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Lions' Lemma, Korn's Inequalities and the Lamé Operator on Hypersurfaces

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Dedicated to my friend and colleague Nikolai Vasilevski on the occasion of his 60*th birthday anniversary*

Abstract. We investigate partial differential equations on hypersurfaces written in the Cartesian coordinates of the ambient space. In particular, we generalize essentially Lions' Lemma, prove Korn's inequality and establish the unique continuation property from the boundary for Killing's vector fields, which are analogues of rigid motions in the Euclidean space. The obtained results, the Lax-Milgram lemma and some other results are applied to the investigation of the basic Dirichlet and Neumann boundary value problems for the Lamé equation on a hypersurface.

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Introduction

Partial differential equations (PDEs) on hypersurfaces and corresponding boundary value problems (BVPs) appear rather often in applications: see [Ha1, §72] for the heat conduction by surfaces, [Ar1, §10] for the equations of surface flow, [Ci1], [Ci3],[Ci4], [Ko2], [Go1] for thin flexural shell problems in elasticity, [AC1] for the vacuum Einstein equations describing gravitational fields, [TZ1, TW1] for the Navier-Stokes equations on spherical domains and spheres, [MM1] for minimal surfaces, [AMM1] for diffusion by surfaces, as well as the references therein. Furthermore, such equations arise naturally while studying the asymptotic behavior of solutions to elliptic boundary value problems in a neighborhood of conical points (see the classical reference [Ko1]).

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By a classical approach differential equations on surfaces are written with the help of covariant and contravariant frames, metric tensors and Christoffel symbols. To demonstrate a difference between a classical and the present approaches, let us consider an example. A surface $\mathscr{S}$ can be given by a local immersion

$$\Theta\,:\,\omega\to\mathscr{S}\,,\quad \omega\subset\mathbb{R}^{n-1}\,, \tag{0.1}$$

which means that the derivatives $\big\{\boldsymbol{g}_k := \partial_k\Theta\big\}_{k=1}^{n-1}$, constituting the *covariant frame* in the space of tangent vector fields to the surface $\mathscr{V}(\mathscr{S})$, are linearly independent. In equivalent formulation that means the Gram matrix $G_{\mathscr{S}}(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}$, $g_{jk} := \langle\boldsymbol{g}_j,\boldsymbol{g}_k\rangle$ has the inverse $G^{-1}_{\mathscr{S}}(\mathscr{X}) = [g^{jk}(\mathscr{X})]_{n-1\times n-1}$, $g^{jk} := \langle\boldsymbol{g}^j,\boldsymbol{g}^k\rangle$, where $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ is the contravariant frame and is biorthogonal to the covariant frame $\langle\boldsymbol{g}_j,\boldsymbol{g}^k\rangle = \delta_{jk}$, $j,k=1,\dots,n-1$. Hereafter

$$f\langle\boldsymbol{U},\boldsymbol{V}\rangle := \sum_{j=1}^{n} U^0_j V^0_j,\qquad \boldsymbol{U}=(U^0_1,\dots,U^0_n)^\top\in\mathbb{R}^n,\quad \boldsymbol{V}=(V^0_1,\dots,V^0_n)^\top\in\mathbb{R}^n$$

denotes the scalar product. The Gram matrix $G_{\mathscr{S}}(\mathscr{X})$ is also called *covariant metric tensor* and is responsible for the *Riemannian metric* on $\mathscr{S}$.

The surface divergence and gradients in classical differential geometry (in intrinsic parameters of the surface $\mathscr{S}$) read as follows:

$$\begin{aligned}
\operatorname{div}_{\mathscr{S}}\boldsymbol{U} &:= \big[\det G_{\mathscr{S}}\big]^{-1/2}\sum_{j=1}^{n}\partial_j\Big\{\big[\det G_{\mathscr{S}}\big]^{1/2}U^j\Big\}\,,\\
\nabla_{\mathscr{S}}f &= \sum_{j,k=1}^{n-1}(g^{jk}\partial_j f)\,\partial_k,\qquad \boldsymbol{U}=\sum_{j=1}^{n-1}U^j\boldsymbol{g}_j
\end{aligned} \tag{0.2}$$

(see [Ta2, Ch. 2, § 3]). The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

A derivative $\partial^{\mathscr{S}}_{\boldsymbol{U}}\,:\,C^1(\mathscr{S})\to C^1(\mathscr{S})$ along some tangential vector field $\boldsymbol{U}\in\mathscr{V}(\mathscr{S})$ is called *covariant* if it is a linear automorphism of the space of tangential vector fields

$$\partial^{\mathscr{S}}_{\boldsymbol{U}}\,:\,\mathscr{V}(\mathscr{S})\longrightarrow\mathscr{V}(\mathscr{S}). \tag{0.3}$$

The covariant derivative of a tangential vector field $\boldsymbol{V}=\sum_{j=1}^{n-1}V^j\boldsymbol{g}_j\in\mathscr{V}(\mathscr{S})$ along a tangential vector field $\boldsymbol{U}=\sum_{j=1}^{n-1}U^j\boldsymbol{g}_j\in\mathscr{V}(\mathscr{S})$ is defined by the formula

$$\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V} := \pi_{\mathscr{S}}\partial_{\boldsymbol{U}}\boldsymbol{V} := \sum_{j,k,m=1}^{n-1}\big[U^jV^k\Gamma^m_{jk}+\delta_{jk}U^j\partial_jV^m\big]\,\boldsymbol{g}_m, \tag{0.4}$$

where $\Gamma^m_{jk}(x)$ are the *Christoffel symbols*

$$\begin{aligned}
\Gamma^m_{jk}(x) &:= \langle\partial_k\boldsymbol{g}_j(x),\boldsymbol{g}^m(x)\rangle = \sum_{q=1}^{n-1}\frac{g^{mq}}{2}\big[\partial_k g_{jq}(x)+\partial_j g_{kq}(x)-\partial_q g_{jk}(x)\big]\\
&:= \Gamma^m_{kj}(x)\,.
\end{aligned} \tag{0.5}$$