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Compact Hausdorff Spaces with Relations and Gleason Spaces

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Abstract

We consider an alternate form of the equivalence between the category of compact Hausdorff spaces and continuous functions and a category formed from Gleason spaces and certain relations. This equivalence arises from the study of the projective cover of a compact Hausdorff space. This line leads us to consider the category of compact Hausdorff spaces with closed relations, and the corresponding subcategories with continuous and interior relations. Various equivalences of these categories are given extending known equivalences of the category of compact Hausdorff spaces and continuous functions with compact regular frames, de Vries algebras, and also with a category of Gleason spaces that we introduce. Study of categories of compact Hausdorff spaces with relations is of interest as a general setting to consider Gleason spaces, for connections to modal logic, as well as for the intrinsic interest in these categories.

Keywords Compact Hausdorff space · Gleason cover · Closed relation · Continuous relation · Interior relation · Compact regular frame · De Vries algebra

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1 Introduction

The category KHaus of compact Hausdorff spaces and the continuous functions between them has been extensively studied, and many well-known equivalences and dual equivalences between KHaus and other well-known categories have been established. Among them is the dual equivalence to the category KRFrm of compact regular frames and the frame homomorphisms between them, and the dual equivalence to the category DeV of de Vries algebras and the de Vries algebra homomorphisms between them. The duality between KHaus and KRFrm is established via the open set functor, and that between KHaus and DeV via the regular open set functor.

In [5] a further equivalence was established between KHaus and what we call here Gle_0 of Gleason spaces and certain relations between them. The idea is to take the projective cover $\pi : \widehat{X} \to X$ of a compact Hausdorff space X, and associate to this a pair $(\widehat{X}, E)$ where E is the kernel of π. These are pairs consisting of an extremally disconnected compact Hausdorff space and an equivalence relation on it with certain properties. In [5] morphisms were defined between Gleason spaces giving a category Gle_0, and this category was shown to be equivalent to KHaus.

The morphisms in [5] were certain relations between Gleason spaces. Their properties were motivated from constructing a duality with DeV and lifting the necessary properties. Composition in Gle_0 was defined through that of DeV, and was not simple relational composition. Here we construct a different category based on Gleason spaces where morphisms are again certain relations between them. These different morphisms have simpler description than in Gle_0, and their composition is given by relational composition. This new category is denoted Gle.

Gleason spaces (X, E) consist of an extremally disconnected compact Hausdorff space X and an equivalence relation E on X with certain properties. Morphisms in Gle_0 and Gle consist of certain relations between Gleason spaces. It becomes natural in our study to work in the generality of relations between compact Hausdorff spaces. Two general types of relations arise as extensions of continuous functions, the closed relations R being those that are closed in the product topology, and the continuous ones that are closed and have $R^{-1}[U]$ open for each open set U. Such continuous relations arise naturally in considering the Vietoris functor on KHaus (see, e.g., [6, Sec. 2]).

This leads naturally to consideration of the category $\mathsf{KHaus}^\mathsf{R}$ of compact Hausdorff spaces and closed relations between them under relational composition, and the category $\mathsf{KHaus}^\mathsf{c}$ of compact Hausdorff spaces and continuous relations between them. We also consider the category $\mathsf{KHaus}^\mathsf{i}$ of compact Hausdorff spaces and interior relations between them. These generalize interior functions, that is, continuous and open functions. These categories are natural to consider also in the context of modal compact Hausdorff spaces [6,7], which have applications in modal logic. But perhaps the strongest motivation for consideration of these categories is their inherent interest.

The category $\mathsf{KHaus}^\mathsf{R}$ was considered in [21]. It was also studied in [19] in the more general setting of stably compact spaces and closed relations. By [21, Thm. 4.3.3], the duality between KHaus and KRFrm generalizes to a duality between $\mathsf{KHaus}^\mathsf{R}$ and the category $\mathsf{KRFrm}^\mathsf{R}$ of compact regular frames and preframe homomorphisms. This is also a consequence of the results of [19].

Here we define the category Gle^R and establish equivalences and dual equivalences between Gle^R and the categories $\mathsf{KHaus}^\mathsf{R}$ and $\mathsf{KRFrm}^\mathsf{R}$. We do similarly for continuous and interior relations. We define categories $\mathsf{KRFrm}^\mathsf{c}$, DeV^c, Gle^c and extend the equivalences and

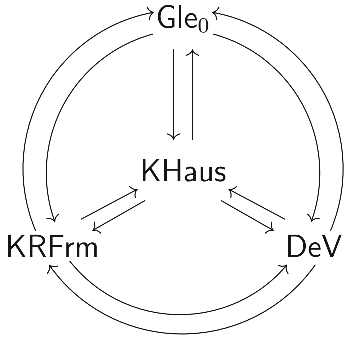


Fig. 1 Equivalences and dual equivalences involving KHaus

dual equivalences of Fig. 1 to equivalences and dual equivalences between these categories and $\mathsf{KHaus}^{\mathsf{c}}$. We also define categories $\mathsf{KRFrm}^{\mathsf{i}}$, $\mathsf{DeV}^{\mathsf{i}}$, $\mathsf{Gle}^{\mathsf{i}}$ and obtain similar equivalences and dual equivalences between these categories and $\mathsf{KHaus}^{\mathsf{i}}$. An analog of $\mathsf{DeV}^{\mathsf{R}}$ is left out of this process since closed relations on a compact Hausdorff space are not determined in a natural way by their behavior on regular open sets.

This paper is arranged in the following way. The second section gives preliminaries and reviews the known equivalences and dual equivalences given in Fig. 1. The third section provides equivalences and dual equivalences related to $\mathsf{KHaus}^{\mathsf{R}}$ and provides obstacles for such an equivalence in the setting of de Vries algebras. The fourth section provides equivalences and dual equivalences related to $\mathsf{KHaus}^{\mathsf{c}}$, and the fifth section does similarly for $\mathsf{KHaus}^{\mathsf{i}}$. The sixth section shows that the equivalences and dual equivalences obtained in earlier sections restrict to known equivalences and dual equivalences among KHaus, KRFrm, and DeV and their subcategories, and produces a new equivalence between KHaus and what we call Gle.

2 Preliminaries

Here we review the categories and known equivalences and dual equivalences between them, as given in Fig. 1. Throughout we use I and C for topological interior and closure.

Definition 2.1 KHaus is the category of compact Hausdorff spaces and continuous functions between them, with composition being function composition.

Recall [17,20] that a *frame* is a complete lattice that satisfies $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$, and a *frame homomorphism* is a function between frames that preserves finite meets and arbitrary joins. An element a of a frame is *compact* if $a \leq \bigvee S$ implies that there is a finite $T \subseteq S$ with $a \leq \bigvee T$, and a frame is *compact* if its top element 1 is compact. Using $\neg a := \bigvee\{x \mid a \wedge x = 0\}$ for the pseudocomplement of a, the *well inside* relation $\prec$ in a frame is defined by $x \prec y$ iff $\neg x \vee y = 1$. We set $\twoheaddownarrow a = \{b \mid b \prec a\}$ and $\twoheaduparrow a = \{b \mid a \prec b\}$. A frame is *regular* if for each a we have $a = \bigvee \twoheaddownarrow a$. For a subset S of a frame, we set $\twoheaddownarrow S = \{b \mid b \prec s \text{ for some } s \in S\}$ and $\twoheaduparrow S = \{b \mid s \prec b \text{ for some } s \in S\}$. An ideal I of a frame is *round* if $I = \twoheaddownarrow I$, and a filter F of a frame is *round* if $F = \twoheaduparrow F$.

Definition 2.2 KRFrm is the category of compact regular frames and frame homomorphisms between them, with composition being function composition.

A *de Vries algebra* [3,9] $(B, \prec)$ is a complete Boolean algebra B with binary relation $\prec$ that satisfies

1. $1 \prec 1$,
2. $a \prec b$ implies $a \leq b$,
3. $a \leq b \prec c \leq d$ implies $a \prec d$,
4. $a \prec b, c$ implies $a \prec b \wedge c$,
5. $a \prec b$ implies $\neg b \prec \neg a$,
6. $a \prec b$ implies there is c with $a \prec c \prec b$ (interpolation)
7. $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

A morphism between de Vries algebras is a function α satisfying (i) $\alpha(0) = 0$, (ii) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, (iii) $a \prec b$ implies $\neg\alpha(\neg a) \prec \alpha(b)$, and (iv) $\alpha(a) = \bigvee\{\alpha(b) \mid b \prec a\}$.

Definition 2.3 DeV is the category of de Vries algebras and their morphisms with composition of de Vries morphisms $\beta\star\alpha$ given by $(\beta\star\alpha)(a) = \bigvee\{\beta\alpha(b) \mid b \prec a\}$.

A binary relation R from X to Y is a subset $R \subseteq X \times Y$. For $A \subseteq X$ and $B \subseteq Y$, the *image of* A *under* R is $R[A] = \{y \mid xRy$ for some $x \in A\}$, and the *preimage of* B *under* R is $R^{-1}[B] = \{x \mid xRy$ for some $y \in B\}$. For $x \in X$ and $y \in Y$ we use $R[x]$ for $R[\{x\}]$ and $R^{-1}[y]$ for $R^{-1}[\{y\}]$ for the image and preimage of these singletons.

A topological space is *extremally disconnected* if the closure of each open set is open. A *Gleason space* [5] is a pair (X, E) consisting of an extremally disconnected compact Hausdorff space X and an equivalence relation E on X that satisfies (i) E is a closed subset of $X \times X$ and (ii) if A is a proper closed subset of X, then so is $E[A]$. The following is the way that a category was constructed from Gleason spaces in [5].

Definition 2.4 Gle_0 is the category whose objects are Gleason spaces, where a morphism between Gleason spaces (X, E) and (X', E') is a relation $R \subseteq X \times X'$ that satisfies

1. $R[x]$ is closed for each $x \in X$,
2. for each clopen B we have $R^{-1}[B]$ is clopen and is the interior of $(R \circ E)^{-1}[B]$,
3. for each $A \subseteq X$, if A is nonempty, then $R[A]$ is nonempty,
4. $R \circ E \circ R^{-1} \subseteq E'$.

The identity morphisms in this category are identity relations. The rule of composition $S\star R$ in Gle_0 is not easily described, and we refer to [5, p. 399].

Remark 2.5 As is customary for functions, which we will treat as special relations, if R is a relation from X to Y, and S is a relation from Y to Z, then we write the composite relation from X to Z as $S \circ R$. It is the set of all ordered pairs (x, z) such that there is y with xRy and ySz.

There are equivalences and dual equivalences among these categories as shown in Fig. 1. We begin with the dual equivalence between KHaus and KRFrm that is known as Isbell duality [2,16,17]. There is a functor $\mathcal{O}$ from KHaus to KRFrm taking a compact Hausdorff space X to its frame $\mathcal{O}X$ of open sets and a continuous function $f : X \to Y$ to the frame homomorphism $f^{-1}[\cdot] : \mathcal{O}Y \to \mathcal{O}X$. There is also a functor $\mathfrak{pt}$ from KRFrm to KHaus taking a frame L to its space of *points*, that is the frame homomorphisms $p : L \to 2$, where the open sets of the topology are the sets $\phi(a) = \{p \mid p(a) = 1\}$ for $a \in L$. This functor $\mathfrak{pt}$ takes a frame homomorphism $f : L \to M$ to the continuous map $\mathfrak{pt}(f) : \mathfrak{pt}\, M \to \mathfrak{pt}\, L$ where $\mathfrak{pt}(f)(p) = p \circ f$.

Theorem 2.6 (Isbell duality) *The open set functor* $\mathcal{O}$ *and the point functor* $\mathfrak{pt}$ *provide a dual equivalence between* KHaus *and* KRFrm.

The dual equivalence between KHaus and DeV is known as de Vries duality [3,9]. There is a functor $\mathcal{RO}$ from KHaus to DeV that takes a compact Hausdorff space X to the complete Boolean algebra of its regular open sets with the relation $\prec$ on this Boolean algebra given by $U \prec V$ if $\mathsf{C}U \subseteq V$. This functor takes a continuous function $f : X \to Y$ to the de Vries homomorphism $\mathcal{RO}(f) : \mathcal{RO}Y \to \mathcal{RO}X$ where $\mathcal{RO}(f)(U) = \mathsf{IC}f^{-1}[U]$.

For a de Vries algebra $(A, \prec)$ and $S \subseteq A$, we set $\Downarrow S = \{a \mid a \prec s \text{ for some } s \in S\}$ and $\Uparrow S = \{a \mid s \prec a \text{ for some } s \in S\}$. An ideal I of A is *round* if $I = \Downarrow I$, and a filter F of A is *round* if $F = \Uparrow F$. Maximal round filters are called *ends*. There is a functor End from DeV to KHaus taking $(A, \prec)$ to the space of its ends topologized by taking as a basis all sets of the form $\{E \mid a \in E\}$ where $a \in A$ and E is an end. For a de Vries morphism $f : A \to B$, $\mathsf{End}(f) : \mathsf{End}\, B \to \mathsf{End}\, A$ takes the end E to $\Uparrow f^{-1}[E]$.

Theorem 2.7 (De Vries duality) *The regular open set functor $\mathcal{RO}$ and the end functor* End *provide a dual equivalence between* KHaus *and* DeV.

It follows from Theorems 2.6 and 2.7 that KRFrm is equivalent to DeV. For a direct point-free proof of this result see [4].

We conclude by outlining the object level correspondence between compact Hausdorff spaces and Gleason spaces. This is extended in [5] to an equivalence between KHaus and Gle_0. For a compact Hausdorff space X, let $\widehat{X}$ be the Stone space of the complete Boolean algebra of regular open sets of X, and let $\pi : \widehat{X} \to X$ be the map sending an ultrafilter $\mathfrak{u}$ of regular open sets of X to its unique limit point in X. Gleason [13] showed that $\pi : \widehat{X} \to X$ is *irreducible*, meaning that it is onto continuous and the image of a proper closed subset is proper, that $\widehat{X}$ is projective in KHaus, and that the pair $(\widehat{X}, \pi)$ is unique up to homeomorphism. We call $\pi : \widehat{X} \to X$ the *Gleason cover* of X.

Theorem 2.8 *Let X be a compact Hausdorff space with Gleason cover $\pi : \widehat{X} \to X$ and let $E = \ker \pi$. Then $\widehat{X}/E$ is compact Hausdorff and there is a homeomorphism $\eta_X : X \to \widehat{X}/E$ given by $\eta_X(x) = \pi^{-1}[x]$.*

Proof This is easy to verify directly, but also follows from general considerations about categories of algebras over a monad on sets. By Manes' Theorem (see, e.g., [17, Sec. III.2]), the category of compact Hausdorff spaces is equivalent to the category of algebras over the ultrafilter monad on the category of sets. Every surjective homomorphism of such algebras is the coequalizer of its kernel pair. □

Theorem 2.9 *If (X, E) is a Gleason space, then the canonical quotient map $\kappa : X \to X/E$ is the Gleason cover of X/E. More precisely, if $\pi : \widehat{X/E} \to X/E$ is the Gleason cover of X/E, then there is a homeomorphism $\epsilon_{(X,E)} : X \to \widehat{X/E}$ such that $\pi \circ \epsilon_{(X,E)} = \kappa$.*[1] *Therefore, for $F = \ker \pi$ we have $\epsilon_{(X,E)} \circ E = F \circ \epsilon_{(X,E)}$.*

Proof Consult [13, Thm. 3.2] for the existence of such a homeomorphism $\epsilon_{(X,E)}$. The statement about the composition of the relations is then a consequence of the commutativity $\pi \circ \epsilon_{(X,E)} = \kappa$. □

3 Compact Hausdorff Spaces and Closed Relations

In this section we consider the category of compact Hausdorff spaces with closed relations, give corresponding definitions of categories of compact regular frames and Gleason spaces, and prove equivalences and dual equivalences for these categories.

[1] This homeomorphism takes an element x to the ultrafilter of all regular open sets of X/E that contain x/E.

Definition 3.1 A *closed relation* R between compact Hausdorff spaces X, Y is a subset $R \subseteq X \times Y$ that is closed in the product topology.

The following is easily proved, and can be found in [5, Lem. 2.12].

Lemma 3.2 *For a relation R from a compact Hausdorff space X to a compact Hausdorff space Y, these are equivalent.*

1. *R is closed.*
2. *For any closed $F \subseteq X$ and $G \subseteq Y$, both $R[F]$ and $R^{-1}[G]$ are closed.*
3. *If $(x, y) \notin R$, then there are open neighborhoods $x \in U$ and $y \in V$ with $R[U] \cap V = \emptyset$.*

This has as a consequence that the composite of closed relations is closed.

Definition 3.3 $\mathsf{KHaus}^\mathsf{R}$ is the category of compact Hausdorff spaces and the closed relations between them with composition being relational composition. The identity morphism on a compact Hausdorff space X is the identity relation on X.

We note that the graph of every continuous function between compact Hausdorff spaces is a closed relation. Hence, KHaus can be viewed as a non-full subcategory of $\mathsf{KHaus}^\mathsf{R}$ that has the same objects as $\mathsf{KHaus}^\mathsf{R}$. Such subcategories are usually called *wide*.

Definition 3.4 $\mathsf{KRFrm}^\mathsf{R}$ is the category of compact regular frames and preframe homomorphisms; that is, maps $\Box : L \to M$ between frames that preserve finite meets, including 1, and directed joins. We note that 0 is not a directed join, so $\Box$ need not preserve it.

Remark 3.5 Preframe homomorphisms between compact regular frames are exactly the Scott continuous maps [11,12] that preserve finite meets. Thus, the category $\mathsf{KRFrm}^\mathsf{R}$ is a natural object of study in its own right.

Clearly KRFrm is a wide subcategory of $\mathsf{KRFrm}^\mathsf{R}$.

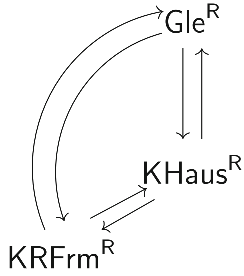
Definition 3.6 Gle^R is the category of Gleason spaces with a morphism between Gleason spaces (X, E) and (X', E') being a closed relation R from X to X' with $R \circ E = R = E' \circ R$. Composition of morphisms is composition of relations and the identity morphism on (X, E) is the relation E.

Remark 3.7 If R is a closed relation between compact Hausdorff spaces X and Y, then its inverse R^{-1} is a closed relation between Y and X by Lemma 3.2. Therefore, $\mathsf{KHaus}^\mathsf{R}$ carries the structure of a *dagger category* (see, e.g., [15]), i. e. it admits a contravariant endofunctor $\dagger : \left(\mathsf{KHaus}^\mathsf{R}\right)^{\mathrm{op}} \to \mathsf{KHaus}^\mathsf{R}$ such that $\dagger$ is identity on objects and $\dagger \circ \dagger$ is the identity functor. It is given by assigning to $R : X \to Y$ its inverse $R^{-1} : Y \to X$. This in particular implies that $\mathsf{KHaus}^\mathsf{R}$ is self-dual. Similar comments hold for Gle^R.

We next discuss the equivalences and dual equivalences depicted in Fig. 2. The dual equivalence between $\mathsf{KHaus}^\mathsf{R}$ and $\mathsf{KRFrm}^\mathsf{R}$ follows from [19,21]. In view of Remark 3.7, this dual equivalence is in fact an equivalence, and $\mathsf{KRFrm}^\mathsf{R}$ is also a dagger category (see Remark 3.17).

Proposition 3.8 *There is a contravariant functor $\mathcal{O}^\mathsf{R}$:$\mathsf{KHaus}^\mathsf{R} \to \mathsf{KRFrm}^\mathsf{R}$ that takes a compact Hausdorff space X to its frame $\mathcal{O}X$ of open sets and a closed relation $R \subseteq X \times Y$ to the function $\Box_R : \mathcal{O}Y \to \mathcal{O}X$ given by $\Box_R U = -R^{-1}[-U] = \{x \mid R[x] \subseteq U\}$.*

Fig. 2 Equivalences and dual equivalences involving $\mathsf{KHaus}^\mathsf{R}$



Proof This follows from [19, Prop. 3.3]. □

Proposition 3.9 *There is a contravariant functor* $\mathfrak{pt}^\mathsf{R} : \mathsf{KRFrm}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$ *that takes a compact regular frame* L *to its space* $\mathfrak{pt}(L)$ *of points and a morphism* $\Box : L \to M$ *to the relation* $\mathfrak{pt}(\Box) = R_\Box$ *where* $q R_\Box p$ *for points* $p : L \to 2$ *and* $q : M \to 2$ *iff* $q \circ \Box \leq p$.

Proof This follows from [19, Prop. 3.4]. □

Theorem 3.10 *The functors* $\mathcal{O}^\mathsf{R}$ *and* $\mathfrak{pt}^\mathsf{R}$ *give a dual equivalence between* $\mathsf{KHaus}^\mathsf{R}$ *and* $\mathsf{KRFrm}^\mathsf{R}$.

Proof See [21, Thm. 4.3.3]. It also follows from [19, Thm. 3.5]. □

We now turn our attention to the equivalence between $\mathsf{KHaus}^\mathsf{R}$ and Gle^R.

Proposition 3.11 *There is a functor* $\mathcal{G}^\mathsf{R} : \mathsf{KHaus}^\mathsf{R} \to \mathsf{Gle}^\mathsf{R}$ *taking a compact Hausdorff space* X *to the pair* $(\widehat{X}, E)$ *where* $\pi : \widehat{X} \to X$ *is the Gleason cover of* X *and* $E = \ker \pi$*. For compact Hausdorff spaces* X_1 *and* X_2 *with Gleason covers* $\pi_1 : \widehat{X}_1 \to X_1$ *and* $\pi_2 : \widehat{X}_2 \to X_2$*, this functor takes a closed relation* R *from* X_1 *to* X_2 *to the relation* $\mathcal{G}^\mathsf{R}(R) = \pi_2^{-1} \circ R \circ \pi_1$.

Proof As described in the preliminaries, the pair $(\widehat{X}, E)$ is a Gleason space. For $\mathcal{G}^\mathsf{R}(R)$ to be a morphism in Gle^R it must be closed and satisfy $\mathcal{G}^\mathsf{R}(R) \circ E_1 = \mathcal{G}^\mathsf{R}(R) = E_2 \circ \mathcal{G}^\mathsf{R}(R)$. Each $\pi_i : \widehat{X}_i \to X_i$ for $i = 1, 2$ is by definition irreducible, which implies that they are continuous functions, hence closed relations. It is clear that the converse of a closed relation is also closed, and the composite of closed relations is closed, thus $\mathcal{G}^\mathsf{R}(R) = \pi_2^{-1} \circ R \circ \pi_1$ is a closed relation. Since $E_1 = \ker \pi_1$ and $E_2 = \ker \pi_2$ we have that

$$E_2 \circ (\pi_2^{-1} \circ R \circ \pi_1) = \pi_2^{-1} \circ R \circ \pi_1 = (\pi_2^{-1} \circ R \circ \pi_1) \circ E_1.$$

So $\mathcal{G}^\mathsf{R}(R)$ is a morphism in Gle^R. Let $R : X_1 \to X_2$ and $S : X_2 \to X_3$ be closed relations between compact Hausdorff spaces, and let $\pi_i : \widehat{X}_i \to X_i$ for $i = 1, 2, 3$ be their Gleason covers. Then $\mathcal{G}^\mathsf{R}(S) \circ \mathcal{G}^\mathsf{R}(R) = (\pi_3^{-1} \circ S \circ \pi_2) \circ (\pi_2^{-1} \circ R \circ \pi_1)$. Since π_2 is onto, $\pi_2 \circ \pi_2^{-1}$ is the identity relation, and this expression reduces to $\pi_3^{-1} \circ (S \circ R) \circ \pi_1 = \mathcal{G}^\mathsf{R}(S \circ R)$. Finally, $\mathcal{G}^\mathsf{R}$ applied to the identity relation on X is the relation $\pi^{-1} \circ \pi = E$ on $\widehat{X}$. Thus, $\mathcal{G}^\mathsf{R}$ preserves identities. So $\mathcal{G}^\mathsf{R}$ is a functor. □

Proposition 3.12 *There is a functor* $\mathcal{Q}^\mathsf{R} : \mathsf{Gle}^\mathsf{R} \to \mathsf{KHaus}^\mathsf{R}$ *that takes a Gleason space* (X, E) *to the quotient space* X/E*. For Gleason spaces* (X_1, E_1) *and* (X_2, E_2) *and a morphism* $R : (X_1, E_1) \to (X_2, E_2)$*, this functor takes* R *to* $\mathcal{Q}^\mathsf{R}(R) : X_1/E_1 \to X_2/E_2$ *where* $\mathcal{Q}^\mathsf{R}(R) = \kappa_2 \circ R \circ \kappa_1^{-1}$ *is the composite using the quotient maps* $\kappa_i : X_i \to X_i/E_i$ *for* $i = 1, 2$.

Proof For a Gleason space (X, E), the relation E is closed, so X/E is a compact Hausdorff space. The morphism R is by definition a closed relation from X_1 to X_2. Since the quotient maps are continuous maps between compact Hausdorff spaces, they also are closed relations, and so are their inverses. Since the composite of closed relations is closed, we have that $\mathcal{Q}^{\mathsf{R}}(R)$ is a closed relation, and hence a morphism of $\mathsf{KHaus}^{\mathsf{R}}$. To see that composition is preserved, suppose R is a morphism from (X_1, E_1) to (X_2, E_2) and S is a morphism from (X_2, E_2) to (X_3, E_3). Then $\mathcal{Q}^{\mathsf{R}}(S) \circ \mathcal{Q}^{\mathsf{R}}(R) = (\kappa_3 \circ S \circ \kappa_2^{-1}) \circ (\kappa_2 \circ R \circ \kappa_1^{-1})$. But $\kappa_2^{-1} \circ \kappa_2 = E_2$ and since R is a morphism, we have $E_2 \circ R = R$. Therefore, $\mathcal{Q}^{\mathsf{R}}(S) \circ \mathcal{Q}^{\mathsf{R}}(R) = \kappa_3 \circ (S \circ R) \circ \kappa_1^{-1} = \mathcal{Q}^{\mathsf{R}}(S \circ R)$. Since $\kappa \circ E \circ \kappa^{-1}$ is the identity relation on X/E, $\mathcal{Q}^{\mathsf{R}}$ preserves identity morphisms, hence is a functor. □

Theorem 3.13 *The functors $\mathcal{G}^{\mathsf{R}}$ and $\mathcal{Q}^{\mathsf{R}}$ give an equivalence between* $\mathsf{KHaus}^{\mathsf{R}}$ *and* $\mathsf{Gle}^{\mathsf{R}}$.

Proof For a compact Hausdorff space X with Gleason cover $\pi : \widehat{X} \to X$ and $E = \ker \pi$, Theorem 2.8 gives a homeomorphism $\eta_X : X \to \widehat{X}/E$ where $\eta_X(x) = \pi^{-1}[x]$. Since η_X is a homeomorphism, it is an isomorphism in $\mathsf{KHaus}^{\mathsf{R}}$. Suppose X_1 and X_2 are compact Hausdorff spaces with Gleason covers $\pi_i : \widehat{X}_i \to X_i$ for $i = 1, 2$ and R is a closed relation from X_1 to X_2. Then $\mathcal{Q}^{\mathsf{R}}\,\mathcal{G}^{\mathsf{R}}(R) = \kappa_2 \circ \pi_2^{-1} \circ R \circ \pi_1 \circ \kappa_1^{-1}$. Using the fact that $\kappa_1^{-1} \circ \eta_{X_1} = \pi_1^{-1}$ and $\kappa_2 \circ \pi_2^{-1} = \eta_{X_2}$, it follows that $\mathcal{Q}^{\mathsf{R}}\,\mathcal{G}^{\mathsf{R}}(R) \circ \eta_{X_1} = \eta_{X_2} \circ R$. Thus, the η_X provide a natural transformation from the identity functor to $\mathcal{Q}^{\mathsf{R}}\,\mathcal{G}^{\mathsf{R}}$.

For a Gleason space (X, E), we have that $\mathcal{G}^{\mathsf{R}}\,\mathcal{Q}^{\mathsf{R}}(X, E)$ is the Gleason space $(\widehat{X/E}, F)$ where F is the kernel of $\pi : \widehat{X/E} \to X/E$. Theorem 2.9 gives a homeomorphism $\epsilon_{(X,E)} : X \to \widehat{X/E}$ taking x to the ultrafilter of regular open sets of X/E that contain x/E. This theorem further provides $F \circ \epsilon_{(X,E)} = \epsilon_{(X,E)} \circ E$. So $\epsilon_{(X,E)}$ is a morphism in $\mathsf{Gle}^{\mathsf{R}}$, and since it is a homeomorphism, it follows that it is an isomorphism in this category.

Suppose $R : (X_1, E_1) \to (X_2, E_2)$ is a morphism in $\mathsf{Gle}^{\mathsf{R}}$. Then for $\kappa_i : X_i \to X_i/E_i$ and $\pi_i : \widehat{X_i/E_i} \to X_i/E_i$ for $i = 1, 2$ we have $\mathcal{G}^{\mathsf{R}}\,\mathcal{Q}^{\mathsf{R}}(R) = \pi_2^{-1} \circ \kappa_2 \circ R \circ \kappa_1^{-1} \circ \pi_1$. Using that $\pi_2^{-1} \circ \kappa_2 = \epsilon_{(X_2,E_2)}$ and $\kappa_1^{-1} \circ \pi_1 = \epsilon_{(X_1,E_1)}^{-1}$ we have $\mathcal{G}^{\mathsf{R}}\,\mathcal{Q}^{\mathsf{R}}(R) \circ \epsilon_{(X_1,E_1)} = \epsilon_{(X_2,E_2)} \circ R$. Thus, the $\epsilon_{(X,E)}$ provide a natural transformation from the identity functor to $\mathcal{G}^{\mathsf{R}}\,\mathcal{Q}^{\mathsf{R}}$. □

Remark 3.14 One might hope to generalize DeV to a category $\mathsf{DeV}^{\mathsf{R}}$ of de Vries algebras with some type of morphisms and rule of composition with the following properties: (1) $\mathsf{DeV}^{\mathsf{R}}$ is dually equivalent to $\mathsf{KHaus}^{\mathsf{R}}$, (2) DeV is a wide subcategory of $\mathsf{DeV}^{\mathsf{R}}$, and (3) the duality between $\mathsf{DeV}^{\mathsf{R}}$ and $\mathsf{KHaus}^{\mathsf{R}}$ restricts to de Vries duality between DeV and KHaus. Below we indicate that there are problems in the way of constructing one.

One such problem is raised by considering the empty relation R from the unit interval $I = [0, 1]$ to itself, and the relation $S = \{(1, 1)\}$ from I to itself. Clearly both R and S are closed relations. One would seek some type of functions $\Box_R$ and $\Box_S$ from the de Vries algebra $\mathcal{RO}\,I$ of regular open subsets of I to itself with $\Box_R(U)$ and $\Box_S(U)$ being some regular open sets formed from U using relational image and inverse image under R and S and set theoretic and topological operations. The only sets one can form by taking $R^{-1}[U]$, $R[U]$, $S^{-1}[U]$, or $S[U]$ for any set $U \subseteq I$ are $\emptyset$ and $\{1\}$. Allowing complementation, arbitrary unions, and arbitrary intersections, interior and closure, $\emptyset$, $\{1\}$, $I \setminus \{1\}$ and I are the only sets one can obtain from these, and the only regular open sets one can obtain in this way are $\emptyset$ and I. So it is difficult to imagine a way to define $\Box_R$ and $\Box_S$ on $\mathcal{RO}\,I$ such that $\Box_R \neq \Box_S$.

Remark 3.15 As we pointed out in Remark 3.7, $\mathsf{KHaus}^{\mathsf{R}}$ is a dagger category. It also has additional structure and properties of interest. The situation is similar to that of the category Rel of sets and relations which is treated in detail in [1, Sec. 3.1] and [14, Sec 9], and the proofs of the following statements amount only to verifying that the relations given in [1,14] are closed. First, for compact Hausdorff spaces X, Y, their topological sum $X \sqcup Y$ serves as both a product and coproduct of X and Y in $\mathsf{KHaus}^{\mathsf{R}}$, and the associated injections and projections have further properties required for the topological sum to provide *biproducts* for $\mathsf{KHaus}^{\mathsf{R}}$. These biproducts are compatible with the dagger † and $\mathsf{KHaus}^{\mathsf{R}}$ is a *dagger biproduct category*. The usual cartesian product of sets gives a monoidal structure $\otimes$ on $\mathsf{KHaus}^{\mathsf{R}}$, and with respect to this monoidal structure, $\mathsf{KHaus}^{\mathsf{R}}$ is a *strongly compact closed category with biproducts* [1, Def. 7.11]. Such categories are of interest for their role in categorical treatments of quantum mechanics.

Remark 3.16 The category $\mathsf{KHaus}^{\mathsf{R}}$ shares other properties with the category Rel of relations. In particular, it comes with a partial order on hom-sets defined by set-theoretic inclusion of closed relations. Order-enriched categories of this particular kind have been axiomatized in slightly different ways in [8,10]. The terminology for the former is *Cartesian bicategories* and for the latter *allegories*. The structure of a general allegory amounts to having meet-semilattice structure on hom-sets and involutions $\hom(X, Y) \to \hom(Y, X)$ that are compatible with composition and satisfy the *modular identity*. For $\mathsf{KHaus}^{\mathsf{R}}$ the meet-semilattice structure corresponds to the intersection of relations and the involution to assigning to R its inverse R^{-1}. The modular identity amounts to $(R \circ S) \cap T \subseteq (R \cap (T \circ S^{-1})) \circ S$ for $S : X \to Y$, $R : Y \to Z$, and $T : X \to Z$.

The category $\mathsf{KHaus}^{\mathsf{R}}$ has the structure of an allegory of special kind, called a *tabular allegory*. In a general allegory, a *map* $f : X \to Y$ is defined as a morphism such that f^{-1} is adjoint to f; that is, $1_X \subseteq f^{-1} \circ f$ and $f \circ f^{-1} \subseteq 1_Y$. In $\mathsf{KHaus}^{\mathsf{R}}$ these are precisely the relations which are graphs of continuous maps. A *tabulation* of a morphism $R : X \to Y$ in an allegory is a pair of maps $f : Z \to X$, $g : Z \to Y$ such that $R = g \circ f^{-1}$ and $(f^{-1} \circ f) \cap (g^{-1} \circ g) = 1_Z$. An allegory is called *tabular* if every morphism admits a tabulation. For $\mathsf{KHaus}^{\mathsf{R}}$ such tabulation for a closed relation $R : X \to Y$ is given by $Z = R$, with the maps $f : R \to X$, $g : R \to Y$ being the projections.

Remark 3.17 Due to the equivalences among $\mathsf{KHaus}^{\mathsf{R}}$, $\mathsf{KRFrm}^{\mathsf{R}}$, and $\mathsf{Gle}^{\mathsf{R}}$, the latter two categories also have the additional structure discussed in the previous two remarks. As we pointed out in Remark 3.7, the dagger structure of $\mathsf{Gle}^{\mathsf{R}}$ is simply given by the relational inverse. For $\mathsf{KRFrm}^{\mathsf{R}}$, each preframe homomorphism is essentially of the form $\mathcal{O}^{\mathsf{R}}(R) = -R^{-1}-$ for some closed relation R between compact Hausdorff spaces. The dagger endofunctor takes this morphism to $\mathcal{O}^{\mathsf{R}}(R^{-1}) = -R-$, again essentially obtained through relational inverse. This can be realized directly. For a preframe homomorphism $h : L \to M$ we define its dagger $g : M \to L$ as follows. For m meet-prime in M, set

$$g(m) = \bigwedge\{n \mid n \text{ meet-prime in } L \text{ and } h(n) \leq m\};$$

and for an arbitrary $x \in M$ set

$$g(x) = \bigwedge\{g(m) \mid m \text{ meet-prime in } M \text{ and } x \leq m\}.$$

We leave it to the reader to verify the details.

4 Compact Hausdorff Spaces and Continuous Relations

In this section we restrict the dualities and equivalences obtained in the previous section to the setting of continuous relations. A new duality involving de Vries algebras is added in this setting. Results in this section are closely linked to the study in [6,7] where continuous relations on compact Hausdorff spaces were studied in relation to the Vietoris construction. In fact, much of what we do here is suggested in [7, Rem. 4.9], although notation there is different.

Definition 4.1 A relation R from a compact Hausdorff space X to a compact Hausdorff space Y is *continuous* if it is closed and for each open set $V \subseteq Y$ we have $R^{-1}[V]$ is open.

A function is a special type of relation. It is well known that a function between compact Hausdorff spaces is closed as a relation iff it is continuous as a function, which occurs iff it is continuous as a relation. So closed relations and continuous relations are both generalizations of continuous functions between compact Hausdorff spaces. Note also that the composition of continuous relations is continuous.

Definition 4.2 $\mathsf{KHaus}^{\mathsf{C}}$ is the category of compact Hausdorff spaces and the continuous relations between them.

Clearly $\mathsf{KHaus}^{\mathsf{C}}$ is a wide subcategory of $\mathsf{KHaus}^{\mathsf{R}}$. We next consider a partner for $\mathsf{KHaus}^{\mathsf{C}}$ in the setting of compact regular frames.

Definition 4.3 A morphism $\Box : L \to M$ in $\mathsf{KRFrm}^{\mathsf{R}}$ is a c-*morphism* if there is a function $\Diamond : L \to M$, called the *companion* of $\Box$, that preserves arbitrary joins and satisfies

$$\Box(a \vee b) \leq \Box a \vee \Diamond b \quad \text{and} \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

Remark 4.4 Such pairs of functions $\Box, \Diamond : L \to L$ on a frame L were studied in [6] as modal operators on a compact regular frame. They were motivated by Johnstone's construction of the Vietoris frame $\mathcal{V}(L)$ [17,18]. It was shown in [6, Rem. 3.7] that $\Box$ and $\Diamond$ are interdefinable via $\Diamond a = \bigvee\{\neg\Box\neg b \mid b \prec a\}$ and $\Box a = \bigvee\{\neg\Diamond\neg b \mid b \prec a\}$. These facts remain valid in our more general setting of c-morphisms between L and M. In [6], pairs $(\Box, \Diamond)$ were taken as basic. Here we take those $\Box$ for which such $\Diamond$ exists as basic since it fits more nicely with considerations of $\mathsf{KRFrm}^{\mathsf{R}}$, but these approaches are clearly equivalent.

Proposition 4.5 *The identity morphism on a frame L is a* c*-morphism, and the composition of* c*-morphisms is a* c*-morphism.*

Proof For $\Box = id_L$ take $\Diamond = id_L$ for its companion. For composition, suppose $\Box_1 : L \to M$ and $\Box_2 : M \to N$ are c-morphisms with $\Diamond_1$ and $\Diamond_2$ their companions. We claim $\Diamond_2 \circ \Diamond_1$ is a companion to $\Box_2 \circ \Box_1$. Since $\Diamond_2 \circ \Diamond_1$ is a composite of maps that preserve arbitrary joins, it preserves arbitrary joins. For the additional conditions note for $i = 1, 2$ that $\Box_i$ preserves order since it preserves finite meets, and $\Diamond_i$ preserves order since it preserves arbitrary joins. Thus,

$$\Box_2\Box_1(a \vee b) \leq \Box_2(\Box_1 a \vee \Diamond_1 b) \leq \Box_2\Box_1 a \vee \Diamond_2\Diamond_1 b,$$
$$\Box_2\Box_1 a \wedge \Diamond_2\Diamond_1 b \leq \Diamond_2(\Box_1 a \wedge \Diamond_1 b) \leq \Diamond_2\Diamond_1(a \wedge b).$$

□

Definition 4.6 $\mathsf{KRFrm}^{\mathsf{C}}$ is the wide subcategory of $\mathsf{KRFrm}^{\mathsf{R}}$ whose morphisms are the c-morphisms.

Before considering the dual equivalence between $\mathsf{KHaus}^{\mathsf{C}}$ and $\mathsf{KRFrm}^{\mathsf{C}}$ we require several facts. First, in any compact regular frame, if $c \prec a$ then $\neg a \prec \neg c$, and by interpolation there is b with $c \prec b \prec a$. Next, for a C-morphism $\Box : L \to M$ with companion $\Diamond$, it was shown in [6, Lem. 3.6] that $c \prec a$ implies $\Box c \prec \Box a$. In [6] this was proved in the setting of $\Box : L \to L$, but the general case requires only trivial modification. We also require the following result that essentially appears in [7, Lem. 5.4], but in a different context. For clarity, we state and prove this in the form required here.

Lemma 4.7 *If $\Box : L \to M$ is a C-morphism with companion $\Diamond$, then for points $p : L \to 2$ and $q : M \to 2$, we have $q \circ \Box \leq p$ iff $p \leq q \circ \Diamond$.*

Proof "$\Rightarrow$" Suppose $q \circ \Box \leq p$ and $p(a) = 1$. Since $a = \bigvee\{c \mid c \prec a\}$ and p is a point, there is $c \prec a$ with $p(c) = 1$. So there is b with $c \prec b \prec a$. Then since $c \wedge \neg c = 0$ and $p(c) = 1$, we have $p(\neg c) = 0$. As $q \circ \Box \leq p$, we have that $q(\Box\neg c) = 0$. Since $c \prec b$ we have $\neg b \prec \neg c$, hence $\Box\neg b \prec \Box\neg c$. This means that $\neg\Box\neg b \vee \Box\neg c = 1$. Therefore, as $q(\Box\neg c) = 0$, we have $q(\neg\Box\neg b) = 1$. Since $\Diamond a = \bigvee\{\neg\Box\neg b \mid b \prec a\}$ and q preserves joins, $q(\Diamond a) = 1$. So $p \leq q \circ \Diamond$.

"$\Leftarrow$" Suppose $p \leq q \circ \Diamond$ and $q(\Box a) = 1$. Since $\Box a = \bigvee\{\neg\Diamond\neg c \mid c \prec a\}$ and q is a point, there is $c \prec a$ with $q(\neg\Diamond\neg c) = 1$. Since $\neg\Diamond\neg c \wedge \Diamond\neg c = 0$, we have $q(\Diamond\neg c) = 0$. By assumption $p \leq q \circ \Diamond$, so $p(\neg c) = 0$. But $c \prec a$ gives $\neg c \vee a = 1$, hence $p(a) = 1$. So $q \circ \Box \leq p$. □

Theorem 4.8 *$\mathsf{KHaus}^{\mathsf{C}}$ is dually equivalent to $\mathsf{KRFrm}^{\mathsf{C}}$.*

Proof By Theorem 3.10, there is a dual equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{KRFrm}^{\mathsf{R}}$ given by the functors $\mathcal{O}^{\mathsf{R}}$ and $\mathfrak{pt}^{\mathsf{R}}$. It is enough to show that these functors restrict to functors between the subcategories $\mathsf{KHaus}^{\mathsf{C}}$ and $\mathsf{KRFrm}^{\mathsf{C}}$. To do this, we must show for R a continuous relation, that $\Box_R$ is a C-morphism; and for $\Box$ a C-morphism, that $R_\Box$ is a continuous relation.

For a continuous relation R from X to Y, $\Box_R : \mathcal{O}Y \to \mathcal{O}X$ is defined by $\Box_R(U) = -R^{-1}[-U]$. To show that $\Box_R$ is a C-morphism, we must show it has a companion. Define $\Diamond_R : \mathcal{O}Y \to \mathcal{O}X$ by $\Diamond_R[U] = R^{-1}[U]$. Since R is continuous, the preimage $R^{-1}[U]$ is open, so $\Diamond_R$ is well defined. Surely $\Diamond_R$ preserves arbitrary unions. The two inequalities in Definition 4.3 are established in [6, Prop. 3.10] for the case of a continuous relation R from X to itself, and carry over to the general case with obvious modification.

Suppose $\Box : L \to M$ is a C-morphism with companion $\Diamond$. Then $R_\Box$ is the relation from $\mathfrak{pt}(M)$ to $\mathfrak{pt}(L)$ given by $q\ R_\Box\ p$ iff $q \circ \Box \leq p$. We have seen in Proposition 3.9 that $R_\Box$ is closed. It remains to show for any open set $\phi(a)$ of $\mathfrak{pt}(L)$ that the preimage $R_\Box^{-1}[\phi(a)]$ is open in $\mathfrak{pt}(M)$. We claim that $R_\Box^{-1}[\phi(a)]$ is the open set $\phi(\Diamond a)$.

By Lemma 4.7, $q\ R_\Box\ p$ iff $p \leq q \circ \Diamond$. If $q \in R_\Box^{-1}[\phi(a)]$, we have $q\ R_\Box\ p$ for some point p of L with $p(a) = 1$. But then $p \leq q \circ \Diamond$ implies that $q(\Diamond a) = 1$, hence $q \in \phi(\Diamond a)$. Conversely, let $q \in \phi(\Diamond a)$. Set $c_q = \bigvee\{b \mid q(\Diamond b) = 0\}$, and note that q preserves arbitrary joins, so c_q is the largest element mapped by $q \circ \Diamond$ to 0. Since $q(\Diamond a) = 1$, we have $a \not\leq c_q$. Because compact regular frames are spatial, there is a point p of L with $p(a) = 1$ and $p(c_q) = 0$. Thus, every element mapped by $q \circ \Diamond$ to 0 is mapped by p to 0, so $p \leq q \circ \Diamond$, giving $q\ R_\Box\ p$. As $p(a) = 1$ we have $p \in \phi(a)$, hence $q \in R_\Box^{-1}[\phi(a)]$. □

We next consider a partner for $\mathsf{KHaus}^{\mathsf{C}}$ in the setting of de Vries algebras. But first a word about our approach. When choosing morphisms for $\mathsf{KRFrm}^{\mathsf{C}}$ we chose certain functions $\Box$ that preserved finite meets, directed joins, and had a companion $\Diamond$ that preserved arbitrary joins. As we mentioned, $\Box$ and $\Diamond$ determined one another. We chose to use $\Box$ to be compatible with $\mathsf{KRFrm}^\mathsf{R}$, but we could have chosen $\Diamond$ to serve as morphisms, or even the pair $(\Box, \Diamond)$ for morphisms. A similar situation arises in creating a category DeV^C. Choosing analogs of $\Box$ for morphisms would make comparison with $\mathsf{KRFrm}^\mathsf{C}$ more natural, but choosing analogs of $\Diamond$ is better compatible with the usual composition in DeV, and is the course we take. The next definition generalizes [6, Def. 4.7].

Definition 4.9 A map $\Diamond$ between de Vries algebras $(A, \prec)$ and $(B, \prec)$ is *de Vries additive* if (i) $\Diamond 0 = 0$ and (ii) $a_1 \prec a_2$ and $b_1 \prec b_2$ imply $\Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$.

A *de Vries multiplicative map* $\Box$ is one with $\Box 1 = 1$ where $a_1 \prec a_2$ and $b_1 \prec b_2$ imply $\Box a_1 \wedge \Box b_1 \prec \Box(a_2 \wedge b_2)$. The two are dual concepts to each other. Namely, $\Box$ is de Vries multiplicative iff $\neg\Box\neg$ is de Vries additive, and $\Diamond$ is de Vries additive iff $\neg\Diamond\neg$ is de Vries multiplicative. We can work with either one of them, and each determines the other since $\neg$ is Boolean negation.

Proposition 4.10 *There is a category* DeV^C *whose objects are de Vries algebras, whose morphisms are functions* $\Diamond : A \to B$ *that are de Vries additive and satisfy* $\Diamond a = \bigvee\{\Diamond b \mid b \prec a\}$*, and with composition defined as* $(\Diamond_2 \star \Diamond_1)a = \bigvee\{\Diamond_2\Diamond_1 b \mid b \prec a\}$.

Proof Clearly identity maps are such morphisms. It remains to show that if $\Diamond_1$ and $\Diamond_2$ are such morphisms, then so is $\Diamond_2 \star \Diamond_1$, and that $\star$ is associative. For this observe that given such a morphism $\Diamond$, de Vries additivity yields that $a \prec b$ implies $\Diamond a \prec \Diamond b$, and $a \leq b$ implies $\Diamond a \leq \Diamond b$ since $\Diamond a = \bigvee\{\Diamond c \mid c \prec a\}$ and $\Diamond c \prec \Diamond b$ implies $\Diamond c \leq \Diamond b$.

Using these observations, it follows that $\Diamond_2 \star \Diamond_1 \leq \Diamond_2\Diamond_1$. Using this, the properties above, and interpolation, it follows that $[\Diamond_3 \star (\Diamond_2 \star \Diamond_1)]a$ and $[(\Diamond_3 \star \Diamond_2) \star \Diamond_1]a$ are both equal to $\bigvee\{\Diamond_3\Diamond_2\Diamond_1 b \mid b \prec a\}$. De Vries additivity of $\Diamond_2 \star \Diamond_1$ follows from that of $\Diamond_2$ and $\Diamond_1$. This implies that if $b \prec a$ then $(\Diamond_2 \star \Diamond_1)b \prec (\Diamond_2 \star \Diamond_1)a$. Using this and $\Diamond_2 \star \Diamond_1 \leq \Diamond_2\Diamond_1$ gives $(\Diamond_2 \star \Diamond_1)a = \bigvee\{(\Diamond_2 \star \Diamond_1)b \mid b \prec a\}$. So $\Diamond_2 \star \Diamond_1$ is a morphism. □

Remark 4.11 Rather than define DeV^C as we have done, we could work with functions $\Box : A \to B$ that are de Vries multiplicative and satisfy $\Box a = \bigwedge\{\Box b \mid a \prec b\}$. The composition in this case would be defined as $(\Box_2 \star \Box_1)a = \bigwedge\{\Box_2\Box_1 b \mid a \prec b\}$. The functions $\Box$ and $\Diamond$ are definable from one another by $\Box = \neg\Diamond\neg$ and $\Diamond = \neg\Box\neg$.

Proposition 4.12 *There is a contravariant functor* $\mathcal{RO}^\mathsf{C} : \mathsf{KHaus}^\mathsf{C} \to \mathsf{DeV}^\mathsf{C}$ *taking a compact Hausdorff space* X *to its de Vries algebra of regular open sets, and taking a continuous relation* R *from* X *to* Y *to the map* $\Diamond_R : \mathcal{RO}Y \to \mathcal{RO}X$ *given by* $\Diamond_R U = \mathsf{I}\,\mathsf{C}\, R^{-1}[U]$.

Proof Clearly $\Diamond_R$ is well defined, and a similar proof to [6, Thm. 5.8] shows that $\Diamond_R$ is a morphism in DeV^C. To show composition is preserved, we must show that for continuous relations R from X to Y and S from Y to Z, that $\Diamond_{S \circ R}\, U = (\Diamond_S \star \Diamond_R)U$. That is,

$$\mathsf{I}\,\mathsf{C}\, R^{-1}S^{-1}[U] \;=\; \bigvee\{\mathsf{I}\,\mathsf{C}\, R^{-1}\,\mathsf{I}\,\mathsf{C}\, S^{-1}[V] \mid \mathsf{C}\, V \subseteq U\}.$$

That the left side is contained in the right follows since $\mathsf{I}\,\mathsf{C}\, R^{-1}S^{-1}[U] = \bigvee\{\mathsf{I}\,\mathsf{C}\, R^{-1}S^{-1}[V] \mid \mathsf{C}\, V \subseteq U\}$ and clearly $\mathsf{I}\,\mathsf{C}\, R^{-1}S^{-1}[V] \subseteq \mathsf{I}\,\mathsf{C}\, R^{-1}\,\mathsf{I}\,\mathsf{C}\, S^{-1}[V]$. For the other containment, suppose $\mathsf{C}\, V \subseteq U$. Since S^{-1} preserves closed and open sets,

$$\mathsf{I}\,\mathsf{C}\, S^{-1}[V] \subseteq \mathsf{I}\,\mathsf{C}\, S^{-1}[\mathsf{C}\, V] = \mathsf{I}\, S^{-1}[\mathsf{C}\, V] \subseteq \mathsf{I}\, S^{-1}[U] = S^{-1}[U].$$

Thus, $\mathsf{I}\,\mathsf{C}\, R^{-1}\,\mathsf{I}\,\mathsf{C}\, S^{-1}[V] \subseteq \mathsf{I}\,\mathsf{C}\, R^{-1}S^{-1}[U]$. □

Proposition 4.13 *There is a contravariant functor* $\mathsf{End}^{\mathsf{c}} : \mathsf{DeV}^{\mathsf{c}} \to \mathsf{KHaus}^{\mathsf{c}}$ *taking a de Vries algebra* $(A, \prec)$ *to its space of ends, and taking a morphism* $\Diamond : A \to B$ *to the relation* $R_\Diamond$ *from* $\mathsf{End}\, B$ *to* $\mathsf{End}\, A$ *given by* $E\, R_\Diamond\, F$ *iff* $\Diamond[F] \subseteq E$.

Proof That $R_\Diamond$ is continuous can be proved as in [6, Thm. 5.2]. To see that composition is preserved, for morphisms $\Diamond_1 : A \to B$ and $\Diamond_2 : B \to C$ we must show that $R_{\Diamond_2 \star \Diamond_1} = R_{\Diamond_1} \circ R_{\Diamond_2}$. We have $E\, R_{\Diamond_2 \star \Diamond_1}\, F$ iff $(\Diamond_2 \star \Diamond_1)[F] \subseteq E$ and $E (R_{\Diamond_1} \circ R_{\Diamond_2}) F$ iff there is an end G with $\Diamond_1[F] \subseteq G$ and $\Diamond_2[G] \subseteq E$.

First suppose $E(R_{\Diamond_1} \circ R_{\Diamond_2})F$. Then $\Diamond_2\Diamond_1[F] \subseteq E$. If $a \in F$, then since F is round, there is $b \in F$ with $b \prec a$. So $\Diamond_2\Diamond_1 b \leq (\Diamond_2 \star \Diamond_1)a$. Since $\Diamond_2\Diamond_1 b \in \Diamond_2\Diamond_1[F] \subseteq E$, we conclude that $(\Diamond_2 \star \Diamond_1)a \in E$. Thus, $(\Diamond_2 \star \Diamond_1)[F] \subseteq E$, and hence $E\, R_{\Diamond_2 \star \Diamond_1}\, F$.

Next suppose $E\, R_{\Diamond_2 \star \Diamond_1}\, F$, so $(\Diamond_2 \star \Diamond_1)[F] \subseteq E$. Since F is a round filter and $\Diamond_1$ is de Vries additive, $\twoheaduparrow \Diamond_1[F]$ is a round filter. Also, since E is an end and $\Diamond_2$ is de Vries additive, $\{b \mid \Diamond_2 \twoheaduparrow b \not\subseteq E\}$ is a round ideal. As $\Diamond_2 \star \Diamond_1 \leq \Diamond_2\Diamond_1$ we have that $\twoheaduparrow \Diamond_1[F]$ is disjoint from $\{b \mid \Diamond_2 \twoheaduparrow b \not\subseteq E\}$. So there is an end G containing $\twoheaduparrow \Diamond_1[F]$ and disjoint from $\{b \mid \Diamond_2 \twoheaduparrow b \not\subseteq E\}$. But then $\Diamond_1[F] \subseteq G$ and $\Diamond_2[G] \subseteq E$. Thus, $E(R_{\Diamond_1} \circ R_{\Diamond_2})F$. □

Theorem 4.14 *The functors* $\mathcal{RO}^{\mathsf{c}}$ *and* $\mathsf{End}^{\mathsf{c}}$ *give a dual equivalence between* $\mathsf{KHaus}^{\mathsf{c}}$ *and* $\mathsf{DeV}^{\mathsf{c}}$.

Proof For a compact Hausdorff space X and de Vries algebra $(A, \prec)$, the natural isomorphisms in de Vries duality are given by $\eta_X : X \to \mathsf{End}(\mathcal{RO}X)$ and $\epsilon_A : A \to \mathcal{RO}(\mathsf{End}\, A)$ where $\eta_X(x) = \{U \mid x \in U\}$ and $\epsilon_A(a) = \{E \mid a \in E\}$. Since η_X is a homeomorphism, it is an isomorphism in $\mathsf{KHaus}^{\mathsf{c}}$, and since ϵ_A is a de Vries isomorphism, it is an isomorphism also in $\mathsf{DeV}^{\mathsf{c}}$. For a continuous relation R from X to Y and morphism $\Diamond : A \to B$ in $\mathsf{DeV}^{\mathsf{c}}$, we must show that the following diagrams commute.

$$
\begin{array}{ccc}
X & \xrightarrow{\quad R \quad} & Y \\
{\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\
\mathsf{End}(\mathcal{RO}X) & \xrightarrow{R_{\Diamond_R}} & \mathsf{End}(\mathcal{RO}Y)
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\quad \Diamond \quad} & B \\
{\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\
\mathcal{RO}(\mathsf{End}(A)) & \xrightarrow{\Diamond_{R_\Diamond}} & \mathcal{RO}(\mathsf{End}(B))
\end{array}
$$

If $x\, R\, y$ then for each regular open V with $y \in V$ we have $R^{-1}[V]$ is an open neighborhood of x, hence $\Diamond_R V = \mathsf{IC}\, R^{-1}[V]$ is in $\eta_X(x)$, so $\eta_X(x)\, R_{\Diamond_R}\, \eta_Y(y)$. If x is not R-related to y then since R is closed, there are regular open neighborhoods U of x and V of y with $U \times V$ disjoint from R. So U is disjoint from $R^{-1}[V]$, and therefore $\mathsf{IC}\, R^{-1}[V] \notin \eta_X(x)$, showing that $\eta_X(x)$ is not $R_{\Diamond_R}$-related to $\eta_Y(y)$.

For the second diagram, let $a \in A$. Consider [6, Thm. 5.2(1)]. This result uses $\varphi(a)$ in place of our $\epsilon_A(a)$, and is in the context of $\Diamond : A \to A$. But with obvious modifications it shows that $R_\Diamond^{-1}\epsilon_A(a) = \bigcup\{\epsilon_B(\Diamond a') \mid a' \prec a\}$. Since $\Diamond_{R_\Diamond} = \mathsf{IC}\, R_\Diamond^{-1}$ and $\Diamond_{R_\Diamond}$ is de Vries additive, it follows that $\bigvee\{\Diamond_{R_\Diamond}\epsilon_A(a') \mid a' \prec a\}$ is equal to $\bigvee\{\epsilon_B(\Diamond a') \mid a' \prec a\}$, and hence from the definition of composition, that $(\Diamond_{R_\Diamond} \star \epsilon_A)(a) = (\epsilon_B \star \Diamond)(a)$. □

For a Gleason space (X, E), we say that $U \subseteq X$ is *saturated* if $E[U] = U$. Note that if R is a morphism in $\mathsf{Gle}^{\mathsf{R}}$ from (X, E) to (X', E'), then $R \circ E = R = E' \circ R$, giving that the preimage of any $S' \subseteq X'$ is saturated. The following is then obvious.

Proposition 4.15 *There is a wide subcategory* $\mathsf{Gle}^{\mathsf{c}}$ *of* $\mathsf{Gle}^{\mathsf{R}}$ *whose morphisms* $R \subseteq X \times X'$ *in addition satisfy that* $R^{-1}[U]$ *is open when* U *is a saturated open of* X'.

Theorem 4.16 $\mathsf{KHaus^c}$ *is equivalent to* $\mathsf{Gle^c}$.

Proof By Theorem 3.13, there is an equivalence between $\mathsf{KHaus^R}$ and $\mathsf{Gle^R}$ given by the functors $\mathcal{G}^\mathsf{R}$ and $\mathcal{Q}^\mathsf{R}$. It is enough to show that these functors restrict to functors between the subcategories $\mathsf{KHaus^c}$ and $\mathsf{Gle^c}$. To do this, we must show for R a continuous relation, that $\mathcal{G}^\mathsf{R}(R)$ is a morphism in $\mathsf{Gle^c}$; and for S a morphism in $\mathsf{Gle^c}$, that $\mathcal{Q}^\mathsf{R}(S)$ is a continuous relation.

$$\begin{array}{ccc} \widehat{X}_1 & \xrightarrow{\mathcal{G}^\mathsf{R}(R)} & \widehat{X}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{R} & X_2 \end{array} \qquad\qquad \begin{array}{ccc} (X_1, E_1) & \xrightarrow{S} & (X_2, E_2) \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_2} \\ X_1/E_1 & \xrightarrow{\mathcal{Q}^\mathsf{R}(S)} & X_2/E_2 \end{array}$$

Suppose R is a continuous relation between compact Hausdorff spaces X_1 and X_2. Then for $i = 1, 2$ we have $\mathcal{G}^\mathsf{R}(X_i) = (\widehat{X}_i, E_i)$ where $\pi_i : \widehat{X}_i \to X_i$ is the Gleason cover and $E_i = \ker \pi_i$, and $\mathcal{G}^\mathsf{R}(R) = \pi_2^{-1} \circ R \circ \pi_1$. Let $U \subseteq X_2$ be saturated open. Then $\mathcal{G}^\mathsf{R}(R)^{-1}[U] = (\pi_1^{-1} \circ R^{-1} \circ \pi_2)[U]$. Since U is saturated and π_2 is a continuous map from $\widehat{X}_2$ onto X_2, hence a quotient map, $\pi_2[U]$ is open. Then continuity of R and π_1 give that $\mathcal{G}^\mathsf{R}(R)^{-1}[U]$ is open.

Suppose that (X_i, E_i) are Gleason spaces for $i = 1, 2$ and S is a morphism in $\mathsf{Gle^c}$ from (X_1, E_1) to (X_2, E_2). We must show that $\mathcal{Q}^\mathsf{R}(S)$ is continuous. Suppose $V \subseteq X_2/E_2$ is open. Then $\mathcal{Q}^\mathsf{R}(S)^{-1}[V] = (\kappa_1 \circ S^{-1} \circ \kappa_2^{-1})[V]$. Since κ_2 is the quotient map for the equivalence relation E_2, we have that $\kappa_2^{-1}[V]$ is saturated open in X_2. Then since S is a morphism in $\mathsf{Gle^c}$ we have $(S^{-1} \circ \kappa_2^{-1})[V]$ is open, and as noted above also saturated. Thus, since κ_1 is a quotient map, $(\kappa_1 \circ S^{-1} \circ \kappa_1^{-1})[V]$ is open. □

We conclude this section by giving an example of a morphism in $\mathsf{Gle^c}$ that is not a continuous relation, and a further remark.

Example 4.17 Let $X = \beta\mathbb{N}$ be the Stone-Čech compactification of the natural numbers and define E on X by letting E be the diagonal on $\mathbb{N}$ and setting xEy for all x, y in the remainder $\mathbb{N}^* := \beta\mathbb{N} \setminus \mathbb{N}$. Then $(X, E) \in \mathsf{Gle^c}$. Now define $R \subseteq X \times X$ by setting $R = E$. Clearly R is closed and $R \circ E = R = E \circ R$. Moreover, if U is a saturated open, then either $U \subseteq \mathbb{N}$ or $\mathbb{N}^* \subseteq U$. In either case $R^{-1}[U] = U$. Therefore, R is a morphism in $\mathsf{Gle^c}$. On the other hand, let U be clopen in X such that $U \cap \mathbb{N}^*, -U \cap \mathbb{N}^* \neq \emptyset$. Then $R^{-1}[U] = (U \cap \mathbb{N}) \cup \mathbb{N}^*$, which is not open. Thus, R is not a continuous relation.

Remark 4.18 Since the notion of a continuous relation is not symmetric, i. e. the inverse of a continuous relation is not continuous in general, the category $\mathsf{KHaus^c}$ possesses neither the structure of a dagger category nor that of an allegory. Nevertheless, an important feature emerges in $\mathsf{KHaus^c}$ that was absent in $\mathsf{KHaus^R}$. Namely, continuous relations $R : X \to Y$ become representable by continuous maps $X \to \mathcal{V}Y$ to the Vietoris space of Y.

By [22, Thm. 5.6], the Vietoris endofunctor on KHaus carries a monad structure. The multiplication transformation $\mathcal{V}\mathcal{V}X \to \mathcal{V}X$ of this monad assigns to a closed set in $\mathcal{V}X$ its union in X. It follows that the Kleisli composition for this monad corresponds to the relational composition in $\mathsf{KHaus^c}$, and hence $\mathsf{KHaus^c}$ is equivalent to the Kleisli category for this monad. For $\mathsf{KHaus}^\mathsf{R}$ there is no similar description since closed (but not necessarily continuous) relations correspond to continuous maps to $\mathcal{V}X$ with the Scott topology, and with this topology $\mathcal{V}X$ is not Hausdorff in general. However, the category of stably compact spaces with closed relations is equivalent to the Kleisli category of a monad [19, Prop. 2.10].

5 Compact Hausdorff Spaces and Interior Relations

In this section we restrict the dualities and equivalences obtained in the previous section for continuous relations to the setting of interior relations.

Definition 5.1 A closed relation R from a compact Hausdorff space X to a compact Hausdorff space Y is *interior* if for each open set $U \subseteq X$ and each open set $V \subseteq Y$, we have $R[U]$ and $R^{-1}[V]$ are open. In other words, R is interior provided both R and R^{-1} are continuous.

A function between topological spaces is called an *interior function* if it is continuous and open. As noted before, a function between compact Hausdorff spaces is continuous iff it is closed when considered as a relation between the spaces. Thus, the interior functions between compact Hausdorff spaces are exactly the interior relations between the spaces that are functions. Noting that the composite of interior relations is an interior relation immediately gives the following.

Proposition 5.2 *The collection of compact Hausdorff spaces with the interior relations between them forms a category* $\mathsf{KHaus}^\mathsf{i}$ *that is a wide subcategory of* $\mathsf{KHaus}^\mathsf{c}$.

Remark 5.3 It would be natural to consider also *open* relations from a compact Hausdorff space X to a compact Hausdorff space Y. These are closed relations R from X to Y where $R[U]$ is an open subset of Y for each open $U \subseteq X$. However, the category of compact Hausdorff spaces and open relations is simply the opposite of the category $\mathsf{KHaus}^\mathsf{c}$ of compact Hausdorff spaces and continuous relations, so little new is gained from this direction.

Definition 5.4 Let $\mathsf{KRFrm}^\mathsf{i}$ be the category whose objects are compact regular frames and whose morphisms are the morphisms $\Box : L \to M$ of $\mathsf{KRFrm}^\mathsf{c}$ that have a left adjoint. We call such morphisms i-*morphisms*.

Clearly $\mathsf{KRFrm}^\mathsf{i}$ is a wide subcategory of $\mathsf{KRFrm}^\mathsf{c}$.

Theorem 5.5 $\mathsf{KHaus}^\mathsf{i}$ *is dually equivalent to* $\mathsf{KRFrm}^\mathsf{i}$.

Proof By Theorem 4.8, there is a dual equivalence between $\mathsf{KHaus}^\mathsf{c}$ and $\mathsf{KRFrm}^\mathsf{c}$ given by the restrictions of the functors $\mathcal{O}^\mathsf{R}$ and $\mathfrak{pt}^\mathsf{R}$. It is enough to show that these functors restrict further to functors between $\mathsf{KHaus}^\mathsf{i}$ and $\mathsf{KRFrm}^\mathsf{i}$. To do this, we must show that if R is an interior relation, then $\Box_R$ is an i-morphism; and if $\Box$ is an i-morphism, then $R_\Box$ is an interior relation. This means we must show that if R is an interior relation, then $\Box_R$ has a left adjoint; and if $\Box$ is a c-morphism with a left adjoint, then $R_\Box$ is an interior relation.

Suppose X, Y are compact Hausdorff spaces and $R \subseteq X \times Y$ is an interior relation. Then $R[\cdot]$ is an order preserving function from $\mathcal{O}X$ to $\mathcal{O}Y$. Let $U \subseteq X$ and $V \subseteq Y$ be open. Since $\Box_R(V) = \{x \mid R[x] \subseteq V\}$, we have $(R \circ \Box_R)(V) \subseteq V$, so $R \circ \Box_R \le id_{\mathcal{O}Y}$. Also, $U \subseteq \{x \mid R[x] \subseteq R[U]\} = (\Box_R \circ R)(U)$, so $id_{\mathcal{O}X} \le \Box_R \circ R$. Thus, R is a left adjoint of $\Box_R$.

Suppose $\Box : L \to M$ is a c-morphism between compact regular frames that has a left adjoint $f : M \to L$. Earlier results provide that $R_\Box$ is a continuous relation from $\mathfrak{pt}M$ to $\mathfrak{pt}L$. An open set of $\mathfrak{pt}M$ is of the form $\phi(a) = \{q \in \mathfrak{pt}M \mid q(a) = 1\}$. By definition, $R_\Box[\phi(a)] = \{p \in \mathfrak{pt}L \mid q \circ \Box \leq p$ for some q with $q(a) = 1\}$. We claim that $R_\Box[\phi(a)] = \phi(f(a))$, so $R_\Box$ is open, hence interior.

Suppose $p \in R_\Box[\phi(a)]$, so there is $q \in \phi(a)$ with $q \circ \Box \leq p$. Then $q(\Box f(a)) \leq p(f(a))$. But f being a left adjoint of $\Box$ gives that $id_L \leq \Box \circ f$, hence $a \leq \Box f(a)$. Therefore, $1 = q(a) \leq q(\Box f(a)) \leq p(f(a))$, showing $p \in \phi(f(a))$. For the other containment, suppose $p \in \phi(f(a))$. Since p is a point, it preserves arbitrary joins, so there is a largest element $c \in L$ with $p(c) = 0$. As $p(f(a)) = 1$, we have $f(a) \nleq c$. Because f is the left adjoint of $\Box$, we have $a \nleq \Box c$. But then there is a point q of M with $q(a) = 1$ and $q(\Box c) = 0$. For any d with $p(d) = 0$, we have $d \leq c$, hence $\Box d \leq \Box c$, so $q(\Box d) \leq q(\Box c) = 0$. Therefore, $q \circ \Box \leq p$. Thus, $p \in R_\Box[\phi(a)]$. □

In the setting of maps between Boolean algebras, there is another feature to adjoints. Suppose A and B are Boolean algebras and $f : A \to B$ and $g : B \to A$ have f left adjoint to g and so g right adjoint to f. Let $h : B \to A$ be the dual of g given by $h(b) = \neg g(\neg b)$. Then f, h have the following property which defines them to be *conjugates*:

$$f(a) \wedge b = 0 \quad \text{iff} \quad a \wedge h(b) = 0.$$

It is easily seen that g is right adjoint to f iff f, h are conjugates. Thus, if a map has a conjugate, it is unique. The property of being conjugates is symmetric, and each map involved in a pair of conjugates preserves arbitrary joins.

Proposition 5.6 *There is a wide subcategory* $\mathsf{DeV}^{\mathsf{i}}$ *of* $\mathsf{DeV}^{\mathsf{c}}$ *whose morphisms are the morphisms* $\Diamond$ *of* $\mathsf{DeV}^{\mathsf{c}}$ *that have conjugates that belong to* $\mathsf{DeV}^{\mathsf{c}}$.

Proof Suppose $\Diamond_1 : A \to B$ and $\Diamond_2 : B \to C$ are morphisms in $\mathsf{DeV}^{\mathsf{c}}$ that have conjugates $h_2 : C \to B$ and $h_1 : B \to A$ that also belong to $\mathsf{DeV}^{\mathsf{c}}$. It is easily seen that the usual function composites $\Diamond_2\Diamond_1$ and h_1h_2 are conjugates. We must show that the composites $\Diamond_2 \star \Diamond_1$ and $h_1 \star h_2$ in $\mathsf{DeV}^{\mathsf{c}}$ are conjugates. Earlier results show that they indeed belong to $\mathsf{DeV}^{\mathsf{c}}$.

In the following, we make use of the infinite distributive law in the complete Boolean algebras A and C, the definition of composition $\star$, that $\Diamond_2\Diamond_1$ and h_1h_2 are conjugates, and the fact that $a = \bigvee\{a' \mid a' \prec a\}$. If $(\Diamond_2 \star \Diamond_1)(a) \wedge c = 0$, then $\bigvee\{\Diamond_2\Diamond_1 a' \mid a' \prec a\} \wedge c = 0$, so $(\Diamond_2\Diamond_1 a') \wedge c = 0$ for each $a' \prec a$. This gives $a' \wedge (h_1h_2c) = 0$ for each $a' \prec a$, hence $a' \wedge (h_1h_2c') = 0$ for each $a' \prec a$ and $c' \prec c$. So $a' \wedge \bigvee\{h_1h_2c' \mid c' \prec c\} = 0$ for each $a' \prec a$, giving $a' \wedge (h_1 \star h_2)(c) = 0$. Since this is true for each $a' \prec a$, we have $\bigvee\{a' \mid a' \prec a\} \wedge (h_1 \star h_2)(c) = 0$, hence $a \wedge (h_1 \star h_2)(c) = 0$. Showing that $a \wedge (h_1 \star h_2)(c) = 0$ implies $(\Diamond_2 \star \Diamond_1)(a) \wedge c = 0$ is similar. □

Theorem 5.7 *There is a dual equivalence between* $\mathsf{KHaus}^{\mathsf{i}}$ *and* $\mathsf{DeV}^{\mathsf{i}}$.

Proof Theorem 4.14 provides a dual equivalence between $\mathsf{KHaus}^{\mathsf{c}}$ and $\mathsf{DeV}^{\mathsf{c}}$. We must show that the functors giving this equivalence restrict to functors between $\mathsf{KHaus}^{\mathsf{i}}$ and $\mathsf{DeV}^{\mathsf{i}}$. So if X and Y are compact Hausdorff spaces and $R \subseteq X \times Y$ is an interior relation, we must show that $\Diamond_R : \mathcal{RO}Y \to \mathcal{RO}X$ given by $\Diamond_R U = \mathsf{IC}\, R^{-1}[U]$ has a conjugate that belongs to $\mathsf{DeV}^{\mathsf{c}}$; and if $\Diamond : (A, \prec) \to (B, \prec)$ is a morphism in DeV^{i} with a conjugate that belongs to $\mathsf{DeV}^{\mathsf{c}}$, we must show that the relation $R_\Diamond$ from $\mathsf{End}\, B$ to $\mathsf{End}\, A$ given by $E\, R_\Diamond\, F$ iff $\Diamond[F] \subseteq E$ is interior.

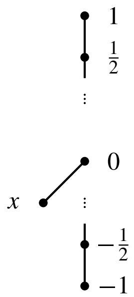
Suppose $R \subseteq X \times Y$ is interior and let $h : \mathcal{RO}X \to \mathcal{RO}Y$ be given by $h(V) = \mathsf{IC}\, R[V]$. Note that since R is interior, so is R^{-1}. Thus, $h = \Diamond_{R^{-1}}$, and hence belongs to $\mathsf{DeV}^{\mathsf{c}}$. Suppose

$U \in \mathcal{RO}X$ and $V \in \mathcal{RO}Y$ with $\Diamond_R V \wedge U = 0$. Then $\mathsf{IC}\, R^{-1}[V] \cap U = \emptyset$, and as $R^{-1}[V]$ is open, $R^{-1}[V] \cap U = \emptyset$. Thus, no element of U is R-related to an element of V, hence $V \cap R[U] = \emptyset$. Since V, $R[U]$ are open, we have $V \cap \mathsf{C}\, R[U] = \emptyset$, so $V \cap \mathsf{IC}\, R[U] = \emptyset$, and hence $V \wedge \Diamond_{R^{-1}} U = 0$. Showing that $V \wedge \Diamond_{R^{-1}} U = 0$ implies $\Diamond_R V \wedge U = 0$ follows by symmetry. Thus, $\Diamond_{R^{-1}}$ is a conjugate of $\Diamond_R$ that belongs to $\mathsf{DeV^c}$.

Suppose $\Diamond : A \to B$ is a morphism in $\mathsf{DeV^c}$ that has a conjugate $\Diamond' : B \to A$ that belongs to $\mathsf{DeV^c}$. Then $R_\Diamond$ is the relation from $\mathsf{End}\, B$ to $\mathsf{End}\, A$ defined by $E\, R_\Diamond\, F$ iff $\Diamond[F] \subseteq E$, and $R_{\Diamond'}$ is the relation from $\mathsf{End}\, A$ to $\mathsf{End}\, B$ defined by $F\, R_{\Diamond'}\, E$ iff $\Diamond'[E] \subseteq F$. We claim that $E\, R_\Diamond\, F$ iff $F\, R_{\Diamond'}\, E$, so $R_{\Diamond'} = R_\Diamond^{-1}$. Results regarding $\mathsf{DeV^c}$ then provide that both $R_\Diamond$ and $R_\Diamond^{-1}$ are continuous, and hence $R_\Diamond$ is interior.

Suppose $E \in \mathsf{End}\, B$ and $F \in \mathsf{End}\, A$ with $\Diamond[F] \subseteq E$. We show $\Diamond'[E] \subseteq F$. Let $a \in \Diamond'[E]$. Then $a = \Diamond' b$ for some $b \in E$. A basic property of ends [9, Thm. I.2.2] provides that if $x \prec y$, then either $\neg x$ or y belongs to the end. If $b' \prec b$, de Vries additivity of $\Diamond'$ gives $\Diamond' b' \prec \Diamond' b$. Therefore, if $a = \Diamond' b$ does not belong to F, then $\neg\Diamond' b' \in F$ for each $b' \prec b$. The assumption that $\Diamond[F] \subseteq E$ then gives that $\Diamond\neg\Diamond' b' \in E$ for each $b' \prec b$. Thus, $\Diamond\neg\Diamond'\neg\neg b' \in E$ for each $b' \prec b$. Letting $\Box' = \neg\Diamond'\neg$, we have that $\Box'$ is right adjoint to $\Diamond$, and hence $\Diamond\Box' \leq id_B$. Since $\Diamond\Box'\neg b' \in E$ for each $b' \prec b$ and $\Diamond\Box'\neg b' \leq \neg b'$, we then have that $\neg b' \in E$ for each $b' \prec b$. So $b' \notin E$ for each $b' \prec b$. Since $b \in E$, this contradicts that E is round. This contradiction shows that $a = \Diamond' b \in F$, and therefore that $\Diamond'[E] \subseteq F$. Symmetry provides that $\Diamond'[E] \subseteq F$ implies $\Diamond[F] \subseteq E$. □

Example 5.8 As follows from Proposition 5.6, morphisms in $\mathsf{DeV^i}$ are morphisms $\Diamond$ in $\mathsf{DeV^c}$ that have a conjugate that is a morphism in $\mathsf{DeV^c}$. It is not the case that if a morphism $\Diamond$ in $\mathsf{DeV^c}$ has a conjugate h, then h is a morphism in $\mathsf{DeV^c}$. Let $S = \{1/n \mid n \in \mathbb{Z} \setminus \{0\}\}$ and consider $S \cup \{x\}$ where $x \notin S$ and $S \cup \{x\}$ is given the discrete topology. Let X be the one-point compactification of $S \cup \{x\}$. Clearly X is compact Hausdorff. Let 0 be the compactification point. Define $R \subseteq X \times X$ to be the union of the usual partial ordering on $S \cup \{0\}$ and $\{(x, 1/n) \mid n \geq 1\} \cup \{(x, 0), (x, x)\}$. One verifies that R is a continuous relation from X to X that is not interior. Then $\Diamond_R$ is a morphism in $\mathsf{DeV^c}$ that has a conjugate $h = \mathsf{IC}\, R$, but the conjugate is not a morphism in $\mathsf{DeV^i}$.



Recall that for (X, E) a Gleason space, $U \subseteq X$ is *saturated* if $E[U] = U$. Note that if R is a morphism in $\mathsf{Gle^R}$ from (X, E) to (X', E'), then $R \circ E = R = E' \circ R$, giving that the image of any $S \subseteq X$ is saturated. The following is then obvious.

Proposition 5.9 *There is a wide subcategory* $\mathsf{Gle^i}$ *of* $\mathsf{Gle^c}$ *whose morphisms* $R \subseteq X \times X'$ *in addition satisfy that* $R[U]$ *is open when* U *is a saturated open of* X.

Theorem 5.10 $\mathsf{KHaus^i}$ *is equivalent to* $\mathsf{Gle^i}$.

Proof Theorem 4.16 provides that $\mathsf{KHaus^c}$ is equivalent to $\mathsf{Gle^c}$. We need only show that the functors involved in this equivalence restrict to functors between $\mathsf{KHaus^i}$ and $\mathsf{Gle^i}$. So if X_1 and X_2 are compact Hausdorff spaces and $R \subseteq X_1 \times X_2$ is an interior relation, we must show that the relation $\pi_2^{-1} \circ R \circ \pi_1$ from $\widehat{X_1}$ to $\widehat{X_2}$ has the image of a saturated open set open. Also, if R is a morphism in $\mathsf{Gle^i}$ between Gleason spaces (X_1, E_1) and (X_2, E_2), we must show that the relation $\kappa_2 \circ R \circ \kappa_1^{-1}$ from X_1/E_1 to X_2/E_2 has the image of an open set open.

Suppose $R \subseteq X_1 \times X_2$ is an interior relation. Then $R^{-1} \subseteq X_2 \times X_1$ is a continuous relation. So our earlier results provide that the preimage under $\pi_1^{-1} \circ R^{-1} \circ \pi_2$ of a saturated open subset $U \subseteq X_1$ is open. But $(\pi_1^{-1} \circ R^{-1} \circ \pi_2)^{-1}[U] = (\pi_2^{-1} \circ R \circ \pi_1)[U]$ is open, as required.

Suppose R is a morphism in $\mathsf{Gle^c}$ from (X_1, E_1) to (X_2, E_2) so that for a saturated open set $U \subseteq X_1$ we have $R[U]$ is open. Then R^{-1} is a morphism in $\mathsf{Gle^c}$. So $\kappa_2 \circ R \circ \kappa_1^{-1}$ and $\kappa_1 \circ R^{-1} \circ \kappa_2^{-1}$ are continuous relations. Since they are inverse to each other, $\kappa_2 \circ R \circ \kappa_1^{-1}$ is an interior relation. □

Remark 5.11 In Remark 3.15 we described further properties of $\mathsf{KHaus^R}$. In particular, it is a self-dual category, and even a strongly compact closed category with biproducts. It is routine to verify that the biproduct injections and projections in $\mathsf{KHaus^R}$ are interior relations, and that the tensor product of interior relations is interior. This yields that $\mathsf{KHaus^i}$ is a monoidal category with biproducts. As before, the dagger on morphisms is given by relational inverse, biproducts by disjoint sums, and the Cartesian product provides an additional symmetric monoidal structure. However, unlike $\mathsf{KHaus^R}$, the monoidal structure is not closed since the evaluation and coevaluation morphisms required for the closed monoidal structure are not interior. Indeed, it is easy to see that for compact Hausdorff spaces X, Y these should be given by the relation between $X \times X \times Y$ and Y defined by the diagonal embedding of $X \times Y$ into $X \times X \times Y \times Y$, so the resulting map $X \times Y \to X \times X \times Y$ is not open, hence the relation is not interior.

Remark 5.12 In Remark 3.16 we saw that $\mathsf{KHaus^R}$ is a tabular allegory. Since a relation $R : X \to Y$ is interior iff its inverse is interior iff both projections $R \to X$ and $R \to Y$ are interior, it is easy to see that $\mathsf{KHaus^i}$ is also a tabular allegory. However, the Vietoris functor does not equip $\mathsf{KHaus^i}$ with the structure of a power allegory because interior relations $R : X \to Y$ no longer correspond to interior maps $X \to \mathcal{V}Y$ to the Vietoris space of Y. For example, while the identity relation $R : X \to X$ is interior, the corresponding map $X \to \mathcal{V}X$, assigning to $x \in X$ the singleton $\{x\} \in \mathcal{V}X$, is not in general an interior map.

6 Restricting to KHaus, KRFrm, and DeV

In this section we restrict the dualities and equivalences established in the previous sections to the setting of compact Hausdorff spaces and continuous functions. The duality between $\mathsf{KHaus^R}$ and $\mathsf{KRFrm^R}$ that restricts to a duality between $\mathsf{KHaus^c}$ and $\mathsf{KRFrm^c}$ further restricts to Isbell duality between KHaus and KRFrm. We show that the duality between $\mathsf{KHaus^c}$ and $\mathsf{DeV^c}$ restricts to de Vries duality between KHaus and DeV, and that the equivalence between $\mathsf{KHaus^R}$ and $\mathsf{Gle^R}$ that restricts to an equivalence between $\mathsf{KHaus^c}$ and $\mathsf{Gle^c}$ further restricts to an equivalence between KHaus and Gle. As a consequence, we obtain dualities and equivalences between the wide subcategory of KHaus whose morphisms are interior functions and the corresponding wide subcategories of KRFrm, DeV, and Gle.

Theorem 6.1 *The dual equivalence between* $\mathsf{KHaus}^{\mathsf{R}}$ *and* $\mathsf{KRFrm}^{\mathsf{R}}$ *given by* $\mathcal{O}^{\mathsf{R}}$ *and* $\mathfrak{pt}^{\mathsf{R}}$ *restricts to Isbell duality between* KHaus *and* KRFrm *that is given by* $\mathcal{O}$ *and* $\mathfrak{pt}$*. Therefore, the dual equivalence between* $\mathsf{KHaus}^{\mathsf{c}}$ *and* $\mathsf{KRFrm}^{\mathsf{c}}$ *also restricts to Isbell duality.*

Proof That the dual equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{KRFrm}^{\mathsf{R}}$ restricts to Isbell duality follows from [19, Cor. 3.7]. Since $\mathsf{KHaus} \subseteq \mathsf{KHaus}^{\mathsf{c}} \subseteq \mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{KRFrm} \subseteq \mathsf{KRFrm}^{\mathsf{c}} \subseteq \mathsf{KRFrm}^{\mathsf{R}}$, we then have that the dual equivalence between $\mathsf{KHaus}^{\mathsf{c}}$ and $\mathsf{KRFrm}^{\mathsf{c}}$ also restricts to Isbell duality. □

Theorem 6.2 *The category* DeV *is a wide subcategory of* $\mathsf{DeV}^{\mathsf{c}}$ *and the dual equivalence between* $\mathsf{KHaus}^{\mathsf{c}}$ *and* $\mathsf{DeV}^{\mathsf{c}}$ *given by* $\mathcal{RO}^{\mathsf{c}}$ *and* $\mathsf{End}^{\mathsf{c}}$ *restricts to de Vries duality between the subcategories* KHaus *and* DeV.

Proof Clearly the objects of DeV are those of $\mathsf{DeV}^{\mathsf{c}}$, and the rule of composition is the same in both categories. By [4, Lem. 2.2], every de Vries morphism is de Vries additive. Thus, DeV is a wide subcategory of $\mathsf{DeV}^{\mathsf{c}}$.

Suppose $\Diamond : A \to B$ is a morphism in DeV. Then $\mathsf{End}^{\mathsf{R}}(\Diamond)$ is the continuous relation $R_\Diamond$ from $\mathsf{End}\, B$ to $\mathsf{End}\, A$ given by $E \, R_\Diamond \, F$ iff $\Diamond[F] \subseteq E$. We have $\Diamond[F] \subseteq E$ iff $F \subseteq \Diamond^{-1}[E]$. Since F is round, this is equivalent to $F \subseteq \Uparrow\Diamond^{-1}[E]$. By de Vries duality, $\Uparrow\Diamond^{-1}[E]$ is an end, yielding that $F = \Uparrow\Diamond^{-1}[E]$. Therefore, $\Uparrow\Diamond^{-1}[E]$ is a unique end F of A satisfying $\Diamond[F] \subseteq E$. Thus, $\mathsf{End}^{\mathsf{R}}(\Diamond) = \mathsf{End}(\Diamond)$.

Let $f : X \to Y$ be a continuous function between compact Hausdorff spaces. Then $\mathcal{RO}^{\mathsf{c}}(f)$ is the map $\Diamond_f : \mathcal{RO}Y \to \mathcal{RO}X$ given by $\Diamond_f U = \mathsf{IC}\, f^{-1}[U]$. This is identical to the definition of $\mathcal{RO}(f)$. Thus, the dual equivalence between $\mathsf{KHaus}^{\mathsf{c}}$ and $\mathsf{DeV}^{\mathsf{c}}$ restricts to de Vries duality between KHaus and DeV. □

We next turn our attention to the equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{Gle}^{\mathsf{R}}$ given by the functors $\mathcal{G}^{\mathsf{R}}$ and $\mathcal{Q}^{\mathsf{R}}$, and its restriction to an equivalence between $\mathsf{KHaus}^{\mathsf{c}}$ and $\mathsf{Gle}^{\mathsf{c}}$. We wish to restrict this further to the setting of KHaus. To do so, we must isolate a wide subcategory of $\mathsf{Gle}^{\mathsf{c}}$ whose morphisms correspond to the $\mathcal{G}^{\mathsf{R}}(f)$ where f is a continuous function between compact Hausdorff spaces.

Definition 6.3 A morphism R in $\mathsf{Gle}^{\mathsf{R}}$ between Gleason spaces (X_1, E_1) and (X_2, E_2) is *functional* if $E_1 \subseteq R^{-1} \circ R$ and $R \circ R^{-1} \subseteq E_2$.

The ordinary notion of adjunction has an analogue when the category is enriched in posets; that is, when homsets carry the structure of a poset. Then for morphisms $f : A \to B$ and $g : B \to A$ we say that f is *left adjoint* to g and g is *right adjoint* to f provided $1_A \leq g \circ f$ and $f \circ g \leq 1_B$.

Proposition 6.4 *A morphism* $R : (X_1, E_1) \to (X_2, E_2)$ *in* $\mathsf{Gle}^{\mathsf{R}}$ *is functional iff it has a right adjoint.*

Proof For $i = 1, 2$ since E_i is the identity on (X_i, E_i), it follows from the definition that if R is functional, then R^{-1} is the right adjoint of R. Conversely, suppose that $S : (X_2, E_2) \to (X_1, E_1)$ is the right adjoint of R. To see that $S = R^{-1}$, let ySx. As xE_1x and $E_1 \subseteq S \circ R$, there is $y' \in X_2$ such that xRy' and $y'Sx$. Therefore, ySx and xRy'. Because $R \circ S \subseteq E_2$, this yields yE_2y'. Thus, xRy' and $y'E_2y$, which gives xRy. Consequently, $S \subseteq R^{-1}$. The reverse inclusion is proved similarly. □

If an equivalence between categories enriched in posets preserves the order of homsets, then morphisms that have right adjoints in one category transfer to morphisms that have right adjoints in the other. For an order-enriched category, its *category of adjunctions* is its wide subcategory whose morphisms have a right adjoint. It follows from [19, Cor. 4.9] that a morphism in $\mathsf{KHaus}^{\mathsf{R}}$ has a right adjoint iff it is a continuous function. (Note that [19] uses $\supseteq$ rather than $\subseteq$ as the order on the homsets.) Thus, KHaus is the category of adjunctions of $\mathsf{KHaus}^{\mathsf{R}}$. We next consider the category of adjunctions in $\mathsf{Gle}^{\mathsf{R}}$.

Definition 6.5 Let Gle be the wide subcategory of $\mathsf{Gle}^{\mathsf{R}}$ whose morphisms are functional (i.e. Gle is the category of adjunctions in $\mathsf{Gle}^{\mathsf{R}}$).

Proposition 6.6 Gle *is a wide subcategory of* $\mathsf{Gle}^{\mathsf{C}}$.

Proof Let $R : (X_1, E_1) \to (X_2, E_2)$ be a morphism in Gle. To see that R is a morphism in $\mathsf{Gle}^{\mathsf{C}}$, let U be saturated open in X_2. We claim that $R^{-1}[U] = -R^{-1}[-U]$. Clearly $-R^{-1}[-U] \subseteq R^{-1}[U]$. Suppose $x \in R^{-1}[U]$. Then there is $y \in U$ with xRy. To see that $R[x] \subseteq U$, let xRz. Since R is functional, yE_2z. As U is saturated, $y \in U$ and yE_2z imply $z \in U$. Therefore, $R[x] \subseteq U$, and hence $x \in -R^{-1}[-U]$. Since R is a closed relation, $-R^{-1}[-U]$ is open, so $R^{-1}[U]$ is open, and so R is a morphism in $\mathsf{Gle}^{\mathsf{C}}$. □

Since the equivalence of Theorem 3.13 between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{Gle}^{\mathsf{R}}$ preserves the order of homsets, it restricts to an equivalence between their categories of adjunctions. Thus, by Theorem 4.16 and Proposition 6.6, we arrive at the following theorem.

Theorem 6.7 *The functors $\mathcal{G}^{\mathsf{R}}$ and $\mathcal{Q}^{\mathsf{R}}$ giving an equivalence between* $\mathsf{KHaus}^{\mathsf{R}}$ *and* $\mathsf{Gle}^{\mathsf{R}}$ *that restricts to an equivalence between* $\mathsf{KHaus}^{\mathsf{C}}$ *and* $\mathsf{Gle}^{\mathsf{C}}$ *further restricts to an equivalence between* KHaus *and* Gle.

Remark 6.8 It is instructive to see more directly why morphisms in Gle correspond to morphisms in KHaus. Suppose $R : (X_1, E_1) \to (X_2, E_2)$ is a morphism in Gle. Then $\mathcal{Q}^{\mathsf{R}}(R) : X_1/E_1 \to X_2/E_2$ is the relation $\kappa_2 \circ R \circ \kappa_1^{-1}$ where $\kappa_i : X_i \to X_i/E_i$ is the quotient map. We know that $\mathcal{Q}^{\mathsf{R}}(R)$ is a continuous relation, so we only need to show that it is a function. Let $x_1 \in X_1$. Since R is functional we have (i) $E_1 \subseteq R^{-1} \circ R$ and (ii) $R \circ R^{-1} \subseteq E_2$. By (i) there is $x_2 \in X_2$ with $x_1 \, R \, x_2$. Thus,

$$x_1/E_1 \;\; \kappa_1^{-1} \;\; x_1 \;\; R \;\; x_2 \;\; \kappa_2 \;\; x_2/E_2.$$

So each element of X_1/E_1 is $\mathcal{Q}^{\mathsf{R}}(R)$-related to an element of X_2/E_2. We must show it is related to only one such element. Suppose $x_1' \in X_1$ and $x_2' \in X_2$ are such that

$$x_1/E_1 \;\; \kappa_1^{-1} \;\; x_1' \;\; R \;\; x_2' \;\; \kappa_2 \;\; x_2'/E_2.$$

Then $x_1 \, E_1 \, x_1'$. So $x_1 \, E_1 \, x_1' \, R \, x_2'$. Since R is a morphism in $\mathsf{Gle}^{\mathsf{R}}$ we have $x_1 \, R \, x_2'$. Then $x_2' \, R^{-1} \, x_1 \, R \, x_2$. By (ii) we have $x_2 \, E_2 \, x_2'$. This gives $x_2/E_2 = x_2'/E_2$. So $\mathcal{Q}^{\mathsf{R}}(R)$ is a function.

Remark 6.9 As we pointed out in Remark 3.7, $\mathsf{KHaus}^{\mathsf{R}}$ has a dagger structure given by $R^\dagger = R^{-1}$. This dagger structure carries to $\mathsf{KRFrm}^{\mathsf{R}}$ and $\mathsf{Gle}^{\mathsf{R}}$. For $\mathsf{Gle}^{\mathsf{R}}$ the dagger is again given by the relational inverse, but the description of the dagger in $\mathsf{KRFrm}^{\mathsf{R}}$ is more involved (see Remark 3.17). It is noteworthy that when a morphism in $\mathsf{KHaus}^{\mathsf{R}}$ has a right adjoint, then it is given by the dagger. This follows from [19, Cor. 4.9] and the observation that for a continuous function $f : X \to Y$ we have $1_X \subseteq f^{-1} \circ f$ and $f \circ f^{-1} \subseteq 1_Y$. This carries over to $\mathsf{Gle}^{\mathsf{R}}$ as is witnessed by Proposition 6.4. It also carries to $\mathsf{KRFrm}^{\mathsf{R}}$, but a direct proof is not obvious.

Remark 6.10 The categories Gle and Gle_0 are equivalent since both are dually equivalent to DeV. The equivalence can be obtained directly as follows. The functor $\mathcal{S} : \mathsf{Gle}_0 \to \mathsf{Gle}$ is identity on the objects and sends a morphism S in Gle_0 to the morphism $R = E \circ S \circ E$ in Gle. The functor $\mathcal{T} : \mathsf{Gle} \to \mathsf{Gle}_0$ is also identity on the objects and sends a morphism R in Gle to the morphism S in Gle_0 given by $x\,S\,y$ iff $x \in \bigcap\{\mathsf{I}R^{-1}[U] \mid y \in U\}$, where U ranges over clopens. Intuitively, since R is not continuous in general, S is a continuous approximation of R.

Remark 6.11 It is natural to consider the intersection of the subcategories KHaus and $\mathsf{KHaus}^{\mathsf{i}}$ of $\mathsf{KHaus}^{\mathsf{R}}$ as they yield the category of compact Hausdorff spaces and interior maps (continuous open maps). Considering our other categories as well, we define:

$$\mathsf{KHaus}^{\mathsf{o}} = \mathsf{KHaus} \cap \mathsf{KHaus}^{\mathsf{i}};$$
$$\mathsf{KRFrm}^{\mathsf{o}} = \mathsf{KRFrm} \cap \mathsf{KRFrm}^{\mathsf{i}};$$
$$\mathsf{DeV}^{\mathsf{o}} = \mathsf{DeV} \cap \mathsf{DeV}^{\mathsf{i}};$$
$$\mathsf{Gle}^{\mathsf{o}} = \mathsf{Gle} \cap \mathsf{Gle}^{\mathsf{i}}.$$

Then Theorems 5.5 and 6.1 yield that $\mathsf{KHaus}^{\mathsf{o}}$ is dually equivalent to $\mathsf{KRFrm}^{\mathsf{o}}$. Similarly, Theorems 5.7 and 6.2 give that $\mathsf{KHaus}^{\mathsf{o}}$ is dually equivalent to $\mathsf{DeV}^{\mathsf{o}}$, and Theorems 5.10 and 6.7 provide an equivalence between $\mathsf{KHaus}^{\mathsf{o}}$ and $\mathsf{Gle}^{\mathsf{o}}$. Thus, $\mathsf{KRFrm}^{\mathsf{o}}$ is equivalent to $\mathsf{DeV}^{\mathsf{o}}$, and both are dually equivalent to $\mathsf{Gle}^{\mathsf{o}}$.

Remark 6.12 By the dual equivalence between $\mathsf{KHaus}^{\mathsf{o}}$ and $\mathsf{KRFrm}^{\mathsf{o}}$, an interior map corresponds to a frame homomorphism having a left adjoint, hence to a complete lattice homomorphism. In general, a continuous map between topological spaces is interior iff the corresponding frame homomorphism $h : L \to M$ has a left adjoint $g : M \to L$ satisfying the *Frobenius equality*

$$g(h(a) \wedge b) = a \wedge g(b).$$

However, if L is regular (or more generally subfit), then the Frobenius equality always holds (see, e.g., [20, V.1.8]). Thus, we do not need to require the Frobenius equality.

We conclude the paper with several tables that list the categories considered, and equivalences and dual equivalences established.

Categories of compact Hausdorff spaces

Category	Morphisms	Appears
$\mathsf{KHaus}^{\mathsf{R}}$	closed relations	3.3
$\mathsf{KHaus}^{\mathsf{c}}$	continuous relations	4.2
$\mathsf{KHaus}^{\mathsf{i}}$	interior relations	5.2
KHaus	continuous maps	2.1
$\mathsf{KHaus}^{\mathsf{o}}$	interior maps	6.11

Categories of Gleason spaces

Category	Morphisms	Appears
$\mathsf{Gle}^{\mathsf{R}}$	closed relations with $R \circ E = R = E' \circ R$	3.6
$\mathsf{Gle}^{\mathsf{c}}$	morphisms in $\mathsf{Gle}^{\mathsf{R}}$ with $R^{-1}[U]$ open for each saturated open U	4.15
$\mathsf{Gle}^{\mathsf{i}}$	morphisms in $\mathsf{Gle}^{\mathsf{c}}$ with $R[U]$ open for each saturated open U	5.9
Gle	functional morphisms	6.5
Gle_0	as in [5]	2.4
$\mathsf{Gle}^{\mathsf{o}}$	functional morphisms in $\mathsf{Gle}^{\mathsf{i}}$	6.11

Categories of compact regular frames

Category	Morphisms	Appears
$\mathsf{KRFrm}^{\mathsf{R}}$	preframe homomorphisms	3.4
$\mathsf{KRFrm}^{\mathsf{C}}$	c-morphisms	4.6
$\mathsf{KRFrm}^{\mathsf{i}}$	i-morphisms	5.4
KRFrm	frame homomorphisms	2.2
$\mathsf{KRFrm}^{\mathsf{O}}$	i-morphisms that are frame homomorphisms	6.11

Categories of de Vries algebras

Category	Morphisms	Appears
$\mathsf{DeV}^{\mathsf{C}}$	de Vries additive $\Diamond : A \to B$ with $\Diamond a = \bigvee\{\Diamond b \mid b \prec a\}$	4.10
$\mathsf{DeV}^{\mathsf{i}}$	morphisms in $\mathsf{DeV}^{\mathsf{C}}$ that have conjugates in $\mathsf{DeV}^{\mathsf{C}}$	5.6
DeV	de Vries morphisms	2.3
$\mathsf{DeV}^{\mathsf{O}}$	de Vries morphisms that have conjugates in $\mathsf{DeV}^{\mathsf{C}}$	6.11

Equivalences

Functors	Appears
$\left(\mathsf{KHaus}^{\mathsf{R}}\right)^{op} \underset{\mathfrak{pt}^{\mathsf{R}}}{\overset{\mathcal{O}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{KRFrm}^{\mathsf{R}}$	3.10
$\mathsf{KHaus}^{\mathsf{R}} \underset{\mathcal{Q}^{\mathsf{R}}}{\overset{\mathcal{G}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{Gle}^{\mathsf{R}}$	3.13
$(\mathsf{KHaus}^{\mathsf{C}})^{op} \underset{\mathfrak{pt}^{\mathsf{R}}}{\overset{\mathcal{O}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{KRFrm}^{\mathsf{C}}$	4.8
$(\mathsf{KHaus}^{\mathsf{C}})^{op} \underset{\mathsf{End}^{\mathsf{C}}}{\overset{\mathcal{RO}^{\mathsf{C}}}{\rightleftarrows}} \mathsf{DeV}^{\mathsf{C}}$	4.14
$\mathsf{KHaus}^{\mathsf{C}} \underset{\mathcal{Q}^{\mathsf{R}}}{\overset{\mathcal{G}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{Gle}^{\mathsf{C}}$	4.16
$\left(\mathsf{KHaus}^{\mathsf{i}}\right)^{op} \underset{\mathfrak{pt}^{\mathsf{R}}}{\overset{\mathcal{O}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{KRFrm}^{\mathsf{i}}$	5.5
$\left(\mathsf{KHaus}^{\mathsf{i}}\right)^{op} \underset{\mathsf{End}^{\mathsf{C}}}{\overset{\mathcal{RO}^{\mathsf{C}}}{\rightleftarrows}} \mathsf{DeV}^{\mathsf{i}}$	5.7
$\mathsf{KHaus}^{\mathsf{i}} \underset{\mathcal{Q}^{\mathsf{R}}}{\overset{\mathcal{G}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{Gle}^{\mathsf{i}}$	5.10
$\mathsf{KHaus}^{op} \underset{\mathfrak{pt}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{KRFrm}$	2.6
$\mathsf{KHaus}^{op} \underset{\mathsf{End}}{\overset{\mathcal{RO}}{\rightleftarrows}} \mathsf{DeV}$	2.7
$\mathsf{KHaus} \underset{\mathcal{Q}^{\mathsf{R}}}{\overset{\mathcal{G}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{Gle}$	6.7
$\mathsf{Gle}_0 \underset{\mathcal{T}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathsf{Gle}$	6.10
$(\mathsf{KHaus}^{\mathsf{O}})^{op} \underset{\mathfrak{pt}^{\mathsf{R}}}{\overset{\mathcal{O}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{KRFrm}^{\mathsf{O}}$	6.11

Functors	Appears
$(\mathsf{KHaus}^{\mathsf{O}})^{\mathrm{op}} \underset{\mathsf{End}^{\mathsf{C}}}{\overset{\mathcal{RO}^{\mathsf{C}}}{\rightleftarrows}} \mathsf{DeV}^{\mathsf{O}}$	6.11
$\mathsf{KHaus}^{\mathsf{O}} \underset{\mathcal{Q}^{\mathsf{R}}}{\overset{\mathcal{G}^{\mathsf{R}}}{\rightleftarrows}} \mathsf{Gle}^{\mathsf{O}}$	6.11

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