Spectral and T_0-Spaces in d-Semantics

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Abstract. In [6] it is shown that if we interpret modal diamond as the derived set operator of a topological space (the so-called d-semantics), then the modal logic of all topological spaces is **wK4**—weak **K4**—which is obtained by adding the weak version $\Diamond\Diamond p \to p \vee \Diamond p$ of the **K4**-axiom $\Diamond\Diamond p \to \Diamond p$ to the basic modal logic **K**.

In this paper we show that the T_0 separation axiom is definable in d-semantics. We prove that the corresponding modal logic of T_0-spaces, which is strictly in between **wK4** and **K4**, has the finite model property and is the modal logic of all spectral spaces—an important class of spaces, which serve as duals of bounded distributive lattices. We also give a detailed proof that **wK4** has the finite model property and is the modal logic of all topological spaces.

1 Introduction

In the appendix to [12], McKinsey and Tarski introduced a new interpretation of modal diamond as the derived set operator of a topological space. Following [1], we call this semantics the *d-semantics*. Thus, we refer to definability and completeness in d-semantics as *d-definability* and *d-completeness*. In [6] (see also [7]) it is shown that the d-logic of all topological spaces is weak **K4**

$$\mathbf{wK4} = \mathbf{K} + (\Diamond\Diamond p \to p \vee \Diamond p),$$

and that **K4** is the d-logic of all T_d-spaces; that is, spaces in which each point is locally closed (open in its own closure). It is well known that the T_d separation axiom is strictly in between T_0 and T_1 (hence, some authors call it the $T_{\frac{1}{2}}$ separation axiom). In [2] we showed that **K4** is the d-logic of all Stone spaces. Since each Stone space is a normal space (T_4-space), it follows that neither of the separation axioms $T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$ is d-definable. On the other hand, it follows from [9, Thm. 3.1.5] that if we enrich the basic modal language with the difference modality, then both T_0 and T_1 separation axioms become d-definable.

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Surprisingly enough, we show that the T_0 separation axiom is d-definable already in the basic modal language. We prove that the corresponding d-logic has the finite model property and is the d-logic of all spectral spaces—an important class of spaces which serve as duals of bounded distributive lattices. Clearly spectral spaces are a generalization of Stone spaces—the duals of Boolean algebras. Thus, our results fit nicely with [6] and [2], where it is shown that **K4** d-defines the T_d separation axiom and is the d-logic of all Stone spaces.

We also give a detailed proof that **wK4** has the finite model property and is the d-logic of all topological spaces.

2 d-Definability of T_0-Spaces

Let X be a topological space and $A \subseteq X$. We let $\mathrm{cl}(A)$ denote the *closure* of A. Recall that $x \in \mathrm{cl}(A)$ iff each open neighborhood U of x has a nonempty intersection with A. We also let $d(A)$ denote the *derived set* of A (the *derivative* of A). Then $x \in d(A)$ iff each open neighborhood U of x has a nonempty intersection with $A - \{x\}$. Clearly $\mathrm{cl}(A) = A \cup d(A)$.

We recall that a topological space X satisfies the T_0 *separation axiom* (is a T_0-*space*) if for each $x, y \in X$, whenever $x \neq y$, there exists an open neighborhood U of x missing y or an open neighborhood V of y missing x.

Definition 1. *Let X be a topological space.*

1. *We call $A \subseteq X$* discrete *if $A \cap d(A) = \emptyset$.*
2. *We call $A, B \subseteq X$* mutually dense *if $A \subseteq \mathrm{cl}(B)$ and $B \subseteq \mathrm{cl}(A)$.*

Theorem 1. *For a topological space X, the following three conditions are equivalent:*

1. *X is a T_0-space.*
2. *For each $A, B \subseteq X$, we have $A \cap d(B \cap d(A)) \subseteq d(A) \cup d(B \cap d(B))$.*
3. *X does not contain two disjoint mutually dense discrete subsets.*

Proof. (1)⇒(2): Suppose that X is a T_0-space and $x \in A \cap d(B \cap d(A))$. We must show that $x \in d(A) \cup d(B \cap d(B))$. Suppose that $x \notin d(A)$. Then $x \in A$ and there exists an open neighborhood U of x such that $U - \{x\} \subseteq X - A$. To show that $x \in d(B \cap d(B))$, let V be an open neighborhood of x. Then $W = U \cap V$ is an open neighborhood of x. Since $x \in d(B \cap d(A))$, there exists $y \in W - \{x\}$ such that $y \in B$ and $y \in d(A)$. As $y \in d(A)$ and x is the only point from A in W, which is an open neighborhood of y, we have $y \in d(x)$. We show that $y \in d(B)$. Let V_1 be an open neighborhood of y. Then $W_1 = V_1 \cap W$ is also an open neighborhood of y. Since $y \in d(x)$, we have $x \in W_1$. As X is a T_0-space and each open neighborhood of y contains x, there exists an open neighborhood U_1 of x such that $y \notin U_1$. Let $U_2 = U_1 \cap W_1$, an open neighborhood of x. Since $x \in d(B \cap d(A))$, there exists $z \in U_2 - \{x\}$ such that $z \in B$. Clearly $y \neq z$. Also, $U_2 \subseteq W_1 \subseteq V_1$. Therefore, $z \in B \cap (V_1 - \{y\})$, and so $y \in d(B)$. Thus,

$y \in B \cap d(B)$. Consequently, in each open neighborhood V of x there exists $y \neq x$ such that $y \in B \cap d(B)$, and so $x \in d(B \cap d(B))$, as required.

(2)⇒(3): Suppose $A, B \subseteq X$ are disjoint mutually dense discrete. Then $A \subseteq \mathrm{cl}(B) = B \cup d(B)$, and as $A \cap B = \emptyset$, we have $A \subseteq d(B)$. Therefore, $A \cap d(B) = A$. Similarly, $B \cap d(A) = B$. Since A, B are discrete, we have $B \cap d(B) = \emptyset$ and $A \cap d(A) = \emptyset$. It follows that $A \not\subseteq d(A)$. On the other hand, $A = A \cap d(B) = A \cap d(B \cap d(A))$ and $d(A) = d(A) \cup d(B \cap d(B))$. Thus, $A \cap d(B \cap d(A)) \not\subseteq d(A) \cup d(B \cap d(B))$.

(3)⇒(1): Suppose X is not T_0. Then there exist distinct $x, y \in X$ that cannot be separated by an open subset of X. Therefore, $\mathrm{cl}(x) = \mathrm{cl}(y)$. Set $A = \{x\}$ and $B = \{y\}$. Then A, B are clearly disjoint, discrete, and mutually dense. □

Definition 2. *Let* $\mathbf{t_0}$ *denote the formula*

$$p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$$

Corollary 1. *Let X be a topological space. Then $X \models \mathbf{t_0}$ iff X is a T_0-space.*

Proof. Let ν be a valuation into X. Then for each formulas φ and ψ, we have $\nu(\varphi \wedge \psi) = \nu(\varphi) \cap \nu(\psi)$, $\nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi)$, and $\nu(\Diamond\varphi) = d(\nu(\varphi))$. Therefore, by Theorem 1, we have:

$$\begin{aligned} X \models \mathbf{t_0} &\text{ iff for each valuation } \nu,\ \nu(p \wedge \Diamond(q \wedge \Diamond p)) \subseteq \nu(\Diamond p \vee \Diamond(q \wedge \Diamond q)) \\ &\text{ iff for each } \nu,\ \nu(p) \cap d(\nu(q) \cap d(\nu(p))) \subseteq d(\nu(p)) \cup d(\nu(q) \cap d(\nu(q))) \\ &\text{ iff for each } A, B \subseteq X,\ A \cap d(B \cap d(A)) \subseteq d(A) \cup d(B \cap d(B)) \\ &\text{ iff } X \text{ is a } T_0\text{-space.} \end{aligned}$$

□

Therefore, $\mathbf{t_0}$ d-defines the class of T_0-spaces.

Definition 3. *Let* $\mathbf{wK4T_0}$ *denote the extension of* $\mathbf{wK4}$ *by the axiom* $\mathbf{t_0}$; *that is,*

$$\mathbf{wK4T_0} = \mathbf{wK4} + \mathbf{t_0}.$$

By Theorem 1, $\mathbf{wK4T_0}$ d-defines the class of T_0-spaces. It is our goal to show that $\mathbf{wK4T_0}$ is in fact the d-logic of all spectral spaces, and hence the d-logic of all T_0-spaces. For this, we need to develop Kripke semantics for $\mathbf{wK4T_0}$.

3 Kripke Semantics for $\mathbf{wK4T_0}$

Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame. We recall that $\mathfrak{F}$ is *transitive* if for all $u, v, w \in W$ we have $uRvRw$ implies uRw and that $\mathfrak{F}$ is *weakly transitive* if for all $u, v, w \in W$ we have $uRvRw$ implies $u = w$ or uRw. It is well known that $\mathfrak{F}$ is a $\mathbf{K4}$-frame iff $\mathfrak{F}$ is transitive. It was shown in [6] that $\mathfrak{F}$ is a $\mathbf{wK4}$-frame iff $\mathfrak{F}$ is weakly transitive.

Let $\mathfrak{F}$ be a $\mathbf{wK4}$-frame. We call a point $w \in W$ *reflexive* if wRw and *irreflexive* otherwise. For $w \in W$, let

$$C(w) = \{w\} \cup \{v \in W : wRvRw\}.$$

We call $C(w)$ the *cluster generated by* w. We also call a subset C of W a *cluster* if $C = C(w)$ for some $w \in W$. A cluster C is *proper* if it consists of more than one point, it is *simple* if it consists of a single reflexive point, and it is *degenerate* if it consists of a single irreflexive point.

Obviously if $\mathfrak{F}$ is a **K4**-frame, then each point of a proper cluster is reflexive. However, in a **wK4**-frame, proper clusters may contain irreflexive points as well. In fact, each **wK4**-frame can be obtained from a **K4**-frame by deleting arbitrarily reflexive arrows in proper clusters.

We show that $\mathbf{t_0}$ is satisfied in a **wK4**-frame $\mathfrak{F}$ iff each proper cluster of $\mathfrak{F}$ contains at most one irreflexive point.

Lemma 1. *Let $\mathfrak{F}$ be a* **wK4***-frame. Then $\mathfrak{F} \models \mathbf{t_0}$ iff for each $w, v \in W$ with $wRvRw$, we have wRw or vRv.*

Proof. [$\Rightarrow$] First suppose that there exist irreflexive $w, v \in W$ such that wRv and vRw. Let $\nu(p) = \{w\}$ and $\nu(q) = \{v\}$. Then we can easily verify that $w \models_\nu p \wedge \Diamond(q \wedge \Diamond p)$ and $w \not\models_\nu \Diamond p \vee \Diamond(q \wedge \Diamond q)$. Therefore, $\mathfrak{F} \not\models_\nu \mathbf{t_0}$.

[$\Leftarrow$] Now suppose that $\mathfrak{F} \not\models \mathbf{t_0}$. Then there exists a valuation ν and a point $w \in W$ such that $w \models_\nu p \wedge \Diamond(q \wedge \Diamond p)$ and $w \not\models_\nu \Diamond p \vee \Diamond(q \wedge \Diamond q)$. Therefore, $w \models_\nu p$, $w \models_\nu \Diamond(q \wedge \Diamond p)$, $w \not\models_\nu \Diamond p$, and $w \not\models_\nu \Diamond(q \wedge \Diamond q)$. Since $w \models_\nu \Diamond(q \wedge \Diamond p)$, there exists $v \in W$ with wRv and $v \models_\nu q \wedge \Diamond p$. As $v \models_\nu \Diamond p$, there exists $u \in W$ such that vRu and $u \models_\nu p$. If $w \neq u$, then as $\mathfrak{F}$ is weakly transitive, wRu. This together with $u \models_\nu p$ implies that $w \models_\nu \Diamond p$, a contradiction. Therefore, $w = u$, and so $wRvRw \models_\nu p$. If wRw, then $w \models_\nu \Diamond p$, a contradiction. Therefore, $w\not{R}w$. Similarly, if vRv, then $v \models_\nu q \wedge \Diamond q$, which contradicts to $w \not\models_\nu \Diamond(q \wedge \Diamond q)$. Thus, $v\not{R}v$, and so we found irreflexive $w, v \in W$ such that $wRvRw$. □

Definition 4. *Let $\mathfrak{F}$ be a* **wK4***-frame. We call $\mathfrak{F}$ a* $\mathbf{wK4T_0}$-frame *if each proper cluster of $\mathfrak{F}$ contains at most one irreflexive point.*

By Lemma 1, a **wK4**-frame $\mathfrak{F}$ is a $\mathbf{wK4T_0}$-frame iff $\mathfrak{F} \models \mathbf{t_0}$. Clearly $\mathbf{wK4T_0}$ defines the class of $\mathbf{wK4T_0}$-frames. Now since both the **wK4**-axiom $\Diamond\Diamond p \to p \vee \Diamond p$ and the $\mathbf{t_0}$-axiom $p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$ are Sahlqvist formulas (see, e.g., [5, Def. 3.41]), it follows that both **wK4** and $\mathbf{wK4T_0}$ are Sahlqvist logics, hence are canonical, and so Kripke complete [5, Thm. 4.42]. Consequently, since the class of **K4**-frames is properly contained in the class of $\mathbf{wK4T_0}$-frames, which in turn is properly contained in the class of **wK4**-frames, we obtain:

Theorem 2. $\mathbf{wK4T_0}$ *is a Kripke complete logic, which is a proper extension of* **wK4** *and is properly contained in* **K4***. Diagrammatically,*

$$\mathbf{wK4} \subsetneq \mathbf{wK4T_0} \subsetneq \mathbf{K4}.$$

In the next section we show that $\mathbf{wK4T_0}$ is actually the logic of finite $\mathbf{wK4T_0}$-frames.

4 Finite Model Property of wK4 and $\mathbf{wK4T_0}$

Let $\mathfrak{F}$ be a Kripke frame. Then it is easy to see that if $\mathfrak{F}$ validates either of **wK4** and $\mathbf{wK4T_0}$, then so does each subframe of $\mathfrak{F}$. Consequently, both **wK4** and $\mathbf{wK4T_0}$ are subframe logics. It is a well-known result of Fine [8] that each subframe logic over **K4** has the finite model property. In a recent paper [4], Fine's theorem was extended to all subframe logics over **wK4**. As a result, we obtain that both **wK4** and $\mathbf{wK4T_0}$ have the finite model property. We point out that the extension of Fine's result to **wK4** is nontrivial. In this section we give a direct proof that both **wK4** and $\mathbf{wK4T_0}$ have the finite model property. Our proof is a modified version of the *filtration method*, and it indicates the difficulties we face when moving from **K4** to **wK4**. In particular, the weakly transitive closure of the least filtration may not be a filtration, and so the proof of the finite model property of **wK4** given in [6] contains a gap, which we will fill below.

Let Σ be a finite set of modal formulas closed under subformulas.

Definition 5. *A* Σ-type $\mathbf{t}$ *is a subset of* Σ *such that:*

(t_1) *For each* $\psi \wedge \xi \in \Sigma$ *we have* $\psi \wedge \xi \in \mathbf{t}$ *iff* $\psi \in \mathbf{t}$ *and* $\xi \in \mathbf{t}$*,*
(t_2) *For each* $\neg\psi \in \Sigma$ *we have* $\neg\psi \in \mathbf{t}$ *iff* $\psi \notin \mathbf{t}$.

We denote the set of all Σ-types by $\mathbf{T}(\Sigma)$. Clearly $\mathbf{T}(\Sigma)$ is finite whenever Σ is finite. If Σ is clear from the context, we call Σ-types simply *types*.

Definition 6. *We call a type* $\mathbf{t}$ irregular *if for some* $\Diamond\psi \in \Sigma$ *we have* $\psi \in \mathbf{t}$ *and* $\Diamond\psi \notin \mathbf{t}$. *If* $\mathbf{t}$ *is not an irregular type, then we call* $\mathbf{t}$ *a* regular *type.*

Let $\mathfrak{M} = \langle \mathfrak{F}, \nu \rangle$ be a **wK4**-model. With each point $w \in W$, we associate its Σ-type $\mathbf{t}(w)$ as follows:

$$\mathbf{t}(w) = \{\psi \in \Sigma : w \models \psi\}$$

It should be clear that reflexive points always have regular types. However, some irreflexive points may still have regular types.

We make use of special sets of types, called cluster-types.

Definition 7. *A set* $\mathbf{c} \subseteq \mathbf{T}(\Sigma)$ *is a* cluster-type *if for each* $\Diamond\psi \in \Sigma$ *and distinct* $\mathbf{t}, \mathbf{s} \in \mathbf{c}$ *we have* $\Diamond\psi \in \mathbf{s}$ *whenever* $\psi \in \mathbf{t}$.

With each $w \in W$, we associate its cluster-type $\mathbf{c}(w)$ as follows:

$$\mathbf{c}(w) = \{\mathbf{t}(v) \ : \ v \in C(w)\}$$

(Here we recall that $C(w)$ is the cluster generated by w.) We denote the set of all cluster-types associated with the model $\mathfrak{M}$ by $\mathbf{C}(\mathfrak{M})$; that is, $\mathbf{C}(\mathfrak{M}) = \{\mathbf{c}(w) \ : \ w \in W\}$. It is easy to see that $\mathbf{C}(\mathfrak{M}) \subseteq \wp(\wp(\Sigma))$, so $\mathbf{C}(\mathfrak{M})$ is finite whenever Σ is finite.

Theorem 3. **wK4** *has the finite model property.*

Proof. It is sufficient to show that each **wK4**-satisfiable formula is satisfiable in a finite **wK4**-frame. Let φ be **wK4**-satisfiable. Then there exists a rooted **wK4**-frame $\mathfrak{F} = \langle W, R\rangle$, with a root $r \in W$, and a valuation ν such that $r \models_\nu \varphi$. Let $\Sigma = \mathbf{Sub}(\varphi)$ be the set of subformulas of φ and $\mathfrak{M} = \langle \mathfrak{F}, \nu\rangle$.

To find a finite **wK4**-frame that satisfies φ, we first consider an auxiliary Kripke frame $\mathfrak{G} = \langle V, S\rangle$ built using the cluster-types of $\mathfrak{M}$. Let $V = \mathbf{C}(\mathfrak{M})$. Clearly V is finite. Suppose $V = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$. For $\mathbf{c}, \mathbf{d} \in V$, we set

$$\mathbf{c}S\mathbf{d} \text{ iff for all } \Diamond\psi \in \Sigma, \text{ if } \psi \in \bigcup \mathbf{d} \text{ or } \Diamond\psi \in \bigcup \mathbf{d}, \text{ then } \Diamond\psi \in \bigcap \mathbf{c}.$$

In other words, $\mathbf{c}S\mathbf{d}$ means that for all $\Diamond\psi \in \Sigma$, if either ψ or $\Diamond\psi$ belong to *some* type in $\mathbf{d}$, then $\Diamond\psi$ belongs to *all* types in $\mathbf{c}$. It is easy to check that S is a transitive relation on V.

Next we build a finite frame $\mathfrak{H} = \langle U, T\rangle$ from $\mathfrak{G} = \langle V, S\rangle$. Let $U = \bigsqcup_{i=1}^{n} \mathbf{c}_i$ be the disjoint union of the cluster-types of $\mathfrak{M}$. Thus, U consists of Σ-types (although the same type may appear in U multiple times since we are taking the disjoint union). As each $\mathbf{c}_i$ is a finite set of types, U is also finite. We define the relation T on U in such a way that the only reflexive points in $\mathfrak{H}$ are regular types, all distinct points from the same cluster-type are T-related, and the types from distinct cluster-types are T-related iff the respective cluster-types are S-related. Formally, let $\mathbf{t} \in \mathbf{c}$ and $\mathbf{s} \in \mathbf{d}$, where $\mathbf{c}, \mathbf{d} \in V$. Set $\mathbf{t}T\mathbf{s}$ iff one of the following three conditions is satisfied:

(i) $\mathbf{c} = \mathbf{d}$, $\mathbf{t} = \mathbf{s}$ and $\mathbf{t}$ is a regular type.
(ii) $\mathbf{c} = \mathbf{d}$ and $\mathbf{t} \neq \mathbf{s}$.
(iii) $\mathbf{c}S\mathbf{d}$.

We show that $\mathfrak{H}$ is a **wK4**-frame. Suppose $\mathbf{t}T\mathbf{s}T\mathbf{u}$ and $\mathbf{t} \neq \mathbf{u}$. We need to show that $\mathbf{t}T\mathbf{u}$. The case when $\mathbf{t}$ and $\mathbf{u}$ come from the same cluster-type in V is trivial by (ii). The cases when either $\mathbf{t}$ and $\mathbf{s}$ or $\mathbf{s}$ and $\mathbf{u}$ come from the same cluster-type is taken care of by (iii). The remaining case, when $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{u}$ come from pairwise distinct cluster-types follows from the transitivity of S and (iii). Thus, T is weakly transitive, and so $\mathfrak{H}$ is a **wK4**-frame.

Define a valuation υ on U by

$$\mathbf{t} \in \upsilon(p) \text{ iff } p \in \mathbf{t}.$$

In other words,

$$\mathbf{t} \models_\upsilon p \text{ iff } p \in \mathbf{t}.$$

We show that this relationship lifts to all members of Σ.

Claim. For all $\psi \in \Sigma$ and $\mathbf{t} \in U$, we have $\mathbf{t} \models_\upsilon \psi$ iff $\psi \in \mathbf{t}$.

Proof of Claim: Induction on the complexity of ψ. The base case is taken care of by the definition of υ. The cases for conjunction and negation follow from Definition 5. We only treat the case when $\psi = \Diamond\xi$.

[$\Leftarrow$] First suppose that $\Diamond\xi \in \mathbf{t}$. Since $\mathbf{t} \in \mathbf{c}_i$ for some $i \in [1,n]$ and $\mathbf{c}_i \in \mathbf{C}(\mathfrak{M})$, there exists $w \in W$ such that $\mathbf{t}(w) = \mathbf{t}$ and $\mathbf{c}(w) = \mathbf{c}_i$. As $\Diamond\xi \in \mathbf{t}$, we have $w \models_\nu \Diamond\xi$. Therefore, there exists $v \in W$ such that wRv and $v \models_\nu \xi$. Clearly $\mathbf{c}(v) = \mathbf{c}_j$ for some $j \in [1,n]$ and $\xi \in \mathbf{t}(v) \in \mathbf{c}_j$. We denote $\mathbf{t}(v)$ by $\mathbf{s}$ and show that $\mathbf{t}T\mathbf{s}$.

- First suppose that $C(w) \neq C(v)$. Then for each $w' \in C(w)$ and $v' \in C(v)$, we have $w'Rv'$. We show that $\mathbf{c}_i S \mathbf{c}_j$. Let $\Diamond\chi \in \Sigma$ and $\chi \in \bigcup \mathbf{c}_j$ or $\Diamond\chi \in \bigcup \mathbf{c}_j$. If $\chi \in \bigcup \mathbf{c}_j$, then there exists $v' \in C(v)$ such that $v' \models_\nu \chi$. Since $w'Rv'$ for each $w' \in C(w)$, we have $w' \models_\nu \Diamond\chi$ for each $w' \in C(w)$. Therefore, $\Diamond\chi \in \bigcap \mathbf{c}_i$. On the other hand, if $\Diamond\chi \in \bigcup \mathbf{c}_j$, then there exists $v' \in C(v)$ such that $v' \models_\nu \Diamond\chi$. This implies that there exists $u \in W$ such that $v'Ru$ and $u \models_\nu \chi$. For each $w' \in C(w)$, we have $w'Rv'Ru$ and $C(w) \neq C(v) = C(v')$. Therefore, $w' \neq u$. As R is weakly transitive, $w'Ru$. Thus, $w' \models \Diamond\chi$, and so $\Diamond\chi \in \bigcap \mathbf{c}_i$. Consequently, $\mathbf{c}_i S \mathbf{c}_j$, which by (iii), gives us $\mathbf{t}T\mathbf{s}$.
- Next suppose that $C(w) = C(v)$ and $\mathbf{t} \neq \mathbf{s}$. Then by (ii), $\mathbf{t}T\mathbf{s}$.
- Finally, suppose that $C(w) = C(v)$ and $\mathbf{t} = \mathbf{s}$. We show that $\mathbf{t}$ is a regular type. Let $\Diamond\chi \in \Sigma$ and $\chi \in \mathbf{t}$. Since $\mathbf{t} = \mathbf{s} = \mathbf{t}(v)$, we have $v \models_\nu \chi$. As wRv, we obtain $w \models_\nu \Diamond\chi$. Therefore, $\Diamond\chi \in \mathbf{t}(w) = \mathbf{t}$. Thus, $\mathbf{t}$ is a regular type, and so by (i), $\mathbf{t}T\mathbf{t} = \mathbf{s}$.

Therefore, $\mathbf{t}T\mathbf{s}$. Since $\xi \in \mathbf{s}$, by the induction hypothesis, $\mathbf{s} \models_\upsilon \xi$. Thus, $\mathbf{t} \models_\upsilon \Diamond\xi$.

[$\Rightarrow$] Now suppose that $\mathbf{t} \models_\upsilon \Diamond\xi$. Then there exists $\mathbf{s} \in U$ such that $\mathbf{t}T\mathbf{s}$ and $\mathbf{s} \models_\upsilon \xi$. By the induction hypothesis, $\xi \in \mathbf{s}$. Let $\mathbf{t} \in \mathbf{c} \in V$ and $\mathbf{s} \in \mathbf{d} \in V$.

- If $\mathbf{c}S\mathbf{d}$, then as $\xi \in \bigcup \mathbf{d}$, by the definition of S, we have $\Diamond\xi \in \mathbf{t}$.
- If $\mathbf{c} = \mathbf{d}$ and $\mathbf{t} \neq \mathbf{s}$, then Definition 7 yields $\Diamond\xi \in \mathbf{t}$.
- If $\mathbf{c} = \mathbf{d}$ and $\mathbf{t} = \mathbf{s}$, then two cases are possible: either $\mathbf{c}S\mathbf{d}$, in which case we use the argument of the first considered case, or $\mathbf{t} = \mathbf{s}$ is regular. In the latter case, by the definition of a regular type, we immediately obtain $\Diamond\xi \in \mathbf{t}$.

Thus, in all possible cases we have $\Diamond\xi \in \mathbf{t}$, which finishes the proof of the claim.

Now, since $r \models_\nu \varphi$ and $\varphi \in \Sigma$, we have $\varphi \in \mathbf{t}(r) \in \mathbf{c}(r) \in V$. Let $\mathbf{t} \in U$ correspond to $\mathbf{t}(r) \in \mathbf{c}(r)$. By the Claim, $\mathbf{t} \models_\upsilon \varphi$, and so the finite **wK4**-frame $\mathfrak{H} = \langle U, T\rangle$ satisfies φ. □

Our next goal is to show that $\mathbf{wK4T_0}$ also has the finite model property.

Theorem 4. $\mathbf{wK4T_0}$ *has the finite model property.*

Proof. It is sufficient to show that each $\mathbf{wK4T_0}$-satisfiable formula φ is satisfiable in a finite $\mathbf{wK4T_0}$-frame. Let φ be a $\mathbf{wK4T_0}$-satisfiable formula. Then there exists a rooted $\mathbf{wK4T_0}$-frame $\mathfrak{F} = \langle W, R\rangle$, with a root $r \in W$, and a valuation ν such that $r \models_\nu \varphi$. Let $\Sigma = \mathbf{Sub}(\varphi)$ be the set of all subformulas of φ and let $\mathfrak{M} = \langle \mathfrak{F}, \nu\rangle$.

Following the proof of Theorem 3, we first build the finite transitive frame $\mathfrak{G} = \langle V, S\rangle$ using the cluster-types of $\mathfrak{M}$, and then the finite weakly transitive frame $\mathfrak{H} = \langle U, T\rangle$ that satisfies φ. We show that $\mathfrak{H}$ is a $\mathbf{wK4T_0}$-frame.

Claim. For each cluster-type $\mathbf{c} \in V$, we have:

1. $\mathbf{c}$ contains at most one irregular type.
2. If $\mathbf{c}S\mathbf{c}$ then $\mathbf{c}$ consists of regular types only.

PROOF OF CLAIM: (1) Suppose $\mathbf{c} = \mathbf{c}(w)$ for some $w \in W$. Then $\mathbf{c} = \{\mathbf{t}(v) : v \in C(w)\}$. Since $C(w)$ contains at most one irreflexive point and reflexive points have regular types, $\mathbf{c}$ contains at most one irregular type.

(2) Let $\mathbf{c}S\mathbf{c}$ and $\mathbf{t} \in \mathbf{c}$. Suppose that $\psi \in \mathbf{t}$ for some $\Diamond\psi \in \Sigma$. Then $\psi \in \bigcup \mathbf{c}$. By the definition of S, we have $\Diamond\psi \in \bigcap \mathbf{c}$. Therefore, $\Diamond\psi \in \mathbf{t}$. Thus, $\mathbf{t}$ is regular, which finishes the proof of the Claim.

Suppose $\mathfrak{H}$ is not a $\mathbf{wK4T_0}$-frame. Then there exist distinct irregular types $\mathbf{t}, \mathbf{s} \in U$ such that $\mathbf{t}T\mathbf{s}$ and $\mathbf{s}T\mathbf{t}$. Let $\mathbf{c} \in V$ be the cluster-type of $\mathbf{t}$ and let $\mathbf{d} \in V$ be the cluster-type of $\mathbf{s}$.

– If $\mathbf{c} = \mathbf{d}$, then $\mathbf{t}, \mathbf{s} \in \mathbf{c}$ are two distinct irregular types in $\mathbf{c}$, contradicting item (1) of the Claim.
– If $\mathbf{c} \neq \mathbf{d}$, then by the definition of T, we have $\mathbf{c}S\mathbf{d}S\mathbf{c}$. Since S is transitive, $\mathbf{c}S\mathbf{c}$. By item (2) of the Claim, all types in $\mathbf{c}$ must be regular. This contradicts the fact that $\mathbf{t} \in \mathbf{c}$ is irregular.

The obtained contradiction proves that $\mathfrak{H}$ is a $\mathbf{wK4T_0}$-frame. Thus, φ is satisfied in the finite $\mathbf{wK4T_0}$-frame $\mathfrak{H}$. □

5 d-Completeness of wK4

It was shown in [6] that **wK4** is the d-logic of all topological spaces. In this section we give an alternative proof of this result. Our strategy will be as follows. Let $\mathbf{wK4} \not\vdash \varphi$. Since **wK4** has the finite model property (see Theorem 3), there exists a finite **wK4**-frame $\mathfrak{F} = \langle W, R\rangle$ refuting φ. We transform $\mathfrak{F}$ into a topological space X which also refutes φ. To describe the construction of X out of $\mathfrak{F}$, we require the following definition, which generalizes a similar definition from [3, Sec. 2].

Definition 8. *Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame and let $(X_w)_{w\in W}$ be a family of topological spaces $\langle X_w, \tau_w\rangle$ indexed by W. Let $X_\oplus = \bigsqcup_{w\in W} X_w$ be the disjoint union of $(X_w)_{w\in W}$. For each $A \subseteq X_\oplus$ and $w \in W$, set $A_w = A \cap X_w$.*

The $\mathfrak{F}$-sum *of $(X_w)_{w\in W}$ (denoted by $\bigoplus_{\mathfrak{F}} X_w$) is defined as $\bigoplus_{\mathfrak{F}} X_w = \langle X_\oplus, \tau_\oplus\rangle$, where $U \in \tau_\oplus$ iff the following two conditions are satisfied for all $w, v \in W$:*

(a) $U_w \in \tau_w$.
(b) If wRv, $w \neq v$, and $U_w \neq \emptyset$, then $U_v = X_v$.

Lemma 2. $\bigoplus_{\mathfrak{F}} X_w = \langle X_\oplus, \tau_\oplus \rangle$ *is a topological space.*

Proof. That $\emptyset, X \in \tau_\oplus$ is obvious. Suppose $\{U_i : i \in I\} \subseteq \tau_\oplus$. We show that $\bigcup U_i \in \tau_\oplus$. We have $(\bigcup U_i)_w = (\bigcup U_i) \cap X_w = \bigcup (U_i \cap X_w) = \bigcup (U_i)_w$. Since $U_i \in \tau_\oplus$, we have $(U_i)_w \in \tau_w$. Therefore, $(\bigcup U_i)_w = \bigcup (U_i)_w \in \tau_w$, and so condition (a) is satisfied for $\bigcup U_i$. Next let $(\bigcup U_i)_w \neq \emptyset$, wRv, and $w \neq v$. Then $(U_j)_w \neq \emptyset$ for some $j \in I$, and using (b) for $U_j \in \tau_\oplus$ we obtain $(U_j)_v = X_v$. Therefore, $(\bigcup U_i)_v = X_v$, and so condition (b) is satisfied for $\bigcup U_i$. Thus, $\bigcup U_i \in \tau_\oplus$.

Now suppose that $U, V \in \tau_\oplus$. We show that $U \cap V \in \tau_\oplus$. We have $(U \cap V)_w = (U \cap V) \cap X_w = (U \cap X_w) \cap (V \cap X_w) = U_w \cap V_w \in \tau_w$. Therefore, condition (a) is satisfied for $U \cap V$. Next let $(U \cap V)_w \neq \emptyset$, wRv, and $w \neq v$. Then $U_w \neq \emptyset$ and $V_w \neq \emptyset$. Therefore, by (b) for $U, V \in \tau_\oplus$, we have $U_v = X_v$ and $V_v = X_v$. It follows that $(U \cap V)_v = X_v$. Thus, condition (b) is satisfied for $U \cap V$, and so $U \cap V \in \tau_\oplus$. □

Let $\mathfrak{F} = \langle W, R \rangle$ be a **wK4**-frame. We recall that $U \subseteq W$ is an *upset* of $\mathfrak{F}$ if $w \in U$ and wRv imply $v \in U$ (*downsets* are defined dually), and that the upsets of $\mathfrak{F}$ form a topology on $\mathfrak{F}$, called the *Alexandroff topology* of $\mathfrak{F}$. We denote the Alexandroff topology of $\mathfrak{F}$ by $\tau_{\mathfrak{F}}$.

We also recall from [1] that a map f from a topological space X to a **wK4**-frame $\mathfrak{F} = \langle W, R \rangle$ is a *d-morphism* if (i) $f : X \to \langle W, \tau_{\mathfrak{F}} \rangle$ is an interior map (continuous and open), (ii) f is r-dense ($f^{-1}(w)$ is dense-in-itself for reflexive $w \in W$), and (iii) f is i-discrete ($f^{-1}(w)$ is discrete for irreflexive $w \in W$). As was shown in [1, Cor. 2.9], onto d-morphisms preserve validity of formulas, or put differently, they reflect refutation of formulas.

Theorem 5 ([6]). **wK4** *is the d-logic of all topological spaces. In fact,* **wK4** *is the d-logic of all finite topological spaces.*

Proof. Since in each topological space X we have $dd(A) \subseteq A \cup d(A)$ for each $A \subseteq X$, it is clear that $\Diamond\Diamond p \to p \vee \Diamond p$ is valid in each topological space. Therefore, **wK4** is sound with respect to all topological spaces. It is left to be shown that each non-theorem of **wK4** can be refuted on a finite topological space. Let **wK4** $\nvdash \varphi$. By Theorem 3, there exists a finite **wK4**-frame $\mathfrak{F} = \langle W, R \rangle$ such that $\mathfrak{F} \not\models \varphi$. For $w \in W$ let $X_w = \{w\}$ denote the one-point space if w is irreflexive, and let $X_w = \{w_1, w_2\}$ denote the two point trivial space (that is, $\tau_w = \{\emptyset, X_w\}$) if w is reflexive. Let $X_\oplus$ be the $\mathfrak{F}$-sum of the family $(X_w)_{w \in W}$. Since $\mathfrak{F}$ and each X_w is finite, so is $X_\oplus$. Let $\pi : X_\oplus \twoheadrightarrow W$ be the canonical map, sending $x \in X_w$ to w. We show that π is a d-morphism.

That π is continuous and i-discrete is obvious. That π is open follows from condition (b) of Definition 8. To see that π is r-dense, let $w \in W$ be reflexive. Then $w_1, w_2 \in X_w = \pi^{-1}(w)$ cannot be separated by an open set of X_w. Therefore, by condition (b) of Definition 8, $\pi^{-1}(w)$ is dense-in-itself. Thus, π is a d-morphism. But then, since $\mathfrak{F} \not\models \varphi$, it follows from [1, Cor. 2.9] that $X_\oplus \not\models \varphi$. Consequently, we found a finite topological space $X_\oplus$ refuting φ, and so **wK4** it the d-logic of (finite) topological spaces. □

6 d-Completeness of $\mathbf{wK4T_0}$

In this final section we show that $\mathbf{wK4T_0}$ is the d-logic of all spectral spaces. It will follow that $\mathbf{wK4T_0}$ is also the d-logic of all T_0-spaces.

Let X be a topological space. We recall that a nonempty subset A of X is *irreducible* if from $A = B \cup C$, with B and C closed, it follows that $A = B$ or $A = C$, and that X is *sober* if each closed irreducible subset of X is the closure of a point. We also recall that X is *coherent* if the compact open subsets of X are closed under finite intersections and form a basis for the topology.

Definition 9 ([10]). *A topological space X is a* spectral space *if X is compact, T_0, sober, and coherent.*

Spectral spaces play an important role in the theory of distributive lattices and commutative rings with identity. An early result of Stone [13] established that spectral spaces are exactly the duals of bounded distributive lattices (Stone's definition was different, but as it turned out later, equivalent to Hochster's definition of a spectral space). Later Hochster [10] showed that each spectral space arises as the Zariski spectrum of a commutative ring with identity. There has been a lot of investigation of spectral spaces (see, e.g., Johnstone's excellent monograph [11]).

Since each spectral space is T_0, it follows from Theorem 1 that $\mathbf{t_0}$ is valid in each spectral space. Thus, $\mathbf{wK4T_0}$ is sound with respect to the class of all spectral spaces. It is our goal to show that $\mathbf{wK4T_0}$ is actually the d-logic of all spectral spaces. For this it is sufficient to show that each non-theorem of $\mathbf{wK4T_0}$ is refuted on a spectral space. In order to establish this we will require a particular spectral space, which we describe below.

Let ω be the first infinite ordinal and let τ be the *cofinite* topology on ω; that is,

$$\tau = \{\emptyset\} \cup \{U \subseteq \omega : U \text{ is a cofinite subset of } \omega\}$$

(remember that cofinite subsets are complements of finite subsets). It is easy to see that (ω, τ) is compact, T_1, and coherent. However, (ω, τ) is not sober because ω is a closed irreducible set, but it is not the closure of any point of ω. Therefore, we modify (ω, τ) as follows:

Definition 10. *Let $X = \omega + 1 = \omega \cup \{\omega\}$ and let $\tau_X = \{\emptyset\} \cup \{U \cup \{\omega\} : U$ is a cofinite subset of $\omega\}$.*

Lemma 3. *(X, τ_X) is a spectral space.*

Proof. Clearly (X, τ_X) is compact. Moreover, each nonempty open subset of X is homeomorphic to X, and so X has a basis of compact open subsets. Since the intersection of any two nonempty open subsets of X is again nonempty, hence homeomorphic to X, it follows that X is coherent. Note that closed subsets of X are $\emptyset, X$, and finite subsets of ω. Therefore, closed irreducible subsets of X are X and points of ω. Clearly each point of ω is its own closure, and $X = \mathrm{cl}(\{\omega\})$. Thus, X is sober. Finally, let $x, y \in X$ with $x \neq y$. If $x, y \in \omega$, then $X - \{y\}$ is an open subset of X containing x and missing y. Suppose that either x or y is ω. If $x = \omega$, then $X - \{y\}$ is an open subset of X containing x and missing y, and if $y = \omega$, then $X - \{x\}$ is an open subset of X containing y and missing x. Consequently, X is T_0, hence a spectral space. □

Remark 1. Since (X, τ_X) is T_0, we have that $(X, \tau_X) \models \mathbf{t_0}$, and so $(X, \tau_X) \models \mathbf{wK4T_0}$. On the other hand, (X, τ_X) is not a T_d-space because ω is not a locally closed point of X. Therefore, $(X, \tau_X) \not\models \mathbf{K4}$.

Let $\mathfrak{F} = \langle W, R \rangle$ be a **wK4**-frame. We build a spectral space out of $\mathfrak{F}$ by substituting each non-degenerate cluster of $\mathfrak{F}$ with a copy of the spectral space $\langle X, \tau_X \rangle$ of Definition 10. Define an equivalence relation $\sim$ on W by

$$w \sim v \text{ iff } C(w) = C(v).$$

Let W_s be the set of equivalence classes of $\sim$ (that is, the set of clusters of $\mathfrak{F}$). Define R_s on W_s by

$$[w] R_s [v] \text{ iff } wRv \text{ and } [w] \neq [v].$$

Clearly R_s is irreflexive transitive. We call $\mathfrak{F}_s = \langle W_s, R_s \rangle$ the *skeleton* of $\mathfrak{F}$. With each $c \in W_s$, we associate the space X_c as follows:

- If $c = C(w)$ is a degenerate cluster, then $X_c = \{w\}$ is the one-point space.
- If $c = C(w)$ is a non-degenerate cluster, then $X_c = \langle X, \tau_X \rangle$ is the spectral space of Definition 10.

Now let $\langle X_\oplus, \tau_\oplus \rangle$ be the $\mathfrak{F}_s$-sum $\bigoplus_{\mathfrak{F}_s} X_c$.

Lemma 4. *$\langle X_\oplus, \tau_\oplus \rangle$ is a spectral space.*

Proof. First we show that $\langle X_\oplus, \tau_\oplus \rangle$ is T_0. Let $x, y \in X_\oplus$ be two distinct points. Let also $\pi(x) = c$ and $\pi(y) = d$, where $\pi : X_\oplus \twoheadrightarrow W_s$ is the canonical map, sending $x \in X_c$ to c. If $c \neq d$, then as R_s is irreflexive transitive, we have $c \not R_s d$ or $d \not R_s c$. Without loss of generality we may assume that $c \not R_s d$. Then $d \notin \{c\} \cup R_s(c)$. Therefore, $x \in \pi^{-1}(\{c\} \cup R_s(c))$ and $y \notin \pi^{-1}(\{c\} \cup R_s(c))$. Thus, $\pi^{-1}(\{c\} \cup R_s(c))$ is an open subset of $X_\oplus$ separating x from y. On the other hand, if $c = d$, then $X_c = \langle X, \tau_X \rangle$. Since $\langle X, \tau_X \rangle$ is a T_0-space, x and y are separated by some $U \in \tau_X$. Obviously $U \cup \pi^{-1}(R_s(c))$ is an open subset of $X_\oplus$ separating x from y.

Next we show that $X_\oplus$ is sober. Let F be a closed irreducible subset of $X_\oplus$. Since F is closed, by condition (b) of Definition 8, $\pi(F)$ is a downset of $\mathfrak{F}_s$. We show that $\pi(F) = \{c\} \cup R_s^{-1}(c)$ for some $c \in W_s$. If not, then there exist two distinct downsets $D_1, D_2 \subsetneq \pi(F)$ such that $D_1 \cup D_2 = \pi(F)$. Therefore, $F \cap \pi^{-1}(D_i) \subsetneq F$ for $i = 1, 2$ and $(F \cap \pi^{-1}(D_1)) \cup (F \cap \pi^{-1}(D_2)) = F$. Clearly $F \cap \pi^{-1}(D_1)$ and $F \cap \pi^{-1}(D_2)$ are closed in $X_\oplus$, which implies that F is not irreducible, a contradiction. Thus, $\pi(F) = \{c\} \cup R_s^{-1}(c)$ for some $c \in W_s$. By condition (b) of Definition 8, $F = F_c \cup G$, where $G = \pi^{-1}(R_s^{-1}(c))$. Clearly F_c is a closed subset of X_c. We show that F_c is irreducible. If $F_c = F_1 \cup F_2$ for some

X_c-closed proper subsets F_1, F_2 of F_c, then both $F_1 \cup G$ and $F_2 \cup G$ are closed subsets of $X_\oplus$ and $F = (F_1 \cup G) \cup (F_2 \cup G)$. This is impossible because F is irreducible. Therefore, F_c is a closed irreducible subset of X_c, and as X_c is sober, F_c is the X_c-closure of some point $x \in X_c$. Thus, F is the $X_\oplus$-closure of x.

Finally, we show that $X_\oplus$ is compact and coherent. For this it is sufficient to show that each open subset U of $X_\oplus$ is compact. Since $\mathfrak{F}_s$ is finite, each open subset U of $X_\oplus$ is a finite union of open sets V_i such that $\pi(V_i) = \{c\} \cup R_s(c)$ for some $c \in W_s$. Therefore, we may assume that $\pi(U) = \{c\} \cup R_s(c)$ for some $c \in W_s$. Let $(U^i)_{i \in I}$ be an open cover of U. By condition (a) of Definition 8, $U_c, U_c^i \in \tau_c$ for each $i \in I$. Clearly U_c is compact in X_c and $(U_c^i)_{i \in I}$ is an open cover of U_c. Thus, there exists a finite subcover $U_c^{i_1}, \ldots, U_c^{i_n}$ of U_c. By condition (b) of Definition 8, $U^{i_1}, \ldots, U^{i_n}$ is a finite subcover of U, and so U is compact. Therefore, $X_\oplus$ is compact and coherent. $\square$

Theorem 6. $\mathbf{wK4T_0}$ *is the d-logic of spectral spaces.*

Proof. Clearly $\mathbf{wK4T_0}$ is sound with respect to spectral spaces. Let $\mathbf{wK4T_0} \nvdash \varphi$. By Theorem 4, there exists a finite $\mathbf{wK4T_0}$-frame $\mathfrak{F} = \langle W, R \rangle$ such that $\mathfrak{F} \not\models \varphi$. Let $\mathfrak{F}_s$ be the skeleton of $\mathfrak{F}$ and let $\langle X_\oplus, \tau_\oplus \rangle$ be the $\mathfrak{F}_s$-sum $\bigoplus_{\mathfrak{F}_s} X_c$. By Lemma 4, $\langle X_\oplus, \tau_\oplus \rangle$ is a spectral space. We show that there is a d-morphism $f : X_\oplus \twoheadrightarrow \mathfrak{F}$.

First we define maps $f_c : X_c \twoheadrightarrow c$ for each $c \in W_s$. Let $c \in W_s$; that is, $c = C(w)$ for some $w \in W$. Clearly $C(w)$ is finite because $\mathfrak{F}$ is finite. If $C(w)$ is degenerate, then $X_c = \{w\}$ is the trivial space and we set f_c to be the identity map. If $C(w)$ is non-degenerate, then $X_c = \langle X, \tau_X \rangle$. We have two cases:

- $C(w) = \{v_1, \ldots, v_n\}$ consists of reflexive points only. Let $V_1, \ldots, V_n$ be an arbitrarily chosen partition of X into n-many infinite sets. Then each V_i is dense in X, and hence each V_i is also dense-in-itself. We set $f_c(x) = v_i$ iff $x \in V_i$.
- $C(w)$ contains an irreflexive point. Since $C(w)$ is non-degenerate and $\mathfrak{F}$ is a $\mathbf{wK4T_0}$-frame, we have that $C(w)$ contains precisely one irreflexive point, say v, and say n-many reflexive points $\{v_1, \ldots, v_n\}$. Let $V_1, \ldots, V_n$ be an arbitrarily chosen partition of $\omega \subseteq X$. Again, each V_i is dense-in-itself. Set $f_c(x) = v_i$ iff $x \in V_i$ and $f(\omega) = v$.

Next we set $f = \bigcup_{c \in W_s} f_c$ and show that $f : X_\oplus \twoheadrightarrow W$ is a d-morphism. That f is continuous is obvious. That f is i-discrete follows from the fact that the f-preimages of irreflexive points of $\mathfrak{F}$ are singletons. That f is r-dense follows from the fact that the f-preimages of reflexive points of $\mathfrak{F}$ are dense-in-itself. We show that f is open. Let U be an open subset of $X_\oplus$. For each $c \in W_s$, if $U_c \neq \emptyset$, then $U_c = X_c$ or $U_c = V \cup \{\omega\}$, where V is a cofinite subset of ω. Therefore, $f_c(U_c) = C(w)$. Now, since $U = \bigcup_{c \in W_s} U_c$ and f distributes over unions, we obtain that if $f(U) \cap C(w) \neq \emptyset$, then $C(w) \subseteq f(U)$. This together with condition (b) of Definition 8 implies that $f(U)$ is an upset of $\mathfrak{F}$. Therefore, f is a d-morphism.

Finally, since d-morphisms reflect refutation of formulas (see [1, Cor. 2.9]), we obtain that $X_\oplus \not\models \varphi$. Thus, each non-theorem of $\mathbf{wK4T_0}$ is refuted on a spectral space, and so $\mathbf{wK4T_0}$ is the d-logic of spectral spaces. □

Corollary 2. $\mathbf{wK4T_0}$ *is the d-logic of* T_0*-spaces.*

Proof. By Theorem 1, $\mathbf{wK4T_0}$ is sound with respect to T_0-spaces, and by Theorem 6, $\mathbf{wK4T_0}$ is complete with respect to T_0-spaces. Thus, $\mathbf{wK4T_0}$ is the d-logic of T_0-spaces. □

Remark 2. We obtained that **wK4** is the d-logic of all topological spaces and $\mathbf{wK4T_0}$ is the d-logic of all T_0-spaces. In fact, **wK4** is the d-logic of all finite topological spaces. On the other hand, $\mathbf{wK4T_0}$ is **not** the d-logic of all finite T_0-spaces. Indeed, it is well known that for finite spaces the T_0 and T_d separation axioms are equivalent. Therefore, the d-logic of all finite T_0-spaces contains **K4**, which is a proper extension of $\mathbf{wK4T_0}$.

Summing up the results of this paper with [6,2], we obtain:

Logic	Defines the class of	Is the d-logic of
wK4	all topological spaces	all topological spaces all finite topological spaces
$\mathbf{wK4T_0}$	all T_0-spaces	all T_0-spaces all spectral spaces
K4	all T_d-spaces	all T_d-spaces all compact Hausdorff spaces all Stone spaces

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