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RELATION BETWEEN THE CONTINUITY OF A GRADIENT OF A FUNCTION AND THE FINITENESS OF ITS STRONG GRADIENT

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Abstract. There exists an absolutely continuous function of two variables who has almost everywhere a finite strong gradient and almost everywhere a discontinuous gradient.

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Introduction

1. For a function of one variable we can indicate two properties which are almost everywhere equivalent, although one property is stronger than the other at an individual point.

To such properties, for example, belong:

(1) continuity and symmetric continuity ([1], p. 266);

(2) differentiability and symmetric differentiability ([1], p. 249; [2], p. 381);

(3) differentiability and existence of a finite upper derivate ([3], pp. 270 and 108).

Next, the continuity of a function of two variables and its continuity with respect to each of variables may almost everywhere be nonequivalent properties ([4], p. 432).

For a function of several variables there is the notion of the strong gradient whose finiteness at an individual point is weaker, in general, than the property of continuity of its gradient at the same point ([5]). In the present article we prove that the above-mentioned nonequivalence is realizable almost everywhere.

2. Let in the neighborhood of the point $x^0 = (x_1^0, \dots, x_n^0)$ be defined the function $f(x)$, $x = (x_1, \dots, x_n)$.

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Definition ([5]). The strong partial derivative of the function f at the point x^0 with respect to the variable x_k, denoted by $f'_{[x_k]}(x^0)$, is called the following limit, finite or of fixed sign infinite,

$$f'_{[x_k]}(x^0) = \lim_{x \to x^0} \frac{f(x) - f(x(x_k^0))}{x_k - x_k^0},$$

where $x(x_k^0) = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n)$.

If there exists $f'_{[x_k]}(x^0)$ for all $k = 1, \dots, n$, then a strong gradient of the function f at the point x^0, symbolically $\operatorname{strgrad} f(x^0)$, is defined by the equality

$$\operatorname{strgrad} f(x^0) = \big(f'_{[x_1]}(x^0), \dots, f'_{x_n}(x^0)\big),$$

under the finiteness of which is meant the finiteness of all its components.

The following facts take place ([5]):

(A) the finiteness of the $\operatorname{strgrad} f(x^0)$ implies the differentiability of the function f at the point x^0, and not vice versa;

(B) the continuity of the $\operatorname{grad} f(x)$ at the point x^0 implies the finiteness of the $\operatorname{strgrad} f(x^0)$, and not vice versa.

Further, the situation described in (A) is realizable almost everywhere ([6]). More exactly, for every $n \geq 2$ there exists the continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ which is differentiable almost everywhere, but no has finite $\operatorname{strgrad} \phi(x)$ almost everywhere.

Result

Theorem. *There exists on a unit square $Q = [0,1]^2$ an absolutly continuous function of two variables $f(x,y)$ such that almost everywhere on Q:*

(1) *the* $\operatorname{strgrad} f(x,y)$ *is finite;*

(2) *the* $\operatorname{grad} f(x,y)$ *is discontinuous.*

Proof. For the bounded and everywhere on $[0,1]$ discontinuous functions $\alpha(x)$ and $\beta(y)$, we consider the corresponding indefinite L-integrals

$$A(x) = \int_0^x \alpha(t)\, dt, \quad B(y) = \int_0^y \beta(\tau)\, d\tau.$$

The function of two variables $f(x,y) = A(x) + B(y)$ is absolutely continuous on Q ([7], pp. 246–247) and almost at all points $(x,y) \in Q$ has the total differential

$$df(x,y) = \alpha(x)\, dx + \beta(y)\, dy.$$

Next, we can consider the function of one variable as the function of two variables which is constant with respect to the second variable. Therefore the strong partial derivative of that new function with respect to the first variable coincides with the derivative of the initial function of one variable.

Consequently, at the points of existence of the tatal differential $df(x, y)$ the strgrad $f(x, y) = (\alpha(x), \beta(y))$ is finite.

On the other hand, the grad $f(x, y) = (\alpha(x), \beta(y))$ is discontinuous almost everywhere.

Hence the strgrad $f(x, y)$ is finite almost everywhere on Q and the function grad $f(x, y)$ is discontinuous almost everywhere on Q. □

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