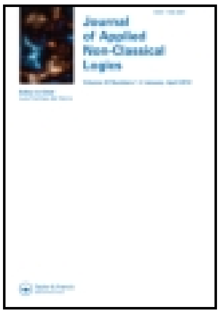

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The modalized Heyting calculus: a conservative modal extension of the Intuitionistic Logic ★

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The modalized Heyting calculus: a conservative modal extension of the Intuitionistic Logic ⋆

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ABSTRACT. *In this paper we define an augmentation mHC of the Heyting propositional calculus HC by a modal operator* $\Box$. *This modalized Heyting calculus mHC is a weakening of the Proof-Intuitionistic Logic KM of Kuznetsov and Muravitsky. In Section 2 we present a short selection of attractive (algebraic, relational, topological and categorical) features of mHC. In Section 3 we establish some close connections between mHC and certain normal extension K4.Grz of the modal system K4. We define a translation of mHC into K4.Grz and prove that this translation is exact, i. e. theorem-preserving and deducibility-invariant. We have established (however, in this note we do not present a proof of this) that the lattice of all extensions of mHC is isomorphic to the lattice of normal extensions of K4.Grz (a generalization of the Kuznetsov and Muravitsky theorem).*

KEYWORDS: *Heyting algebra, modal operator, derivative algebra, provability logic, topological semantics, Kripke frame.*

1. Introduction

In this section we describe an augmentation of the Heyting propositional calculus by a modal operator $\Box$. We do not intend to give a systematic survey, but present a short selection of attractive (algebraic, relational, topological and categorical) features of the *modalized Heyting calculus* (Section 2). We establish some close connections between the modalized Heyting Calculus and some normal extensions of the minimal modal system K (Section 3).

The language of mHC consists of a set of propositional variables, connectives $\wedge$, $\vee$, $\rightarrow$, $\bot$ and a modal operator $\Box$ ("Always").

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DEFINITION 1. — *The* modalized Heyting calculus *mHC is obtained by adding*

(m1) $\Box(p \to q) \to (\Box p \to \Box q)$,

(m2) $p \to \Box p$ *and*

(m3) $\Box p \to (q \vee (q \to p))$

to the Heyting propositional Calculus HC as new *"modal" axiom schemes.*

Note that postulating the formula

(m4) $(\Box p \to p) \to p$

(an intuitionistic version of the Löb principle) as an additional axiom leads to the well-known *Proof-Intuitionistic logic* KM of Kuznetsov-Muravitsky ([KUZ 85, MUR 85]) in which the modality $\Box$, being the "ambassador" of the Box as a Proof of the Gödel-Löb modal system GL, expresses the provability predicate of the classical Peano arithmetic.

The algebraic models of mHC are Heyting algebras with a modal load subject to certain additional identities.

DEFINITION 2. — *A* frontal Heyting algebra *is an algebra* $(H : \wedge, \vee, \to, \bot, \tau)$ *(or,* $(H : \tau)$*, for short) such that* $(H : \wedge, \vee, \to, \bot)$ *is a Heyting algebra and* τ *is a unary operator (*modal operator*) satisfying the following conditions:*

(f1) $\tau(p \wedge q) = (\tau p \wedge \tau q)$,

(f2) $p \leqslant \tau p$,

(f3) $\tau p \leqslant q \vee (q \to p)$ *for every* $p, q \in H$.

We say that a frontal Heyting algebra $(H : \wedge, \vee, \to, \bot, \tau)$ *is a* KM-algebra *(or a* fronton*, for short) if the operator satisfies the additional condition*

(f4) $\tau p \to p \leqslant p$.

The equational class (and the category) of frontal Heyting algebras is denoted by fHA. Before proceeding further let us pause to present a sketch, mixing algebraic and modal viewpoints, of some properties of mHC, and to try to justify our favorite choice of this modal extension mHC of Heyting Calculus HC.

2. Motivations and justifications

Note that in almost all "standard" intuitionistic modal systems known to the author the postulate (m3) only seldom occurs (for notable examples see [GAB 77, GOL 81, SIM 82, TOU 87, TOU 90, WOL 99]; see as well the papers cited in [WOL 99] as a point of entry to the literature on intuitionistic versions of the classical modal systems). The postulate (m2) is not typical, while the postulate (m3) stresses even more "nonstandardness" of the chosen basic system mHC and of its extension KM, which enables one to draw a conventional "demarcation line" between mHC and the standard intuitionistic modal logics.

It seems to us that the modalized Heyting calculus mHC (and, to no less extent, KM) is interesting not only from the point of view of the provability interpretation, but also thanks to its connections to

– The Intuitionistic logic with propositional quantification;

– Topology: Cantor scattered spaces, notions of the limit and isolated point;

– Categorical logic: topoi, whose subobject classifiers (= truth value objects, carrying an internal Heyting algebra structure), as it turns out, are always equipped with a canonical operator τ of fHA-type;

– Intuitionistic temporal logic "Always & Before" possessing rich expressive possibilities.

One would hope that comments on these points (although sketchy) will enable us to support the claim about attractiveness and usefulness of such a "nonstandard" version of intuitionistic modal logic.

The Intuitionistic logic with propositional quantification (alias, the intuitionistic 2nd-order propositional logic)

qHC is obtained by adding to Heyting propositional calculus HC some form of quantification on the propositional variables. One way of introducing propositional quantification into an intuitionistic propositional logic is to specify the characteristic properties of quantification in the form of axioms and rules of inference and add them to the list of the axioms and rules of inference of the Heyting propositional calculus HC. The merits of propositional quantification can be given, for example, by the following *schemata of formulas*, $\forall p\ F(p) \to F(q)$ and $F(q) \to \exists p\ F(p)$, and *rules of inference*,

$$\frac{F(p) \to G}{\exists p\ F(p) \to G} \text{ and } \frac{G \to F(p)}{G \to \forall p\ F(p)}, \text{ where } p \text{ is not free in } G.$$

These basic axioms and rules governing the quantifiers, together with usual axiomatization of the Heyting calculus, give rise to the system qHC which can be regarded as the minimal system corresponding to the *intuitionistic logic with propositional quantification*. Assume the following definition:

$$\Box F(q) := \forall p(p \lor (p \to F(q)))$$

for every formula F of the Intuitionistic logic with propositional quantification qHC (p is not free in F). It is easy to verify that this operator satisfies all axioms (m1), (m2), (m3) of the modalized Heyting Calculus mHC.

For example, using $\vdash F(q) \to (p \lor (p \to F(q)))$ (p not free in F) we obtain

$$\vdash F(q) \to \forall p(p \lor (p \to F(q))),$$

i. e. $F(q) \to \Box F(q)$ (axiom (m1)) by the right rule of qHC. The axiom (m3), i. e. $\forall p(p \lor (p \to F(q))) \to (p \lor (p \to F(q)))$, is an instance of the left axiom of qHC.

Thus we can identify mHC with a certain *fragment* of the qHC and consequently consider the modality $\Box$ as an operator "intrinsically" definable in the Intuitionistic logic with propositional quantification.

Algebraic and topological models

Let X be a topological space. Recall that δA is, by definition, the set of all *accumulation* (alias, *limit*) points of a subset A of X. A point x is said to be a *limit point* of the set A, if every neighborhood of x contains a point of A other than x. The dual operator τ (co-derivative) is defined as follows: τA is the set of all *frontal points* of a subset A of the space X. A point x is said to be a frontal point of a set A, if there is a neighborhood U_x of x such that $U_x \subseteq A \cup \{x\}$. It is not hard to verify that our "favorite example", namely the Heyting algebra $\mathrm{H}(X)$ of all open sets of the topological space X with the co-derivative operator τ forms a frontal Heyting algebra. Indeed, to prove that $\tau B \subseteq A \cup (A \to B)$ for every $A, B \in \mathrm{H}(X)$, suppose that $x \notin A$ but $x \in \tau B$. Then by definition of frontal points there exists a neighborhood U of x such that $U \subseteq B \cup \{x\}$. Since $x \notin A$ we see that $B \cup \{x\} \subseteq B \cup (X - A)$ and $U \subseteq B \cup (X - A)$. Hence $x \in \mathrm{I}(B \cup (X - A)) = A \to B$, where I denotes interior.

A topological space is called *scattered* (Cantor) if it has no dense-in-itself nonempty subset. It is known that each ordinal α can be viewed as a scattered space $\Gamma(\alpha)$ of all ordinals not exceeding α, with its intrinsic, interval topology. The frontal Heyting algebra $\mathrm{H}(X)$ corresponding to $X = \Gamma(\alpha)$ as described above is a KM-algebra for any ordinal α [ESA 00]. In an arbitrary topos (i. e. in a category-theoretic universe for intuitionistic mathematics) the *subobject classifier* (= the object of truth values) Ω carrying an internal Heyting algebra structure, is moreover an internal frontal Heyting algebra in a canonical way, which enables one to interpret in a topos, along with ordinary intuitionistic connectives and quantifiers, also the *modal operator* τ [ESA 00]. Toposes whose subobject classifiers are frontons, the so called *scattered toposes*, are categorical models of a quantifier extension of the Proof-Intuitionistic logic KM [ESA 00].

PROPOSITION 3. — *A frontal Heyting algebra (H, τ) is a fronton if and only if every polynomial $t(x)$ in which the variable x occurs inside the scope of τ possesses a fixed point $p \in H$: $t(p) = p$.*

PROOF. — (*if*) Consider the polynomial $t(.) = \tau(.) \to p$, where $p \in H$. Since there is a fixed point q of t then $q = \tau(q) \to p$. Hence $q \leqslant \tau q \to p$ and $q \wedge \tau q \leqslant p$. Since $q \leqslant \tau q$, we have $q \leqslant p$, hence $\tau q \leqslant \tau p$ and $\tau p \to p \leqslant \tau q \to p$. Using $q = \tau(q) \to p$ we obtain $\tau p \to p \leqslant q$ and $\tau p \to p \leqslant p$.

(*only if*) Using Lemma 1.5 of [SAM 76] it is not hard to prove that in every fronton (H, τ) every polynomial $t(.)$ in which the variable x occurs inside the scope of τ possesses a fixed point. ■

It is not difficult to see that every Heyting algebra H can be "turned into" a frontal one: we can just equip H with the trivial operator τ putting $\tau p = p$ for all $p \in H$. The situation with frontons is however different!

For every element p of a Heyting algebra H consider the subset

$$\{q \in H : q \to p \leqslant q\} = \mathrm{F}_p$$

of the algebra H.

PROPOSITION 4. — *For every p the set F_p is a proper filter of the algebra H.*

PROOF. — 1) Let q be an element of F_p and $q \leqslant r$. Since $r \to p \leqslant q \to p$ and $q \to p \leqslant q$ one has $r \leqslant p \leqslant q$. Hence $r \to p \leqslant r$. 2) Supposing now that $q, r \in \mathrm{F}_p$ we show that $q \wedge r \in \mathrm{F}_p$. Since $q, r \in \mathrm{F}_p$ we have $q \to p \leqslant q$ and $r \to p \leqslant r$, hence $q \wedge (q \to p) = q \to p$ and $r \wedge (r \to p) = r \to p$, i. e. (a) $q \wedge p = q \to p$ and (b)$r \wedge p = r \to p$. Note that $q \wedge r \to p = q \to (r \to p)$: using (b) we have $q \to (r \to p) = q \to (r \wedge p) = (q \to r) \wedge (q \to p) \leqslant q \to p$.

Using (a) we obtain $(q \wedge r) \to p \leqslant q \wedge p \leqslant q$. In a similar way, $q \wedge r \to p = r \to (q \to p) = r \to q \wedge p = (r \to q) \wedge (r \to p) = (r \to q) \wedge (r \wedge q) \leqslant r \wedge q \leqslant r$. Thus $(r \wedge q) \to p \leqslant q$ and $(r \wedge q) \to p \leqslant r$, hence $(r \wedge q) \to p \leqslant r \wedge q$. Note that if $\bot \in \mathrm{F}_p$, i. e. $\bot \to p \leqslant \bot$, then $\top \leqslant \bot$. ■

An intrinsic (τ-less) characterization of frontons is given by

PROPOSITION 5. — *A Heyting algebra H admits a structure of a fronton if and only if the filters F_p are principal for all $p \in H$. Moreover such a structure is then unique.*

PROOF. — (*only if*) Let (H, τ) be a fronton. It is necessary to verify that every filter F_p ($p \in H$) is principal. Since $\tau p \to p \leqslant p$ and $p \leqslant \tau p$ one has $\tau p \to p \leqslant \tau p$, i. e. $\tau p \in \mathrm{F}_p$. Suppose that $s \in \mathrm{F}_p$, i. e. $s \to p \leqslant s$. Then $s \vee (s \to p) \leqslant s$; using axiom (m3) we obtain $\tau p \leqslant s \vee (s \to p)$, hence $\tau p \leqslant s$. Thus we see that $\mathrm{F}_p = [\tau p)$.

(*if*) Suppose now that every filter F_p ($p \in H$) is principal. Consider the operator $\tau : H \to H$ which assigns to every element $p \in H$ the element τp such that $\mathrm{F}_p = [\tau p)$. Note that for every $q \in H$ one has $q \vee (q \to p) \in \mathrm{F}_p$. Indeed we have $(q \vee (q \to p)) \to p = (q \to p) \wedge ((q \to p) \to p) = (q \to p) \wedge p = p$. Since $p \leqslant p \to q$, we have $p \leqslant q \vee (q \to p)$. Thus $(q \vee (q \to p)) \to p \leqslant q \vee (q \to p)$, i. e. $q \vee (q \to p) \in \mathrm{F}_p$ for every $q \in H$. Recall that if some $q \in \mathrm{F}_p$, i. e. $p \leqslant (q \to p)$, then $q \vee (q \to p) = q$. In particular $\tau p \leqslant \tau q \vee (\tau q \to p)$. Moreover $q \to p \leqslant q$, i. e. $(q \to p) \wedge p = q \to p$ implies

$(*)$ $p \wedge q = q \to p$.

From the remark made above it follows that $\tau p = \bigwedge\{q \vee (q \to p) : q \in H\}$. It can now be shown that the map τ satisfies the axioms (f1)–(f4). Indeed since $\tau p \in \mathrm{F}_p$, i. e. $\tau p \to p \leqslant \tau p$, using $(*)$ we have $\tau p \to p \leqslant p$ (axiom f4). Since $p \leqslant q \vee (q \to p)$ for every q, we obtain $p \leqslant \bigwedge\{q \vee (q \to p) : q \in H\} = \tau p$ (axiom f2). It is clear that $\tau p = \bigwedge\{q \vee (q \to p) : q \in H\} \leqslant q \vee (q \to p)$ (axiom f3). Definition of the operator τ implies directly the axiom f1. Thus the algebra (H, τ) is a fronton.

Finally to show uniqueness, suppose that (H, τ') is a fronton. Using the axiom f3 we have $\tau p \leqslant \tau' q \vee (\tau' q \to p)$; by (f4) one sees that $\tau' p \to p = p$, hence $\tau p \leqslant \tau' p \vee p$. From the axiom f2 we see that $\tau' p \vee p = p$, hence $\tau p \leqslant \tau' p$. In a similar manner we may obtain $t' p \leqslant \tau p$. Thus $\tau p = \tau' p$ for every $p \in H$. ■

All finite Heyting algebras, all Heyting algebras over well-founded Kripke frames, all Heyting algebras $\mathrm{H}(X)$ of open subsets of scattered spaces X (and consequently, in particular, of ordinal spaces!) are frontons [ESA 00]. However most Heyting algebras are not frontons! Note that although the Rieger-Nishimura lattice (= the free cyclic Heyting algebra) is a fronton (each filter F_p is principal), *no* other nontrivial free finitely generated Heyting algebra has this property. Despite that fact *every* intermediate logic is determined by its frontons. More precisely, an algebraic reformulation of the well-known result of Kuznetsov [KUZ 85], underlies

PROPOSITION 6 ([MUR 85]). — *Every variety of Heyting algebras is generated by its frontons.*

Relational semantics: transits

Let us call a Kripke frame $(W, <)$ a *transit*, if its reflexive closure $\leqslant$ (i. e. $x \leqslant y \iff x = y$ or $x < y$) is a partial order.

Thus any transit "automatically" gives rise to an ordinary intuitionistic Kripke frame $(W, \leqslant)$: in a definition of the forcing relation $\models$ let us single out two characteristic items:

$$x \models p \to q \qquad \text{iff } \forall y (x \leqslant y \;\&\; y \models p \;\Rightarrow\; y \models q);$$

$$x \models \Box p \qquad \text{iff } \forall y (x < y \;\Rightarrow\; y \models p).$$

The modalized Heyting calculus mHC is characterized by the class of transits.

Proof-intuitionistic logic KM requires in addition *conversely well-founded* transits (i. e. those satisfying ascending chain condition).

Let $(W, <)$ be a transit and let A be an arbitrary cone in it (i. e. $x \in A$ and $x < y$ implies $y \in A$); action of the modal operator $\Box$ on A can be described in the following way: $\Box A = A \cup \max(W - A)$, where the symbol "-´´ denotes the set-theoretic difference operator and $\max B$ denotes the set of all maximal points of a set B, i. e. $x \in \max B$ iff $\neg \exists y (y \in B$ and $x < y)$. Thus to each cone A the operator $\Box$ "builds", as an "architectural" fronton, the set of maximal points of its complement. Such "behavior" of the modal operator has inspired our "frontal" terminology.

Few words about the canonical (= *descriptive*) frames. Let (H, τ) be any frontal Heyting algebra, and let $(W, \subseteq)$ be the descriptive frame of the Heyting algebra H, i. e. W is the set of all prime filters of H ordered by the inclusion relation $\subseteq$. Using axioms of mHC it is not difficult to see that the additional relation $<$ induced on W by the modal operator τ, namely the relation

(†) $x < y \iff (\forall p \in H)(\tau p \in x \Rightarrow p \in y)$,

satisfies the following conditions:

1) If $x < y$, then $x \subseteq y$,

2) If $x \subset y$ (i. e. $x \subseteq y$ and $x \neq y$), then $x < y$ and

3) Reflexive closure of the relation $<$ coincides with the inclusion relation $\subseteq$.

(1) Indeed, suppose that $x \subset y$ and $p \in x$. Since $p \leqslant \tau p$, one has $\tau p \in x$ and then by definition $p \in y$. Thus $x \subseteq y$. (2) Suppose that $x \subset y$ (i. e. $x \subseteq y$ and $x \neq y$), then $q \in y$ and $q \notin x$ for some q. Suppose now that $x < y$ is not true, i. e. $\tau p \in x$ and $p \notin y$ for some p. Since $q \notin x$, we have $q \to p \notin x$, as otherwise we would have $q \to p \in y$ (as $x \subset y$). Since moreover $q \in y$, this would imply $p \in y$, which contradicts our assumptions. Thus we obtain $q \notin x$ and $q \to p \notin x$, hence $q \vee q \to p \notin x$; but $\Box p \in x$, and since $\Box p \leqslant q \vee q \to p$, we would have $q \vee q \to p \in x$, contradiction. Thus we have $x < y$.

Thus, in the canonical frames the "modal" accessibility relation $<$ is obtained from the inclusion relation $\subseteq$ by "removing the loops" from some points in W.

Temporal intuitionistic logic

As usual we will say that an operator $\Diamond$ on a Heyting lattice H is *adjoint* to an operator $\Box$ if for any elements $a, b \in H$ one has $\Diamond a \leqslant b$ iff $a \leqslant \Box b$.

Recall that existence of an adjoint implies its uniqueness. Let us adopt the following

DEFINITION 7. — *A Heyting algebra* $(H, \Box, \Diamond)$ *equipped with operators* $\Box$, $\Diamond$ *is called* temporal, *if* $(H, \Box)$ *is a frontal Heyting algebra and the operator* $\Diamond$ *is adjoint to* $\Box$.

In corresponding enriched calculi (let us mark them with letter t—tHC and tKM) the adjoint operators $\Box$ ("Always") and $\Diamond$ ("Before") have distinct "flavor" of temporal connectives. For example, in the "temporal reading" of the Kripke semantics $(W, <, \models)$ definition of the forcing relation $\models$ looks like this:

(1) $x \models \Box p$ iff $(\forall y)(x < y \Rightarrow y \models p)$, i. e. in future it will always hold p (Always(p));

for the adjoint operator one has:

(2) $x \models \Diamond p$ iff $(\exists y)(y < x \;\&\; y \models p)$, i. e. there already has been a precedent for p (Before(p)).

The *temporal Heyting Calculus* tHC is defined on the basis of mHC with additional axioms for the "adjoint" modality $\Diamond$; namely

t1) $p \to \Box\Diamond p$,

t2) $\Diamond\Box p \to p$,

t3) $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$,
t4) $\Diamond\bot \to \bot$

and an additional rule:

$$\frac{p \to q}{\Diamond p \to \Diamond q}.$$

REMARK. — Let $(H, \Box, \Diamond)$ be a temporal Heyting algebra and let $(W, \subseteq)$ be the descriptive frame of the Heyting algebra H. Let $<$ be the relation on W as above (see (†)) and define a *new* relation R on W using the adjoint operator $\Diamond$: xRy iff $(\forall p \in H)(p \in x \;\Rightarrow\; \Diamond p \in y)$. It is not difficult to see that xRy iff $x < y$. Indeed if xRy and $\Box p \in x$ then $\Diamond\Box p \in y$. Since $\Diamond\Box p \leqslant p$ we obtain $p \in y$, thus $x < y$. If $x < y$ and $p \in x$ then since $p \leqslant \Box\Diamond p$ we obtain $\Box\Diamond p \in x$. Using $x < y$ we have $\Diamond p \in y$. Thus xRy.

This temporal enrichment tHC of the system mHC is a certain specimen of temporal intuitionistic logics. In connection with computer science applications of classical temporal logics with operators alluding to the "past´´, let us mention the work [LAR 95], where richness of expressive possibilities of such logical systems is stressed. As for us here we will restrict ourselves to some remarks. In the calculi tHC and tKM one can express some useful properties both of the points themselves (*stages*) of the Kripke semantics and their global properties. For example, the following rule is a propositional version of the first order principle of *descent induction* ([KLE 52]): $\forall x\,[P(x) \;\Rightarrow\; \exists y(y < x \;\&\; P(y))] \Rightarrow \neg P(x)$.

A point $x \in W$ of a descriptive Kripke model $(W, <, \models)$ is called p-*critical* if on the stage x the formula p is *not* forced, but $(\forall y)(x < y \;\Rightarrow\; y \models p)$; we will say that the point x (= prime filter or, if wished, prime intuitionistic theory) is *critical,* if it is p-critical for some propositional formula p.

An "adjoint" property is the *creativity* property. We will say that a point $x \in W$ is p-creative if $x \models p$ but the formula p has not been forced on any earlier stage y (i. e. for $y < x$). Thus the point x is creative if on the stage x at least one new fact has been established. In terms of the adjoint modality $\Diamond$ this can be expressed as follows: the point x is creative if on stage x certain formula is not just established (i. e. $x \models p$) but also its *precedentlessness* is established too (i. e. it is not the case that $x \models \Diamond p$). In general not all points of the model are critical or creative. For example creativity of the descriptive ordered model implies its well-foundedness. Nevertheless there is always "a sufficient amount" of critical and creative points in descriptive models W of the calculus tKM: the set W_0 of all critical points of the model W is *topologically dense* in W or, in order-theoretic terms, the set of critical points of any clopen (= formula-induced) set is *cofinal* in it. □

3. Exact embedding of the mHC-calculus into the modal system K4.Grz: an algebraic consideration

In his famous 1932 short note Gödel described an interpretation of the Heyting Calculus HC in the Lewis's modal system S4. Gödel's result thus takes the following form:

(1) if HC $\vdash p$, then S4 $\vdash \mathrm{g}(p)$,

where the modal formula $\mathrm{g}(p)$ is formed from the intuitionistic formula p according to Gödel's translation rules. In addition, Gödel conjectured that the converse of (1) also holds, that is

(2) HC $\vdash p$ iff S4 $\vdash \mathrm{g}(p)$.

This conjecture was later verified by McKinsey and Tarski [MCK 44].

Solovay [SOL 76] characterized a modal system GL corresponding to formal provability in PA. The Gödel-Löb modal system GL (alias, the Provability logic) adequately reflects behavior of the formalized Provability Predicate in Peano Arithmetic PA. GL is the result of adding the Löb axiom $\Box(\Box p \rightarrow p) \rightarrow \Box p$ to K4. Solovay defines an *arithmetical realization* of modal formulas of the system GL and proves its arithmetical completeness. Using more technical terminology, we say that an arithmetical realization of modal formulas is an assignment * to each atom p of an arithmetic sentence p^* which commutes with non-modal connectives and $(\Box p)^* = \mathrm{Pr}(\ulcorner p^* \urcorner)$, where $\mathrm{Pr}(.)$ is the standard provability predicate for the Peano Arithmetic PA and $\ulcorner p^* \urcorner$ is the code numeral of p^*.

Arithmetical completeness of GL [SOL 76]: GL $\vdash p$ iff under all arithmetical realizations * the sentence p^* is provable in PA.

Grzegorczyk [GRZ 67] axiomatically defined a modal system S4.Grz, which is a proper normal extension of the system S4 and proved that HC could be embedded (via the Gödel translation g) in the system S4.Grz, i. e.

(3) HC $\vdash p$ iff S4.Grz $\vdash \mathrm{g}(p)$.

S4.Grz is the system that results when the schema

(Grz) $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p$

is added to the modal system S4.

It is appropriate to mention here that the system S4.Grz is the largest modal system in which HC can be embedded by the Gödel embedding g ([ESA 79]); moreover, the lattice Lat(HC) of all intermediate logics is isomorphic to the lattice Lat(S4.Grz) of all normal extensions of the system S4.Grz (Blok-Esakia, 1976).

Define a transformation s (= *splitting map*) of the set of modal formulas into itself stipulating that s commutes with Boolean connectives and $\mathrm{s}(\Box p) = \mathrm{s}(p) \wedge \Box\, \mathrm{s}(p)$.

It is appropriate to mention here a well-known fact which was obtained independently by Boolos, Goldblatt and Kuznetsov:

S4.Grz $\vdash p$ if and only if K4.Grz $\vdash \mathrm{s}(p)$.

We shall now define a proper normal extension K4.Grz of the system K4 and observe that S4.Grz could be embedded (via the splitting map s) in the modal system K4.Grz. K4.Grz is the system that results when the schema

(Grz) $\Box(\Box(p \to \Box p) \to p) \to \Box p$

is added to the modal system K4.

K4.Grz axiomatizes those properties of $\mathrm{Pr}(.)$ that do not depend on the Gödel's Diagonal Lemma. It is not hard to verify that GL is a *proper* extension of K4.Grz. Indeed for example a Boolean algebra with the identity operator as $\Box$ is a model for K4.Grz but not for GL. Note also that the same example gives a model for mHC but not for KM.

The following proposition presents our key observation.

PROPOSITION 8. — *S4.Grz* $\vdash p$ *iff K4.Grz* $\vdash \mathrm{s}(p)$.

Moreover, the system K4.Grz is the *least normal extension* of K4 for which this proposition is true.

This observation was inspired by an intimate connection existing between K4.Grz- and S4.Grz-algebras (see below, Main Lemma).

Before proceeding further let us focus attention on certain provability interpretation of the modal system K4.Grz. We assume that the reader is familiar with the conception of the Provability as a modality, i. e. as a modal operator $\Box$ acting on propositional formulas. Suppose we modify the notion of arithmetical realization by amending a recursive clause for the box $\Box$, namely: $(\Box p)^* = (A \to p^*) \wedge \mathrm{Pr}(\ulcorner p^* \urcorner)$, where the parameter A is a given sentence in the language of Peano arithmetic PA.

With algebraic nomenclature at hand, this notion of "reincarnation" is easily translatable into the language of GL-algebras. Let $(B, \wedge, \vee, \to, \bot, \Box)$ be an arbitrary GL-algebra (for example, the Lindenbaum Sentence algebra for PA) and $e \in B$; we define a new (polynomially definable) modal operator $[e]$ on the Boolean algebra B by $[e]p := (e \to p) \wedge \Box p$ for every $p \in B$. (The notion of *polynomial* used here is simply that from universal algebra: polynomials are functions arising from constant functions and the identity function by means of the Boolean operations and $\Box$). We note some observations regarding this reincarnation. Denote by $(B, [e])$ the Boolean algebra B endowed with the operator $[e]$ and note that the modal algebra $(B, [e])$ is a K4-algebra, satisfying additional condition $[e]([e](p \to [e]p) \to p) \leqslant [e]p$, i. e. the algebra $(B, [e])$ *is a K4.Grz-algebra*. We note some particular cases which illustrate the general picture:

1. If $e = \bot$ then the modal operator $[e]$ coincides with $\Box$;

2. If $e = \neg\bot$ then $[e]p$ represents the "demonstrability" predicate $\mathrm{Dem}(\ulcorner p^* \urcorner) = p^* \wedge \mathrm{Pr}(\ulcorner p* \urcorner)$ and the algebra $(B, [e])$ is a S4.Grz-algebra;

3. If $e \neq \bot$ and $e \leqslant \neg\Box\bot$ then the modal version $\neg\Box\bot \to \neg\Box\neg\Box\bot$ of the Gödel's Second incompleteness theorem is still *valid* in the algebra $(B, [e])$ while the Löb axiom is *refutable*.

4. If $e \neq \bot$ and $e \leqslant \Box\bot$, then a modal version of the Gödel's Second incompleteness theorem is *refutable* in the algebra $(B, [e])$.

We recall the relevant definition of some notions concerning algebraic semantics of certain *classical* modal systems.

In *Appendix I. Derivative algebra* of the paper [MCK 44], McKinsey and Tarski initiated an investigation of the fundamental topological operation of derivation from a purely algebraic (and/or modal) point of view. On p.182 of [MCK 44] the authors say: *"Like the topological operation of closure, other topological operations can be treated in an algebraic way. This may be especially interesting in regard to those operations which are not definable in terms of closure... An especially important notion is that of the derivative of a point set A which will be denoted by δA".*

Thus, Derivative algebras $(B : \wedge, \vee, \neg, \delta)$ are Boolean algebras with an unary operation δ, which captures algebraic properties of the topological derivation. Recall that δA is, by definition, the set of all *accumulation* (alias, *limit*) points of a subset A of a topological space X, where a point x is said to be a limit point of a set A, if every neighborhood of x contains a point of A other than x.

DEFINITION 9. — *We say that a Boolean algebra B is a* Derivative algebra *with respect to the operation δ, if*

1) $\delta\bot = \bot$,

2) $\delta(a \vee b) = \delta a \vee \delta b$,

3) $\delta\delta a \leqslant a \vee \delta a$.

REMARK. — It must be pointed out that we *weaken* the definition of Derivative algebra [MCK 44] slightly; namely, we postulate the condition (3) instead of $\delta\delta a \leqslant \delta a$. We justify this weakening by noting that there are topological spaces, in which the condition $\delta\delta a \leqslant \delta a$ is *not valid* (for example, spaces with anti-discrete topology). □

With the operator δ is associated a dual operator τ (co-derivative) defined by $\tau a := \neg\delta\neg a$, i. e. τA is the set of all frontal points of a subset A of a topological space X. Using the usual intuitively obvious relations between closure and derivative operations in topological spaces the *closure* of a set can be defined in terms of the *derivative*, namely, $\mathrm{c}\, A = A \cup \delta A$. If we introduce a corresponding definition into derivative algebra (namely, $\mathrm{C}\, a := a \vee \delta a$), we can easily show that the derivative algebra (B, δ) becomes a closure algebra (B, C) with respect to the operation C just defined. Note that the *interior* operator I can be defined as follows: $\mathrm{I}\, a := a \wedge \tau a$.

We will use whichever of δ (resp., C) and τ (resp., I) is rhetorically the most convenient. As an immediate consequence of the definition 9 we have a corollary.

COROLLARY 10. — *In any Derivative algebra (B, δ) the operator* C *satisfies the well-known Kuratowski axioms:*

1) $a \leqslant \mathrm{C}\, a$*,*

2) $\mathrm{C}\, a = \mathrm{C}\,\mathrm{C}\, a$*,*

3) $\mathrm{C}(a \vee b) = \mathrm{C}\, a \vee \mathrm{C}\, b$*,*

4) $\mathrm{C}\,\bot = \bot$.

We recall that an element $a \in B$ is called *open* if $a \leqslant \tau a$ (i. e. $\mathrm{I}\, a = a$) and *closed* if $\delta a \leqslant a$ (i. e. $\mathrm{C}\, a = a$). The following simple Lemma will be useful below.

LEMMA 11. — *In any Derivative algebra* (B, δ) *one has* $\tau a \leqslant b \vee \mathrm{I}(\neg b \vee a)$.

PROOF. — Using monotonicity of τ, $a \leqslant \neg b \vee a$ implies $\tau a \leqslant \tau(\neg b \vee a) \leqslant b \vee \tau(\neg b \vee a) = (b \vee \neg b \vee a) \wedge (b \vee \tau(\neg b \vee a)) = b \vee [(\neg b \vee a) \wedge \tau(\neg b \vee a)] = b \vee \mathrm{I}(\neg b \vee a)$. Thus we have $\tau a \leqslant b \vee \mathrm{I}(\neg b \vee a)$. ■

Derivative algebras are algebraic models of a slightly weakened version wK4 of the modal system K4; namely, wK4 = K + $p \wedge \Box p \;\rightarrow\; \Box\Box p$, where the system K (named after Kripke) is the minimal normal modal logic whose axioms are all Boolean tautologies and all expressions of the form $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ and whose rules are modus ponens and necessitation. The diamond $\Diamond$ as usual means the dual $\neg\Box\neg$ of $\Box$.

Recall that a relational semantics for the system K is based on the notion of a Kripke frame, that is, a pair (X, R) where X is a nonempty set ("of possible worlds") and R is a binary relation on X ("accessibility relation"). A valuation is a function f assigning to each propositional letter p a subset $f(p)$ of X ("the set of worlds in which p is true"). The valuation is then extended to all formulas via the obvious definitions for Boolean connectives, together with $x \in f(\Diamond p)$ iff $\exists y \in X$ such that xRy and $y \in f(p)$. A formula p is valid in (X, R) iff $f(p) = X$. For detailed exposition of Modal Logic we refer the reader to the comprehensive textbook [CHA 97] or to any other source on Modal Logic. Relational semantics for the system wK4 is based on the notion of Kripke frame with a *weakly transitive* accessibility relation. This terminology was inspired by the following "historic" discussion: "*This is continuation of the discussion initiated in the papers XXIV 185(1,2). In spite of disagreements on the way, the polemic ends with all parties agreeing that notion of weak-transitivity of a relation* R*, characterized by* $x \neq y \;\&\; xRy \;\&\; yRz \;\Rightarrow\; xRz$ *must be distinguished from that of strong transitivity, characterized by* $xRy \;\&\; yRz \;\Rightarrow\; xRz$" (Church, [CHU 60]).

The reason for our favoring of the system wK4 and weak-transitivity as follows.

PROPOSITION 12 ([ESA 01]). —

(a) Relational completeness of wK4: wK4 $\vdash p$ *iff* p *is valid in* every *weakly transitive Kripke frame;*

(b) Topological completeness of wK4: wK4 $\vdash p$ *iff* p *is valid* in every *topological space; in other words, wK4 is the Logic of topological spaces (under reading the diamond-modality* $\Diamond$ *as the* derivative *operation* δ*).*

For the system K4 we need to impose some *restriction* on topological spaces. Recall that X is said to be a $\mathrm{T_d}$-*space* if every singleton subset of X is an intersection of an open and a closed subset. This separation axiom, introduced by Aull and Thron [AUL 62], proved to play important role in the context of lattice-equivalence of topological spaces.

Recall that the system K4 is obtained by adding $\Box p \to \Box\Box p$ to K as a new axiom schema.

PROPOSITION 13 ([ESA 04]). — *Topological completeness of K4: K4* $\vdash p$ *iff* p *is valid in* every $\mathrm{T_d}$-space.

Recall that a topological space X is called *irresolvable* (Hewitt) if each pair of its dense subsets has nonempty intersection. A space is said to be *hereditary irresolvable* (HI-space, for short) if each subspace of X is irresolvable. Various aspects of HI-spaces have been investigated in [BEZ 03]. It is appropriate to mention here the following related interesting result which may be found in [GAB 05]:

Topological completeness of K4.Grz: for every formula p, K4.Grz $\vdash p$ iff p is valid in every HI-space; in other words, K4.Grz is the Modal Logic of topological HI-spaces (under reading the diamond modality $\Diamond$ as the derivative operation δ).

In what follows we need the following simple, but useful, observation.

Main Lemma

A K4-algebra (B, δ) *satisfies the equation*

(a) $\delta a = (\delta a - \delta(\delta a - a))$, *i. e.* $(B, \delta) \in$ *K4.Grz*

iff the corresponding S4-algebra (B, C) *satisfies the equation*

(b) $\mathrm{C}\, a = (\mathrm{C}\, a - \mathrm{C}(\mathrm{C}\, a - a))$, *i. e.* $(B, C) \in$ *S4.Grz*

(where, as above, $\mathrm{C}\, a := a \vee \delta a$*).*

For the sake of completeness we present a proof.

PROOF. — Taking into account monotonicity of the operators δ and C it is only necessary to verify that the following conditions are equivalent:

(c) $\delta a \leqslant \delta(a - \delta(\delta a - a))$

and

(d) $\mathrm{C}\, a \leqslant \mathrm{C}(a - \mathrm{C}(\mathrm{C}\, a - a))$.

($\Leftarrow$) We eliminate step by step the closure operator C in the condition (d).

1. $\mathrm{C}\, a - a = \neg a \wedge (a \vee \delta a) = \neg a \wedge \delta a$.

2. $a - \mathrm{C}(\mathrm{C}\, a - a) = a - ((\delta a - a) \vee \delta(\delta a - a))$
$= a - (\delta a - a) - \delta(\delta a - a)) = a - \delta(\delta a - a) \wedge (\neg\delta a \vee a) = a - \delta(\delta a - a)$.

Thus the condition (d) is equivalent to the condition

(d*) $a \vee \delta a \leqslant (a - \delta(\delta a - a)) \vee \delta(a - \delta(\delta a - a))$.

Applying monotonicity and additivity of the derivative operator δ we obtain $\delta a \vee \delta\delta a \leqslant \delta(a - \delta(\delta a - a)) \vee \delta\delta(a - \delta(\delta a - a))$. From the K4-axiom $\delta\delta a \leqslant \delta a$, we see that $\delta a \leqslant \delta(a - \delta(\delta a - a))$.

($\Rightarrow$) We notice that $\delta a - a \leqslant \delta a$ implies $\delta(\delta a - a) \leqslant \delta\delta a \leqslant \delta a$. Thus we have $\delta(\delta a - a) \leqslant \delta a$ and $\neg\delta a \leqslant \neg\delta(\delta a - a)$. Multiplying both sides through by a we obtain

(e) $a - \delta a \leqslant a - \delta(\delta a - a)$.

The formula (e) together with (c) implies $\delta a \vee (a - \delta a) \leqslant (a - \delta(\delta a - a)) \vee \delta(a - \delta(\delta a - a))$. Using the equation $\delta a \vee (a - \delta a) = a \vee \delta a$, we have $a \vee \delta a \leqslant (a - \delta(\delta a - a)) \vee \delta(a - \delta(\delta a - a))$, i. e. the condition (d*) which is equivalent to (d). ∎

Let us return to the variety fHA.

Let $(B : \wedge, \vee, \neg, \delta)$ be an arbitrary Derivative algebra and $H = \{a \in B : a \leqslant \tau a\}$.

It is easy to show that $(H : \wedge, \vee, \rightarrow, \bot)$ is a Heyting algebra, where $(H : \wedge, \vee)$ is a sublattice of the Boolean lattice $(B : \wedge, \vee)$, and $p \rightarrow q = \mathrm{I}(\neg p \vee q)$ for $p, q \in H$.

Notice that if $p \in H$ then $\tau p \in H$. Indeed, suppose $p \in H$, i. e. $p \leqslant \tau p$; by monotonicity of τ we have $\tau p \leqslant \tau(\tau p)$, i. e. $\tau p \in H$.

THEOREM 14. — *The above algebra* $(H : \wedge, \vee, \rightarrow, \bot, \tau)$ *is a frontal Heyting algebra.*

PROOF. — To see that (H, τ) is a frontal Heyting algebra it is only necessary to verify that the axiom (3) $\tau p \leqslant q \vee (q \rightarrow p)$ is satisfied. By the Lemma 11 we see that this axiom is simply an instance of $\tau a \leqslant b \vee \mathrm{I}(\neg b \vee a)$ for $a, b \in H$. ∎

Thus we associate with every Derivative algebra (B, δ) a frontal Heyting algebra (H, τ) of all open elements of (B, δ). We call the algebra (H, τ) the *Heyting core* (*H-core*, for short) of (B, δ). This assignment of the Heyting core (H, τ) to each Derivative algebra (B, δ) can be expanded to yield a *functor* F from the category of derivative algebras DA to the category of frontal Heyting algebras fHA. Indeed, it is easy to see that restriction of a homomorphism of Derivative algebras to H-cores is a frontal homomorphism.

DEFINITION 15. — *Let* (B, δ) *be a Derivative algebra and let* (H, τ) *be its H-core. We say that* (B, δ) *is a* stencil *Derivative algebra (or simply, a* stencil*) if the Boolean part* B *of* (B, δ) *is generated (as a Boolean algebra) by the subset* H*, i. e. every element of* B *is a finite Boolean combination of elements of* H.

In the following theorem we show that every frontal Heyting algebra can (and henceforth will) be identified with the Heyting core of a suitable Derivative algebra.

This theorem is a modest generalization of a related result of McKinsey and Tarski concerning closure algebras [MCK 46].

We obtain a representation for arbitrary frontal Heyting algebra; this representation is functorial and is extended to a full duality.

THEOREM 16. — *Let (H, τ) be a frontal Heyting algebra. There exists a Derivative algebra (B, τ^*) such that (H, τ) is (isomorphic to) the Heyting core of (B, τ^*) and $\tau^* p = \tau p$ for $p \in H$.*

PROOF. — First of all we note that for every Heyting algebra H there exists a map $\tau : H \to H$ such that the algebra (H, τ) is a frontal Heyting algebra. Thus if we set $\tau p = p$ for all $p \in H$ then we see that our "modal packing" is conservative over the variety of Heyting algebras. We know that for every Heyting algebra there exists an Interior algebra $(\mathrm{B}(H), \mathrm{I})$ containing (an isomorphic copy of) H as the sublattice of all its open elements, and generated as a Boolean algebra by the set H. Such algebras $(\mathrm{B}(H), \mathrm{I})$ are called *stencil algebras* [ESA 79]. Note that every element a of the stencil algebra $(\mathrm{B}(H), \mathrm{I})$ can be represented in the form $a = \bigwedge_i (\neg p_i \vee q_i)$ for suitable $p_i, q_i \in H$ and besides $\mathrm{I}\, a = \bigwedge_i (p_i \to q_i)$. Furthermore [ESA 85] the algebra $(\mathrm{B}(H), \mathrm{I})$ is a K4.Grz-algebra, i. e. the interior operator I satisfies the additional equation: $\mathrm{I}(\neg \mathrm{I}(\neg a \vee \mathrm{I}\, a) \vee \mathrm{I}\, a) = \mathrm{I}\, a$ (cf. the "dual equation" $\mathrm{C}\, a = (\mathrm{C}\, a - \mathrm{C}(\mathrm{C}\, a - a))$ of the main lemma).

Now we define an operator τ^* on the algebra $\mathrm{B}(H)$ in terms of operators I and τ by means of the equality $\tau^* a = \tau\, \mathrm{I}\, a$ (for $a \in \mathrm{B}(H)$). It is not hard to see that the Boolean algebra becomes a Derivative algebra with respect to the operator τ^* just defined. It is clear from the definition of τ^* that $\tau^* p = \tau p$ for every $p \in H$. ■

LEMMA 17. — *In the stencil algebra $(\mathrm{B}(H), \tau^*)$ the following relation between the interior operator I and the derivative τ^* holds:*

$$\mathrm{I}\, a = a \wedge \tau^* a.$$

PROOF. — Using the definition $\tau^* a = \tau\, \mathrm{I}\, a$ and the axiom (2) $p \leqslant \tau p$ of frontal Heyting algebras we obtain $\mathrm{I}\, a \leqslant \tau\, \mathrm{I}\, a$, consequently $\mathrm{I}\, a \leqslant a \wedge \tau\, \mathrm{I}\, a$, i. e. $\mathrm{I}\, a \leqslant a \wedge \tau^* a$. Thus it only remains to verify that $a \wedge \tau^* a \leqslant \mathrm{I}\, a$, i. e. $a \wedge \tau\, \mathrm{I}\, a \leqslant \mathrm{I}\, a$. Substituting $\mathrm{I}\, a$ for p and $\mathrm{I}(\neg a \vee \mathrm{I}\, a)$ for q in the axiom (3) $\tau p \leqslant q \vee (q \to p)$ yields $\tau\, \mathrm{I}\, a \leqslant \mathrm{I}(\neg a \vee \mathrm{I}\, a) \vee \mathrm{I}(\neg \mathrm{I}(\neg a \vee \mathrm{I}\, a) \vee \mathrm{I}\, a)$. Using the Grz-axiom $\mathrm{I}(\neg \mathrm{I}(\neg a \vee \mathrm{I}\, a) \vee \mathrm{I}\, a) = \mathrm{I}\, a$ we obtain $\tau\, \mathrm{I}\, a \leqslant \mathrm{I}(\neg a \vee \mathrm{I}\, a) \vee \mathrm{I}\, a$. Since $\mathrm{I}(\neg a \vee \mathrm{I}\, a) \leqslant \neg a \vee \mathrm{I}\, a$ we have $\tau\, \mathrm{I}\, a \leqslant \neg a \vee \mathrm{I}\, a$. Multiplying both sides through by a we obtain $a \wedge \tau\, \mathrm{I}\, a \leqslant a \wedge (\neg a \vee \mathrm{I}\, a) = \mathrm{I}\, a$. Thus $a \wedge \tau\, \mathrm{I}\, a \leqslant \mathrm{I}\, a$. Moreover it is clear that for every $a \in B$, $a \leqslant \tau^* a$ iff $a \in H$. ■

THEOREM 18. — *The stencil derivative algebra $(\mathrm{B}(H), \tau^*)$ is a K4.Grz-algebra, that is, the derivative algebras of the form $(\mathrm{B}(H), \tau^*)$ satisfy the conditions*

(a) $\tau^* a \leqslant \tau^* \tau^* a$

and

(b) $\delta^* a = \delta^* a - \delta^*(\delta^* a - a)$.

PROOF. — To see that (a) $\tau^* a \leqslant \tau^*\tau^* a$ holds for all $a \in \mathrm{B}(H)$ we notice that $\tau : H \to H$, i. e. $\tau\,\mathrm{I}\,a \leqslant \mathrm{I}(\tau\,\mathrm{I}\,a)$. Using the axiom (m2) of the definition 1 we obtain $\mathrm{I}\,\tau\,\mathrm{I}\,a \leqslant \tau(\mathrm{I}\,\tau\,\mathrm{I}\,a)$, hence $\tau\,\mathrm{I}\,a \leqslant \tau\,\mathrm{I}\,\tau\,\mathrm{I}\,a$, i. e. $\tau^* a \leqslant \tau^*\tau^* a$. (b) Recall that the Interior algebra $(\mathrm{B}(H), \mathrm{I})$ is a S4.Grz-algebra. Since our Derivative algebra is also a K4-algebra and $\mathrm{I}\,a = a \wedge \tau^* a$ (Lemma 17), the Main Lemma applies. ■

Thus we associate with every frontal Heyting algebra (H, τ) a Derivative algebra $(\mathrm{B}(H), \tau^*)$. The following lemma shows that this assignment of the "Boolean embrace" $\mathrm{B}(H)$ to every Heyting algebra H can be expanded to yield a functor G from the category of frontal Heyting algebras fHA to the category of Derivative algebras DA.

LEMMA 19. — *Let (H_1, τ_1), (H_2, τ_2) be frontal Heyting algebras and $h : H_1 \to H_2$ a frontal homomorphism (i. e. a Heyting algebra homomorphism commuting with "modal loads" τ_1, τ_2. There exists a unique extension $h^+ : \mathrm{B}(H_1) \to \mathrm{B}(H_2)$ of h to a homomorphism of Derivative algebras from $(\mathrm{B}(H_1), \tau_1^*)$ to $(\mathrm{B}(H_2), \tau_2^*)$.*

PROOF. — We know ([BLO 75], see also [ESA 85]) that there exists a unique extension h^+ of h to a homomorphism of the corresponding Interior algebras, from $(\mathrm{B}(H_1), \mathrm{I}_1)$ to $(\mathrm{B}(H_2), \mathrm{I}_2)$. It follows from the definition of the operator τ^* and the fact that h is a frontal homomorphism, that $h^+\tau_1^* a = \tau_2^* h^+ a$ for every $a \in \mathrm{B}(H_1)$. ■

Now note that for a frontal algebra (H, τ) the corresponding stencil algebra $G(H, \tau) = (\mathrm{B}(H), \tau^*)$ has a universal property of being the "best possible" Derivative algebra obtainable from (H, τ), in the following sense: there is an embedding $H \hookrightarrow \mathrm{B}(H)$ identifying (H, τ) with the Heyting core of $(\mathrm{B}(H), \tau^*)$ such that for any other Derivative algebra (B', τ') and any fHA homomorphism $f : (H, \tau) \to F(B', \tau')$ to the Heyting core of (B', τ') there is a unique extension $f^* : (\mathrm{B}(H), \tau^*) \to (B', \tau')$ of f to a Derivative algebra homomorphism.

In category-theoretic terms this means that the functors F and G constructed above form an adjoint pair, with G left adjoint to F. Moreover the fact that the fHA (H, τ) is isomorphic to the Heyting core of $G(H, \tau) = (\mathrm{B}(H), \tau^*)$ means in this language that the adjunction structure unit natural transformation $(H, \tau) \to FG(H, \tau)$ is an isomorphism. It is a well known fact in abstract category theory that this happens if and only if the functor G is a full embedding and in this case the adjoint pair $G \dashv F$ restricts to an equivalence between fHA and the full image of F, i. e. the full subcategory of DA consisting of stencil algebras.

COROLLARY 20. — *The equational category of frontal Heyting algebras (resp., frontons) is equivalent to the category of stencil Derivative algebras (resp. stencil GL-algebras).*

Using the composite of the Gödel translation g and of the splitting map s we inductively define a translation $\#$ from formulas of the modalized Heyting Calculus mHC to formulas of the modal system K4.Grz setting $\#(p) = p \wedge \Box p$ if p is a propositional variable; $\#$ commutes with $\wedge$, $\vee$, $\bot$, $\Box$ and

$$\#(p \to q) = (\#p \to \#q) \wedge \Box(\#p \to \#q).$$

With the above algebraic considerations in mind it is not hard to see validity of the following

COROLLARY 21. — *mHC* $\vdash p$ *iff K4.Grz* $\vdash \# p$.

It is easy to see that as a by-product of this corollary we obtain an exact embedding of the Heyting Calculus HC into the modal system K4.Grz.

Finally let us note that this corollary can be further strengthened: the lattice Lat(mHC) of all extensions of mHC is isomorphic to the lattice Lat(K4.Grz) of all normal extensions of the modal system K4.Grz. However, a proof of this result requires additional considerations as the above algebraic machinery does not suffice for it.

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