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Asymmetric Factorizations of Matrix Functions on the Real Line

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Dedicated to I.B. Simonenko on the occasion of his 70th birthday

Abstract. We indicate a criterion for some classes of continuous matrix functions on the real line with a jump at infinity to admit both, a classical right and an asymmetric factorization. It yields the existence of generalized inverses of matrix Wiener-Hopf plus Hankel operators and provides precise information about the asymptotic behavior of the factors at infinity and of the solutions to the corresponding equations at the origin.

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1. Introduction

In 1968, I.B. Simonenko published his celebrated paper *Some general questions in the theory of the Riemann boundary problem* [Si] that gave rise to intensive studies on Riemann problems, singular integral and Toeplitz operators, etc. including the concepts of generalized factorization [ClGo], Φ-factorization [LiSp] and Wiener-Hopf factorization [BöSi]. In that paper, I. Simonenko gave a rather general definition of factorization of matrices with measurable functions as entries. He proved equivalence of generalized factorization with the solvability of the corresponding systems of singular integral operators and gave many properties of generalized factorization. The paper [Si] continuous to influence the investigations almost four decades already.

Among the pioneering works on the subject one should mention contributions by T. Carleman, N. Wiener and H. Hopf, F. Gakhov, N. Muskhelishvili, M. Krein,

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I. Gohberg, I. Simonenko and many others. See also [GoKaSp] for a survey on matrix-valued functions factorization.

Different types of matrix factorizations revealed to be a powerful tool for solving explicitly many problems, e.g. in mathematical physics. Recent work on applications in diffraction theory [CaSpTe1, CaSpTe3] initiated a detailed investigation of Wiener–Hopf plus Hankel operators in spaces of Bessel potentials and their theoretical background.

The present paper continues the investigation started in [CaSpTe2, CaSp]. Some related results on factorization of matrix symbols of pseudodifferential operators are exposed in [ChDu, Sh]. Corresponding work for the circle instead of $\mathbb{R}$ and the factorization theory for Toeplitz plus Hankel operators can be found in [Eh]. The present environment is designed for further applications in mathematical physics as started in [CaSpTe1].

Here we devote particular attention to factorization of matrix-valued functions with discontinuity at infinity, which plays a crucial role in solving some problems of mathematical physics. We establish a criterion for such matrix-valued functions on the real line admit, both, an asymmetric and a classical right factorization. It yields the existence of generalized inverses of *matrix convolution type operators with symmetry* [CaSpTe2] (or Wiener–Hopf plus/minus Hankel operators), and provides precise information about the asymptotic behavior of the factors at infinity, and of the solutions to the corresponding equations at the origin.

2. Classical factorization

Let $\mathscr{A}$ be a bounded matrix-valued function which belongs to the Zygmund space $\mathscr{Z}^\mu(\overline{\mathbb{R}})$ or to the algebra $\mathscr{H}_0^\mu(\overline{\mathbb{R}})$, $\mu > 0$ (see Appendix, § A.2) and is supposed to be elliptic:

$$\inf_{x\in\mathbb{R}} |\det \mathscr{A}(x)| > 0. \tag{2.1}$$

The limits $\mathscr{A}(+\infty)$ and $\mathscr{A}(-\infty)$ might differ (in contrast to the case $\mathscr{A} \in \mathscr{Z}^\mu(\mathbb{R})$ or $\mathscr{A} \in \mathscr{H}_0^\mu(\mathbb{R})$ when these limits coincide) and we consider the *Jordan normal decomposition* of the matrix

$$\mathscr{A}_\infty := [\mathscr{A}(+\infty)]^{-1}\mathscr{A}(-\infty) = \mathscr{K}\,\Lambda_{\mathscr{A}_\infty} B_{\mathscr{A}_\infty}(1)\,\mathscr{K}^{-1}\,. \tag{2.2}$$

Here $\Lambda_{\mathscr{A}_\infty}$ is a diagonal matrix of eigenvalues of $\mathscr{A}_\infty$, $B_{\mathscr{A}_\infty}(1)$ is upper triangular with entries 1 on the main diagonal and $\mathscr{K}$ is an elliptic ($\det \mathscr{K} \neq 0$) transformation matrix (see Appendix, § A.1 for details).

Let $\lambda_1,\dots,\lambda_\ell$ be all eigenvalues of the matrix $\mathscr{A}_\infty$ with the Riesz indices $m_1,\dots, m_\ell$, respectively (i.e., λ_j defines m_j linearly independent associated vectors for $\mathscr{A}_\infty$; see [Ga]) and

$$\delta_j := \frac{1}{2\pi i}\log\lambda_j, \quad \gamma < \Re e\ \delta_j \le \gamma+1\,, \quad j=1,\dots,\ell \tag{2.3}$$

for some $\gamma \in \mathbb{R}$.