows are being proved to exist given the rest. Thus, the two squares represent the first two applications of Lemma 4.7, and the transition from the right vertical arrow of the first square to the left vertical arrow of the second one is an application of Lemma 5.2. □

Lemma 8.6 *(Lifting). Suppose $X = ([0,\lambda],\tau_1,\dots,\tau_n)$ is an ordinal lme-space. Then there is an ordinal lme-space $Y = ([1,\omega^\lambda],\sigma_0,\sigma_1,\dots,\sigma_n)$ such that $r : ([1,\omega^\lambda],\sigma_i) \twoheadrightarrow ([0,\lambda],\tau_i)$ is a d-map, for each $i = 1,\dots,n$.*

Such a Y can be called a *lifting* of the space X, since it is similar to X w.r.t. higher topologies (starting from the second one rather than the first).

Proof. Topology σ_0, being an ℓ-maximal ℓ-extension of the order topology, has the same rank function. Therefore, $r : ([1,\omega^\lambda],\sigma_0) \twoheadrightarrow ([0,\lambda],\tau_\leftarrow)$ is a d-map. By Lemma 5.2 we obtain that $r : ([1,\omega^\lambda],\sigma_0^+) \twoheadrightarrow ([0,\lambda],\tau_<)$ is a d-map, as well. Since τ_1 is an ℓ-maximal ℓ-extension of the order topology, we are now in a position to apply Lemma 8.5. So, we obtain an lme-space $([1,\omega^\lambda],\sigma_1,\dots,\sigma_n)$ based on σ_0^+ such that $r : ([1,\omega^\lambda],\sigma_i) \twoheadrightarrow ([0,\lambda],\tau_i)$ is a d-map, for each $i = 1,\dots,n$. It follows that $Y = ([1,\omega^\lambda],\sigma_0,\sigma_1,\dots,\sigma_n)$ is as required. □

9. Proof of main lemma

Now we provide the key construction proving Lemma 6.8 above. Its closest relative is the construction in [9], however it is now complicated by the fact that the topologies of a suitable lme-space are constructed along with a J_n-morphism to a given Kripke frame.

Proof. For each J_n-tree $(T,R_0,\dots,R_n)$ with a root a we are going to build an ordinal lme-space $X = ([1,\lambda],\tau_0,\dots,\tau_n)$ and a J_n-morphism $f : X \twoheadrightarrow T$ such that $f^{-1}(\{a\}) = \{\lambda\}$. Such J_n-morphisms will be called *suitable*. The construction goes by induction on n with a subordinate induction on the R_0-height of T, which is denoted $ht_0(T)$.

If $n = 0$ we let τ_0 be the interval topology and notice that on any $\lambda < \omega^\omega$ this topology is ℓ-maximal (since there are no points of limit rank). From the topological completeness proofs for the Gödel–Löb logic it is known (see [1,14,13]) that there is an ordinal $\lambda < \omega^\omega$ and a suitable d-map from $[1,\lambda]$ onto (T,R_0). This map is constructed by induction on $ht_0(T)$.

If $ht_0(T) = 0$, then T consists of a single point a. We put $\lambda = 1$ and $f(1) = a$. If $ht_0(T) = m > 0$ let $a_1,\dots,a_l$ be the children of the root a, and let T_i denote the subtree generated by a_i, for $1 \leqslant i \leqslant l$. By the induction hypothesis, there are ordinals $\kappa_1,\dots,\kappa_l$ and suitable d-maps[7] $g_i : [1,\kappa_i] \twoheadrightarrow T_i$, for each $i = 1,\dots,l$. Let $\kappa := \kappa_1 + \dots + \kappa_l$, then $[1,\kappa]$ can be identified with the topological sum $\bigsqcup_{i=1}^l [1,\kappa_i]$. Let $g : [1,\kappa] \twoheadrightarrow \bigsqcup_{i=1}^l T_i$ be defined by

[7] Recall that J_0-morphisms are just d-maps.

$$g(\alpha) := g_i(\beta), \quad \text{if } \alpha = \kappa_1 + \cdots + \kappa_{i-1} + \beta,\ \beta \in [1, \kappa_i].$$

Then g is clearly a d-map.

We now let $\lambda := \kappa\omega$ and let $f : [1, \lambda] \twoheadrightarrow T$ be defined by

$$f(\alpha) := \begin{cases} g(\beta), & \text{if } \alpha = \lambda n + \beta \text{ where } n < \omega,\ \beta \in [1, \kappa], \\ a, & \text{if } \alpha = \lambda. \end{cases}$$

It is then easy to verify that f is, indeed, a suitable d-map. This accounts for the case $n = 0$.

For the induction step suppose the lemma is true for each J_k-tree with $k < n$. Let $T = (T, R_0, \ldots, R_n)$ be a J_n-tree with the root a. We prove our claim by induction on the R_0-height of T.

Case 1: $ht_0(T) = 0$, in other words $R_0 = \varnothing$. Let $T_1 := (T, R_1, \ldots, R_n)$. By the induction hypothesis there is a suitable J_{n-1}-morphism $f_1 : X_1 \twoheadrightarrow T_1$ where $X_1 = ([1, \lambda_1], \tau_1, \ldots, \tau_n)$. We note that X_1 is isomorphic to $([0, \mu], \tau_1, \ldots, \tau_n)$, for some μ (obviously, $\mu = \lambda_1$ if λ_1 is infinite). By the Lifting lemma there is an ordinal lme-space $X = ([1, \lambda], \sigma_0, \sigma_1, \ldots, \sigma_n)$ such that $\lambda = \omega^\mu$ and

$$r : \big([1, \lambda], \sigma_i\big) \twoheadrightarrow \big([0, \mu], \tau_i\big)$$

is a d-map, for each $i \in [1, n]$. It follows that $f := r \circ f_1$ is a suitable J_n-morphism. In fact, it is immediate that conditions (j_1), (j_2) are met and that (j_3), (j_4) are satisfied for each $k \geqslant 1$. Let us consider (j_3) for $k = 0$.

Since R_0 is empty, the only 1-hereditary root of T is in fact the unique 0-hereditary root a, thus $R_0^*(a) = T \setminus \{a\}$. Then clearly $f^{-1}(R_0^*(a)) = [1, \lambda)$ and $f^{-1}(R_0^*(a) \cup \{a\}) = [1, \lambda]$, both of which are σ_0-open. Thus (j_3) is met.

Condition (j_4) for $k = 0$ boils down to the fact that $f^{-1}(a)$ is discrete. However, $f^{-1}(a)$ is the singleton $\{\lambda\}$. Thus (j_4) is also met and $f : X \twoheadrightarrow T$ is the required J_n-morphism.

Case 2: $ht_0(T) = m > 0$. Let $a_1, \ldots, a_l$ be the immediate R_0-successors of a which are hereditary 1-roots. Denote $T_i = \{a_i\} \cup R_0^*(a_i)$ for $i \in [1, l]$ and $T_0 = \{a\} \cup R_1^*(a)$. Note that $T = \bigcup_{i=0}^l T_i$. Furthermore, for each $i \in [1, l]$ the subframe T_i of T is a J_n-tree of R_0-height less than m. By the induction hypothesis there exist ordinal lme-spaces $S_i = ([1, \kappa_i], \xi_0^i, \ldots, \xi_n^i)$ and suitable J_n-morphisms $g_i : S_i \to T_i$. Let $\kappa := \kappa_1 + \cdots + \kappa_l$, then $[1, \kappa]$ can be identified with the disjoint union $\bigsqcup_{i=1}^l [1, \kappa_i]$. Let $\xi_0, \ldots, \xi_n$ be the topologies of the corresponding topological sum, that is, $\xi_j = \bigsqcup_{i=1}^l \xi_j^i$, and let $g : [1, \kappa] \to \bigsqcup_{i=1}^l T_i$ be the disjoint union of g_i, i.e. $g = \bigsqcup_{i=1}^l g_i$. Notice that $\bigsqcup_{i=1}^l T_i$ is identified with $R_0(a)$. It is easy to see that $X = ([1, \kappa], \xi_1, \ldots, \xi_n)$ is an ordinal lme-space and that $g : [1, \kappa] \to R_0(a)$ is a J_n-morphism.

Now consider the 1-sheet $(T_0, R_1, \ldots, R_n)$. By the induction hypothesis (for n) there is an ordinal lme-space $Y_0 = ([1, \lambda_0], \tau_1, \ldots, \tau_n)$ and a suitable J_{n-1}-morphism $g_0 : Y_0 \twoheadrightarrow T_0$. Let $Y = ([1, \omega^{\lambda_0}], \sigma_0, \sigma_1, \ldots, \sigma_n)$ be an ordinal lme-space defined as in Case 1 and let $h : Y \twoheadrightarrow (T_0, \varnothing, R_1, \ldots, R_n)$ be the corresponding suitable J_n-morphism.

We now consider the d-product $Z := X \otimes_d Y$ of these ordinal lme-spaces. Note that $iso(Y) = \{\alpha + 1\colon \alpha < \kappa_0\}$ and $dY = \mathrm{Lim} \cap [1, \kappa_0]$ where $\kappa_0 := \omega^{\lambda_0}$. Hence, we can identify Z with an ordinal lme-space $([1, \lambda], \theta_0, \ldots, \theta_n)$ where $\lambda := \kappa \cdot \kappa_0$ and $X_{\alpha+1} = [\kappa\alpha + 1, \kappa(\alpha + 1)]$, for all $\alpha < \kappa_0$. Hence, $Z_0 = \bigsqcup_{\alpha<\kappa_0} X_{\alpha+1}$ and $Z_1 = \{\kappa\lambda\colon \lambda \in \mathrm{Lim}, \lambda \leqslant \kappa_0\}$. The associated projection maps $\pi_0 : Z_0 \twoheadrightarrow X$ and $\pi_1 : Z \twoheadrightarrow Y$ are defined by formulas $\pi_1(\kappa\lambda) = \lambda$ and $\pi_0(\kappa\alpha + \beta) = \beta$, where $\lambda \in \mathrm{Lim}$, $\lambda \leqslant \kappa_0$, $\beta \in [1, \kappa]$, $\alpha < \kappa_0$.

We define the required J_n-morphism $f : Z \twoheadrightarrow T$ as follows:

$$f(z) := \begin{cases} g(\pi_0(z)), & \text{if } z \in Z_0, \\ h(\pi_1(z)), & \text{if } z \in Z_1. \end{cases}$$

We have to check that f satisfies (j_1)–(j_4). Recall that for $k \geqslant 1$ the space (Z, θ_k) is homeomorphic to the topological sum of $Z_0 \simeq \bigsqcup_{\alpha<\kappa_0} (X, \xi_k)$ and $Z_1 \simeq (Y, \sigma_k)$. Then both $\pi_0 : (Z_0, \theta_k{\restriction}Z_0) \twoheadrightarrow (X, \xi_k)$ and

$\pi_1 : (Z_1, \theta_k \restriction Z_1) \twoheadrightarrow (Y, \sigma_k)$ are d-maps. Since both g and h are J_n-morphisms, it follows that conditions (j_1)–(j_4) are satisfied for all $k \geqslant 1$. We must only check (j_2)–(j_4) for $k = 0$.

Recall that the topology θ_0 on a d-product $X \otimes_d Y$ is generated by the base of open sets $\{i_y(V)\colon V \in \tau_X, y \in iso(Y)\}$ and $\{\pi_1^{-1}(U)\colon U \in \tau_Y\}$. Hence, in order to check (j_2) it is sufficient to show that the image under f of any such basic open set is open. Since $i_y(V) \subseteq Z_0$ and $\pi_0(i_y(V)) = V$ we obtain that $f(i_y(V)) = g(V)$ is open (g is a J_n-morphism). On the other hand, if U is nonempty, then $f(\pi_1^{-1}(U)) = h(U) \cup g(X) = h(U) \cup R_0(a)$. This holds because every nonempty open subset of Y, in particular U, has a point y of rank 0. Then $X_y \subseteq \pi_1^{-1}(U)$ and hence $f(\pi_1^{-1}(U)) \supseteq f(X_y) = g(X)$. Clearly, both $h(U)$ and $R_0(a)$ are open in T. Hence, f satisfies (j_2).

Condition (j_3) follows from the fact that both π_0 and π_1 are continuous. Indeed, if w is a hereditary 1-root of T, then either $w = a$ or $w \in R_0(a)$. In the former case $R_0^*(w) = T \setminus \{a\}$ and hence $f^{-1}(R_0^*(w)) = [1, \lambda)$ is open. Similarly, $f^{-1}(R_0^*(w) \cup \{w\}) = Z$ is open.

If $w \in R_0(a)$ then both $R_0^*(w)$ and $R_0^*(w) \cup \{w\}$ are contained in $R_0(a)$. Since g is a J_n-morphism, $g^{-1}(R_0^*(w))$ is open. Then $f^{-1}(R_0^*(w)) = \pi_0^{-1}(g^{-1}(R_0^*(w)))$ is open, by the continuity of π_0. The argument for $R_0^*(w) \cup \{w\}$ is similar. Hence, condition (j_3) is met.

To check condition (j_4) assume w is a hereditary 1-root of T. If $w = a$ then $f^{-1}(\{w\})$ is the singleton $\{\lambda\}$. If $w \in R_0(a)$ then $g^{-1}(\{w\})$ is discrete as a subspace of X, since g is a J_n-morphism. We know that $\pi_0 : Z_0 \twoheadrightarrow X$ is both continuous and pointwise discrete. Hence, $f^{-1}(\{w\}) = \pi_0^{-1}(g^{-1}(\{w\}))$ is discrete in Z_0 and thereby in Z (Z_0 is open in Z). This shows (j_4).

Thus, we have checked that $f : Z \twoheadrightarrow T$ is a suitable J_n-morphism, which completes the proof of Lemma 6.8 and thereby of Theorem 6.6. □

10. Further results

After a preliminary version of this paper has appeared as a preprint [10] some interesting further developments took place that we briefly mention here.

Topological semantics of polymodal provability logic has been extended to the language with transfinitely many modalities. A logic $\mathbf{GLP}_\Lambda$ having modalities $[\alpha]$, for all ordinals $\alpha < \Lambda$, is introduced in [6]. It was intended for the proof-theoretic analysis of predicative theories and is currently being actively investigated for that purpose.

David Fernandez and Joost Joosten undertook a thorough study of the variable-free fragment of that logic mostly in connection with the arising ordinal notation systems (see [19,20] for a sample). In particular, they found a suitable generalization of Icard's polytopological space and showed that it is complete for that fragment [21]. Fernandez [18] also proved topological completeness of the full $\mathbf{GLP}_\Lambda$ by extending the results of the present paper.

The ordinal GLP-space is easily generalized to transfinitely many topologies $(\tau_\alpha)_{\alpha<\Lambda}$ by letting τ_0 be the left topology, $\tau_{\alpha+1} := \tau_\alpha^+$ and, for limit ordinals λ, τ_λ be the topology generated by all τ_α such that $\alpha < \lambda$. This space is a natural model of $\mathbf{GLP}_\Lambda$ and has been studied quite recently by Joan Bagaria [2] and further by Bagaria, Magidor and Sakai [3]. In particular, the three authors proved that in L the limit points of τ_{n+2} are Π_n^1-indescribable cardinals. The question posed in [12] whether the non-discreteness of τ_{n+2} is equiconsistent with the existence of Π_n^1-indescribable cardinals still seems to be open. So is the more difficult problem whether **GLP** is complete for the ordinal polytopological space (under suitable set-theoretic assumptions).

The reader can also consult our recent paper [11] for a general survey of topological semantics of provability logic and more information on ordinal GLP-spaces.

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Appendix A. Proof of Proposition 2.1

The correspondence between Magari frames and scattered topological spaces is due to Esakia [16]. A frame (X, δ) is called a *Magari frame* if it satisfies the following identities, for any $A, B \subseteq X$:

(i) $\delta(A \cup B) = \delta A \cup \delta B$; $\delta\varnothing = \varnothing$;
(ii) $\delta A = \delta(A \setminus \delta A)$.

It is well-known and easy to see that (X, δ) is Magari iff (X, δ) validates the axioms of Gödel–Löb logic **GL** (corresponding to Axioms (i)–(iii) of **GLP**). We notice that any such operator δ is monotone, that is, $A \subseteq B$ implies $\delta A \subseteq \delta B$. In addition, $\delta\delta A \subseteq \delta A$ holds in any Magari frame, since the formula $\Diamond\Diamond p \to \Diamond p$ is a theorem of **GL**.

Lemma A.1. *If (X, τ) is a scattered topological space then (X, d_τ) is a Magari frame.*

Proof. The validity of (i) is obvious, whereas (ii) means that any limit point of A is a limit point of the set $iso(A)$ of isolated points of A. Let $x \in d_\tau A$ and let U be an open neighborhood of x. $U \cap A \setminus \{x\}$ is not empty, hence it has an isolated point y. Then $y \in iso(A)$ as well. □

Suppose $(X, \tau_0, \tau_1, \ldots)$ is a GLP-space. To prove part (i) of Proposition 2.1 observe that Axioms (i)–(iii) of **GLP** are satisfied in $(X, d_0, d_1, \ldots)$ by the previous corollary. Axiom (iv) is clearly valid since $\tau_n \subseteq \tau_{n+1}$.

To check Axiom (v) consider a set of the form $d_n(A)$. Since X is a GLP-space, $d_n(A)$ is open in τ_{n+1}. Hence, every $x \in d_n(A)$ cannot be a τ_{n+1}-limit point of $X \setminus d_n(A)$, that is, $x \in \tilde{d}_{n+1}(d_n A)$. In other words, $d_n(A) \subseteq \tilde{d}_{n+1}(d_n A)$, for any A, that is, Axiom (v) is valid.

To prove part (ii) of Proposition 2.1 we first remark that, if (X, δ) is a Magari frame, then the operator $c(A) := A \cup \delta A$ satisfies the Kuratowski axioms of the topological closure. This defines a topology on X in which any set A is closed iff $A = c(A)$ iff $\delta A \subseteq A$. (Alternatively, one can check that the collection of all sets U satisfying $U \subseteq \tilde{\delta} U$ is a topology.)

Lemma A.2. *Suppose (X, δ) is Magari. Then, for all $x \in X$,*

(i) $x \notin \delta(\{x\})$;
(ii) $x \in \delta A \iff x \in \delta(A \setminus \{x\})$.

Proof. (i) By Axiom (iii), $\delta\{x\} \subseteq \delta(\{x\} \setminus \delta\{x\})$. If $x \in \delta\{x\}$ then $\delta(\{x\} \setminus \delta\{x\}) \supseteq \delta(\{x\} \setminus \{x\}) = \delta\varnothing = \varnothing$. Hence, $\delta\{x\} = \varnothing$, a contradiction.

(ii) $x \in \delta A$ implies $x \in \delta((A \setminus \{x\}) \cup \{x\}) = \delta(A \setminus \{x\}) \cup \delta\{x\}$. By (i), $x \notin \delta\{x\}$, hence $x \in \delta(A \setminus \{x\})$. The other implication follows from the monotonicity of δ. □

Lemma A.3. *Suppose (X, δ) is Magari and τ is the associated topology. Then $\delta = d_\tau$.*

Proof. Let $d = d_\tau$; we show that, for any set $A \subseteq X$, $dA = \delta A$. Notice that, for any B, $cB = dB \cup B = \delta B \cup B$.

Assume $x \in \delta A$ then

$$x \in \delta\big(A \setminus \{x\}\big) \subseteq c\big(A \setminus \{x\}\big) \subseteq d\big(A \setminus \{x\}\big) \cup \big(A \setminus \{x\}\big).$$

Since $x \notin A \setminus \{x\}$, we obtain $x \in d(A \setminus \{x\})$. By the monotonicity of d, $x \in dA$.

Similarly, if $x \in dA$ then $x \in d(A \setminus \{x\})$. Hence,

$$x \in c\big(A \setminus \{x\}\big) = \delta\big(A \setminus \{x\}\big) \cup \big(A \setminus \{x\}\big).$$

Since $x \notin A \setminus \{x\}$ we obtain $x \in \delta A$. □

Lemma A.4. *Suppose (X, δ) is Magari and τ is the associated topology. Then (X, τ) is scattered.*

Proof. Since δ is Löb we know that $\delta = d_\tau$. We show that any nonempty subspace $A \subseteq X$ has an isolated point.

Suppose not, then $iso(A) = A \setminus \delta A = \varnothing$. Then $\delta A = \delta(A \setminus \delta A) = \delta\varnothing = \varnothing$. Then $A = A \setminus \delta A = \varnothing$. □

Now we prove part (ii). Let $(X, \delta_0, \delta_1, \ldots)$ be a neighborhood frame satisfying **GLP**. Then each of the frames (X, δ_n) is Magari, hence it defines a scattered topology τ_n on X for which $\delta_n = d_{\tau_n}$. Recall that $U \in \tau_n$ iff $U \subseteq \tilde{\delta}_n(U)$. We only have to show that the last two conditions of a GLP-space are met.

Suppose $U \in \tau_n$, then $U \subseteq \tilde{\delta}_n(U) \subseteq \tilde{\delta}_{n+1}(U)$ by Axiom (iv). Hence, $U \in \tau_{n+1}$. Thus, $\tau_n \subseteq \tau_{n+1}$.

Similarly, by Axiom (v) for any set A we have $\delta_n(A) \subseteq \tilde{\delta}_{n+1}(\delta_n(A))$. Hence, $d_{\tau_n}(A) = \delta_n(A) \in \tau_{n+1}$. Thus, $(X, \tau_0, \tau_1, \ldots)$ is a GLP-space.

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