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THE MODAL LOGIC OF STONE SPACES: DIAMOND AS DERIVATIVE

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Abstract. We show that if we interpret modal diamond as the derived set operator of a topological space, then the modal logic of Stone spaces is **K4** and the modal logic of weakly scattered Stone spaces is **K4G**. As a corollary, we obtain that **K4** is also the modal logic of compact Hausdorff spaces and **K4G** is the modal logic of weakly scattered compact Hausdorff spaces.

§1. Introduction. Topological semantics of modal logic was first developed by McKinsey & Tarski (1944), who suggested two interpretations of modal diamond $\Diamond$: one as the closure operator and another as the derived set operator of a topological space. They showed that if we interpret $\Diamond$ as the closure operator, then the modal logic of all topological spaces is Lewis' well-known modal system **S4**. The main result of McKinsey & Tarski (1944) states that **S4** is in fact the modal logic of any dense-in-itself metrizable space.

On the other hand, if we interpret $\Diamond$ as the derived set operator, then the modal logic of all topological spaces is **wK4**—weak **K4**—which is obtained from the basic normal modal logic **K** by adding $\Diamond\Diamond p \to (p \vee \Diamond p)$ as a new axiom (Esakia, 2004). Moreover, $\mathbf{K4} = \mathbf{K} + \Diamond\Diamond p \to \Diamond p$ is the modal logic of all T_D-spaces (Esakia, 2004) and $\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$ is the modal logic of all scattered spaces (Esakia, 1981). Further results in this direction can be found in Shehtman (1990), Bezhanishvili *et al.* (2005), and Gabelaia (2004).

In this paper we are interested in the modal logic of compact Hausdorff zero-dimensional spaces, also known as Stone spaces. The interest in Stone spaces stems from the celebrated Stone duality, which establishes duality (dual equivalence) between the category of Boolean algebras and Boolean algebra homomorphisms and the category of Stone spaces and continuous maps. Under Stone duality atomless Boolean algebras correspond to dense-in-itself Stone spaces, atomic Boolean algebras correspond to weakly scattered Stone spaces, and superatomic Boolean algebras correspond to scattered Stone spaces. It follows from Abashidze (1988) that the modal logic of scattered Stone spaces is **GL**. In Shehtman (1990), the McKinsey–Tarski technique was adopted to show that $\mathbf{K4D} = \mathbf{K4} + \Diamond\top$ is the modal logic of any dense-in-itself zero-dimensional metrizable space. It follows that **K4D** is the modal logic of dense-in-itself Stone spaces. To this we add that the modal logic of all Stone spaces is **K4** and the modal logic of weakly scattered Stone spaces is $\mathbf{K4G} = \mathbf{K4} + \neg\Box\bot \to \neg\Box\neg\Box\bot$. As a corollary, we obtain that the modal logic

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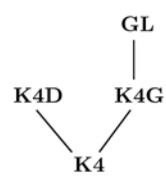
of compact Hausdorff spaces is **K4** and the modal logic of weakly scattered compact Hausdorff spaces is **K4G**.

§2. Preliminaries.

In this paper we will be interested in the following modal logics:

1. $\mathbf{K4} = \mathbf{K} + \Diamond\Diamond p \to \Diamond p$;
2. $\mathbf{K4D} = \mathbf{K4} + \Diamond\top$;
3. $\mathbf{K4G} = \mathbf{K4} + \neg\Box\bot \to \neg\Box\neg\Box\bot$; and
4. $\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$.

These logics are related to each other by the following diagram:



It is well known that **K4** is the modal logic of transitive frames, that **K4D** is the modal logic of transitive serial frames, and that **GL** is the modal logic of dually well-founded frames. These three logical systems are well known in the literature (see, e.g., Chagrov & Zakharyaschev, 1997). On the other hand, **K4G** is a relatively new system introduced in Esakia (2002). Its main importance lies in its capability to express modally Gödel's second incompleteness theorem (a consistent logical system cannot prove its own consistency).

Each of the four modal logics is complete with respect to its relational semantics. We briefly recall some basic facts about relational semantics which will be used subsequently. Let $\mathfrak{F} = (W, R)$ be a **K4**-frame; that is, $\mathfrak{F}$ is *transitive* (wRv and vRu imply wRu). Then $\mathfrak{F}$ is a **K4D**-frame if in addition it is *serial* (i.e., for each $w \in W$ there exists $v \in W$ such that wRv). We call $w \in W$ a *reflexive* point if wRw; otherwise we call w an *irreflexive* point. Let

$$C(w) = \{w\} \cup \{v \in W : wRv \text{ and } vRw\}.$$

We call $C(w)$ the *cluster generated by* w; we also call a subset C of W a *cluster* if $C = C(w)$ for some $w \in W$. Let C be a cluster of W. We call C *proper* if it consists of more than one element, *simple* if it consists of a single reflexive point, and *degenerate* if it consists of a single irreflexive point. We call $w \in W$ a *maximal* point if wRv implies $w = v$, and a *quasimaximal* point if wRv implies vRw. Clearly each maximal point is quasimaximal, but not vice versa.

Now, $\mathfrak{F}$ is a **GL**-frame iff $\mathfrak{F}$ is *dually well founded* (i.e., for each nonempty subset V of W there exists $v \in V$ such that vRu for no $u \in V$); and $\mathfrak{F}$ is a ***K4G**-frame* iff $\mathfrak{F}$ is a **K4**-frame and for each $w \in W$, either w is an irreflexive maximal point or there exists an irreflexive maximal point $v \in W$ such that wRv.

We say that $w \in W$ is a *root* of $\mathfrak{F}$ if wRv for each $v \in W - \{w\}$, and that $\mathfrak{F}$ is *rooted* if there exists a root in $\mathfrak{F}$. Note that a root may not be unique. In fact, if w is a root, then each element of $C(w)$ is also a root.

The next proposition states that all four modal logics of our interest have the finite model property.

PROPOSITION 2.1.

1. **K4** *is the modal logic of finite rooted transitive frames.*
2. **K4D** *is the modal logic of finite rooted transitive serial frames.*
3. **GL** *is the modal logic of finite rooted transitive irreflexive frames.*
4. **K4G** *is the modal logic of finite rooted* **K4G***-frames.*

Proof. For (1) and (2) see, for example, Chagrov & Zakharyaschev (1997, corollary 5.3.2); and for (3) see, for example, Chagrov & Zakharyaschev (1997, theorem 5.46). We sketch a proof that **K4G** is the modal logic of finite rooted **K4G**-frames, using the standard filtration argument through a well-chosen set of formulas. If **K4G** $\nvdash \varphi$, then φ is refuted on the canonical model $\mathcal{M}_{\mathbf{K4G}}$ of **K4G**. Since **K4** is a canonical logic and the formula $\neg\Box\bot \to \neg\Box\neg\Box\bot$ contains no propositional letters, the underlying frame of $\mathcal{M}_{\mathbf{K4G}}$ is a **K4G**-frame. Consider the standard transitive filtration (see, e.g., Chagrov & Zakharyaschev, 1997, pp. 141–145) of $\mathcal{M}_{\mathbf{K4G}}$ through the set

$$\Sigma = \{\psi \ : \ \psi \text{ is a subformula of } \varphi \wedge (\neg\Box\bot \to \neg\Box\neg\Box\bot)\}.$$

Since the underlying frame of $\mathcal{M}_{\mathbf{K4G}}$ is a **K4G**-frame, it is not difficult to see that the finite refutation frame obtained by such a filtration has all quasimaximal clusters degenerate. Indeed, let x be an arbitrary element in the filtrated model. Then x can be identified with a maximal consistent subset of Σ. Suppose x is not an irreflexive maximal point. Then x must contain $\neg\Box\bot$. We also have that $\neg\Box\bot \to \neg\Box\neg\Box\bot \in x$. Therefore, by Modus Ponens, $\neg\Box\neg\Box\bot \in x$. But then x is related to some y in the filtrated model with $\Box\bot \in y$. This implies that y is an irreflexive maximal point of the filtrated model. Thus, the underlying frame of the filtrated model is a finite **K4G**-frame. That φ can be refuted on a finite rooted **K4G**-frame is now straightforward. $\Box$

Let X be a topological space and $A \subseteq X$. We recall that $x \in X$ is a *limit point* of A if for each open neighborhood U of x we have $A \cap (U - \{x\}) \neq \emptyset$. Let $d(A)$ denote the set of limit points of A; $d(A)$ is called the *derived set* of A. It is obvious that the closure of A is A union $d(A)$; that is, $\mathrm{cl}(A) = A \cup d(A)$.

We also recall that a *valuation* of the basic modal language in a topological space X is a map ν from the set of propositional letters into the powerset of X. Given a valuation ν and $x \in X$, we define the *satisfaction relation* by induction:

1. $x \models_\nu p$ iff $x \in \nu(p)$;
2. $x \models_\nu \varphi \wedge \psi$ iff $x \models_\nu \varphi$ and $x \models_\nu \psi$;
3. $x \models_\nu \neg\varphi$ iff not $x \models_\nu \varphi$; and
4. $x \models_\nu \Diamond\varphi$ iff for each open neighborhood U of x there exists $y \in U - \{x\}$ such that $y \models_\nu \varphi$.

It follows that

2a. $x \models_\nu \varphi \vee \psi$ iff $x \models_\nu \varphi$ or $x \models_\nu \psi$

and that

4a. $x \models_\nu \Box\varphi$ iff there exists an open neighborhood U of x such that $y \models_\nu \varphi$ for each $y \in U - \{x\}$.

Given a topological space X, a valuation ν, and a formula φ, we say that φ is *true* in X if $x \models_\nu \varphi$ for each $x \in X$ and that φ is *valid* if φ is true under any valuation. If φ is valid in X, then we write $X \models \varphi$.

Let $L(X) = \{\varphi : X \models \varphi\}$. Then it is well known (and easy to verify) that $L(X)$ is a modal logic, called the *modal logic of* X. Given a class K of topological spaces, let $L(K) = \bigcap\{L(X) : X \in K\}$. Obviously $L(K)$ is a modal logic, called the *modal logic of* K.

Let X be a topological space. We recall that X is a T_D*-space* if each point of X is the intersection of an open subset and a closed subset of X. Alternatively, X is a T_D-space iff $dd(A) \subseteq d(A)$ for each $A \subseteq X$. We also recall that $x \in X$ is an *isolated* point if $\{x\}$ is an open subset of X. Let $\mathrm{iso}(X)$ denote the set of isolated points of X. Then X is called *dense-in-itself* if $\mathrm{iso}(X) = \emptyset$. Alternatively, X is dense-in-itself iff $d(X) = X$.

We say that a subset A of X is *dense* if $\mathrm{cl}(A) = X$, that X is *weakly scattered* if $\mathrm{iso}(X)$ is dense in X, and that X is *scattered* if each subspace of X is weakly scattered.

The next proposition is well known. It shows that three of the four logics we are interested in are all modal logics of natural classes of topological spaces.

PROPOSITION 2.2.

1. *Esakia (2004)* **K4** *is the modal logic of* T_D*-spaces.*
2. *Shehtman (1990)* **K4D** *is the modal logic of dense-in-itself* T_D*-spaces.*
3. *Esakia (1981)* **GL** *is the modal logic of scattered spaces.*

On the other hand, it will follow from our results that **K4G** is the modal logic of weakly scattered T_D-spaces.

A particularly important class of topological spaces is that of compact Hausdorff spaces. Since each Hausdorff space is T_D, it follows that the modal logic of compact Hausdorff spaces contains **K4**.

We recall that a subset A of a topological space is *clopen* if it is both closed and open, and that X is *zero-dimensional* if clopen subsets of X form a basis for the topology. Compact Hausdorff zero-dimensional spaces are often called *Stone spaces*. They play an important role in the theory of Boolean algebras as it follows from Stone duality that the category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of Stone spaces and continuous maps. Under Stone duality, atomless Boolean algebras correspond to dense-in-itself Stone spaces, atomic Boolean algebras correspond to weakly scattered Stone spaces, and superatomic Boolean algebras correspond to scattered Stone spaces.

It follows from Shehtman (1990) that **K4D** is the modal logic of any dense-in-itself zero-dimensional metrizable space. In particular, **K4D** is the modal logic of the Cantor space **C**. Since **C** is a dense-in-itself Stone space, it follows that the modal logic of dense-in-itself Stone spaces is **K4D**. In addition, it follows from Abashidze (1988) that **GL** is the modal logic of any ordinal $\alpha \geq \omega^\omega$ (viewed as a topological space in the interval topology). In particular, **GL** is the modal logic of $\omega^\omega + 1$. Since $\omega^\omega + 1$ is a scattered Stone space, it follows that **GL** is the modal logic of scattered Stone spaces.

In this paper we show that **K4** is the modal logic of all Stone spaces and **K4G** is the modal logic of weakly scattered Stone spaces. As a consequence, we obtain that **K4** is also the modal logic of all compact Hausdorff spaces and **K4G** is the modal logic of weakly scattered compact Hausdorff spaces. Consequently, **K4G** is also the modal logic of weakly scattered T_D-spaces. Thus, we obtain the following picture:

1. **K4** = the modal logic of T_D-spaces = the modal logic of compact Hausdorff spaces = the modal logic of Stone spaces;

2. **K4D** = the modal logic of dense-in-itself T_D-spaces = the modal logic of dense-in-itself compact Hausdorff spaces = the modal logic of dense-in-itself Stone spaces;
3. **K4G** = the modal logic of weakly scattered T_D-spaces = the modal logic of weakly scattered compact Hausdorff spaces = the modal logic of weakly scattered Stone spaces; and
4. **GL** = the modal logic of scattered spaces = the modal logic of scattered compact Hausdorff spaces = the modal logic of scattered Stone spaces.

§3. Modal logic of dense-in-itself Stone spaces: a new proof. As we pointed out in the previous section, **K4D** is the modal logic of the Cantor space. In this section we give a new and simplified proof of this result by adopting the technique developed in Aiello *et al.* (2003) for proving completeness of **S4** with respect to the Cantor space (when $\Diamond$ is interpreted as the closure operator).

We proceed as follows. By Proposition 2.1(2), **K4D** is complete with respect to finite rooted **K4D**-frames. Therefore, if **K4D** $\nvdash \varphi$, then there exists a finite rooted **K4D**-frame $\mathfrak{F} = (W, R)$ such that $\mathfrak{F} \not\models \varphi$. Since $\mathfrak{F}$ is a **K4D**-frame, each quasimaximal point of $\mathfrak{F}$ is reflexive. Figuratively speaking, $\mathfrak{F}$ is *top-reflexive*.

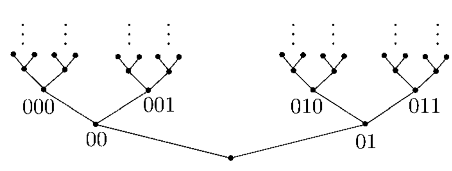
We recall that $U \subseteq W$ is an *upset* of W if $w \in U$ and wRv imply $v \in U$, and that the collection of upsets of W forms a topology τ_R on W, called an *Alexandroff topology* (in which the intersection of any family of open subsets is again open). We also recall from Bezhanishvili *et al.* (2005) that a map f from a topological space (X, τ) into (W, R) is a *d-morphism* if:

(i) f is continuous ($V \in \tau_R$ implies $f^{-1}(V) \in \tau$),
(ii) f is open ($U \in \tau$ implies $f(U) \in \tau_R$),
(iii) f is i-discrete (w an irreflexive point of W implies $f^{-1}(w)$ is a discrete subspace of X), and
(iv) f is r-dense (w a reflexive point of W implies $f^{-1}(w)$ is a dense-in-itself subspace of X),

and that onto d-morphisms preserve validity of formulas; or put differently, they reflect refutation. Therefore, in order to refute φ on the Cantor space $\mathbf{C}$, it is sufficient to construct a d-morphism from $\mathbf{C}$ onto W.

LEMMA 3.1. *For each finite rooted* **K4D***-frame* $\mathfrak{F}$*, there exists a d-morphism* $f : \mathbf{C} \twoheadrightarrow \mathfrak{F}$ *from the Cantor space* $\mathbf{C}$ *onto* $\mathfrak{F}$*.*

Proof. We view $\mathbf{C}$ as the collection of infinite paths of the infinite binary tree $\mathfrak{T}_2$.



The topology on $\mathbf{C}$ is defined as follows. For each finite path X of $\mathfrak{T}_2$, let

$$B_X = \{\sigma \in \mathbf{C} : X \text{ is an initial segment of } \sigma\}.$$

Then $\{B_X : X \text{ is a finite path of } \mathfrak{T}_2\}$ is a basis for the topology on $\mathbf{C}$.

Now we label $\mathfrak{T}_2$ by nodes of $\mathfrak{F}$ as follows. First let us fix some enumeration of $W = \{w_1, \ldots, w_n\}$. Let $w \in W$, $R(w) = \{v \in W : wRv\}$, and $R^+(w) = \{w\} \cup R(w)$. Since $\mathfrak{F}$ is a **K4D**-frame, $R(w) \neq \emptyset$. We label the root of $\mathfrak{T}_2$ by a root of $\mathfrak{F}$; if a node t of $\mathfrak{T}_2$ is labeled by $w \in W$, then we label the whole left path of $\mathfrak{T}_2$ starting at t by w, and we label the right-son of t by the first unused node of $R(w)$ in the enumeration of W (if all nodes were already used, we start over at the least node in $R(w)$ in the enumeration of W).

Let σ be an infinite path of $\mathfrak{T}_2$. If σ is going infinitely to the left, then there is a $w \in W$ such that each node of σ is labeled by w starting from some node on. In this case we say that w *stabilizes* σ. Else there exists a cluster C of W such that every node of σ is labeled by an element of C starting from some node on. In this case we say that σ *keeps cycling* in C. We pick any $w_C \in C$ and define $f : \mathbf{C} \to W$ as follows:

$$f(\sigma) = \begin{cases} w & \text{if } w \text{ stabilizes } \sigma, \\ w_C & \text{if } \sigma \text{ keeps cycling in } C. \end{cases}$$

It is left to be shown that f is an onto d-morphism. That f is onto is obvious from the definition of f. To see that f is open, let X be a finite path of $\mathfrak{T}_2$ and the end of X be labeled by w. Then, by the definition of f, we have $f(B_X) \subseteq R^+(w)$. Conversely, if $v \in R^+(w)$, then there exists a finite path Y extending X whose end is labeled by v. Let $\sigma = (Y, 0, 0, \ldots)$. Then $\sigma \in B_Y \subseteq B_X$ and $f(\sigma) = v$. Thus, $f(B_X) = R^+(w)$, and so f is open.

To see that f is continuous, let $w \in W$. We let

$$U = \bigcup \{B_X : X \text{ is a finite path of } \mathfrak{T}_2 \text{ whose end is labeled by } v \in R^+(w)\},$$

and show that $f^{-1}(R^+(w)) = U$. We have $\sigma \in f^{-1}(R^+(w))$ iff $f(\sigma) \in R^+(w)$, and $\sigma \in U$ iff there exists a finite path X of $\mathfrak{T}_2$ whose end is labeled by $v \in R^+(w)$. It is easily seen that if $\sigma \in U$, then $f(\sigma) \in R^+(w)$, and so $U \subseteq f^{-1}(R^+(w))$. For the converse inclusion, note that $f(\sigma) \in R^+(w)$ iff there is a $v \in R^+(w)$ such that either v stabilizes σ or σ keeps cycling in $C(v)$. In either case we can find a finite path X of $\mathfrak{T}_2$ whose end is labeled by $v \in R^+(w)$. Thus, $f^{-1}(R^+(w)) \subseteq U$, and so $f^{-1}(R^+(w)) = U$. It follows that f is continuous.

To see that f is i-discrete, let w be an irreflexive point of W and $\sigma \in f^{-1}(w)$. Then $f(\sigma) = w$ and there exists a finite initial segment X of σ whose end is labeled by w. Note that all finite paths $Y = (X, \ldots, 1, \ldots)$ have ends labeled with some $v \in R(w)$. Since w is irreflexive, $w \notin R(w)$. Therefore, the only infinite path in B_X that contains infinitely many points labeled with w is $(X, 0, 0, \ldots)$. Thus, $B_X \cap f^{-1}(w) = \{\sigma\} = \{(X, 0, 0, \ldots)\}$, and so $f^{-1}(w)$ is a discrete subspace of $\mathbf{C}$.

To see that f is r-dense, let w be a reflexive point of W and $\sigma \in f^{-1}(w)$. Suppose $\sigma \in B_X$ for some finite initial segment X of σ whose end is labeled by v. Then vRw and so we can find a finite initial segment Y of σ such that Y contains X as an initial segment and whose end is labeled by w. But w is reflexive, hence $w \in R(w)$. Therefore, there are at least two infinite paths having Y as an initial segment that belong to $f^{-1}(w)$: One is $(Y, 0, 0, \ldots)$ and the other is of the form $(Y, 0, 0, \ldots, 1, 0, 0, \ldots)$, where the number of 0s after Y is precisely the number required for w to come up again as a label in the enumeration of $R(w)$. It follows that $B_X \cap f^{-1}(w)$ contains at least one infinite path other than σ. Thus, there are no isolated points in the subspace $f^{-1}(w)$ of $\mathbf{C}$, and so

$f^{-1}(w)$ is a dense-in-itself subspace of $\mathbf{C}$. Consequently, $f : \mathbf{C} \twoheadrightarrow W$ is an onto d-morphism. $\square$

COROLLARY 3.2. **K4D** *is the modal logic of the Cantor space, hence* **K4D** *is the modal logic of dense-in-itself Stone spaces.*

Proof. Since $\mathbf{C}$ is a dense-in-itself T_D-space, **K4D** is sound with respect to $\mathbf{C}$. To see completeness, let $\mathbf{K4D} \nvdash \varphi$. By Proposition 2.1(2), there exists a finite rooted **K4D**-frame $\mathfrak{F} = (W, R)$ such that $\mathfrak{F} \not\models \varphi$. By Lemma 3.1, there exists a d-morphism from $\mathbf{C}$ onto W. Therefore, $\mathbf{C} \not\models \varphi$, and so $\mathbf{K4D} = L(\mathbf{C})$. Thus, **K4D** is the modal logic of dense-in-itself Stone spaces. $\square$

§4. Trees, ordinals, and compactifications. In this section we discuss connections between trees, ordinals, and compactifications, thus providing the necessary background for our main results, which will be discussed in the next section.

Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. For $w \in W$ we recall that

$$R^{-1}(w) = \{v \in W : vRw\} \text{ and } R(w) = \{v \in W : wRv\}.$$

Also, for $U \subseteq W$ let

$$R^{-1}(U) = \bigcup\{R^{-1}(w) : w \in U\} \text{ and } R(U) = \bigcup\{R(w) : w \in U\}.$$

For $w, v \in W$ we write $w\vec{R}v$ whenever wRv and $v \not{R}\, w$. We say that w is of *depth n* if there exists a sequence $w_0\vec{R}, \dots, \vec{R}w_n$ with $w = w_0$ and for each other sequence $v_0\vec{R}, \dots, \vec{R}v_k$ with $w = v_0$ we have $k \leq n$. We also say that $\mathfrak{F}$ is of *depth n* if there is $w \in W$ of depth n and no other element of $\mathfrak{F}$ has greater depth.

Let $\mathfrak{F} = (W, R)$ be a rooted **K4**-frame. We call $\mathfrak{F}$ a *quasitree* if for each $u, v \in R^{-1}(w)$ we have that $u \neq v$ implies uRv or vRu. If in addition $\mathfrak{F}$ has no proper clusters, then we call $\mathfrak{F}$ a *tree*. We call a tree $\mathfrak{T}$ *reflexive* if each element of $\mathfrak{T}$ is reflexive, and *irreflexive* if each element of $\mathfrak{T}$ is irreflexive. In addition, we call a finite quasitree $\mathfrak{F}$ *top-irreflexive* if each quasimaximal element of $\mathfrak{F}$ is irreflexive. Then we have the following strengthening of Proposition 2.1:

THEOREM 4.1.

1. **K4** *is the modal logic of finite quasitrees.*
2. **K4D** *is the modal logic of finite top-reflexive quasitrees.*
3. **GL** *is the modal logic of finite irreflexive trees.*
4. **K4G** *is the modal logic of finite top-irreflexive quasitrees.*

Proof. Parts of Theorem 4.1 are well known. We sketch a uniform construction akin to the standard finite unraveling argument to treat all four cases. Suppose $\mathfrak{F} = (W, R)$ is a finite rooted transitive frame. Let $C = \{C_1, \dots, C_n\}$ be the set of clusters of $\mathfrak{F}$. We set $C_i \leq C_j$ if $i = j$ or there exist $w \in C_i$ and $v \in C_j$ with wRv. Then $\mathfrak{C} = (C, \leq)$ is a finite partially ordered set. By the standard unraveling of $\mathfrak{C}$ (see, e.g., Chagrov & Zakharyaschev, 1997, theorem 2.19), we obtain a finite tree $\mathfrak{T}$ of clusters. The points of $\mathfrak{T}$ are finite paths $(x_1, \dots, x_k)$ of $\mathfrak{C}$, where $x_i \in C$ and $x_i < x_j$ whenever $i < j$, ordered by the relation "is an initial segment of." We substitute each point $(x_1, \dots, x_k)$ of $\mathfrak{T}$ by the cluster x_k to obtain a finite quasitree $\mathfrak{G}$. Then it can be easily checked that:

- $\mathfrak{G}$ maps p-morphically onto $\mathfrak{F}$,
- $\mathfrak{G}$ is top-reflexive whenever $\mathfrak{F}$ is top-reflexive,

- $\mathfrak{G}$ is top-irreflexive whenever $\mathfrak{F}$ is top-irreflexive,
- $\mathfrak{G}$ is irreflexive whenever $\mathfrak{F}$ is irreflexive.

The theorem follows as p-morphisms reflect refutation. □

Now we proceed to link trees with appropriate spaces.

LEMMA 4.2. *For each finite irreflexive tree $\mathfrak{T}$ with the root r there exists a limit ordinal ω^{n+1} and an onto d-morphism $g : \omega^{n+1} \twoheadrightarrow \mathfrak{T}$ such that $g^{-1}(r) = \{\omega^n \cdot k \ : \ 0 < k < \omega\}$.*

Proof. Let $\mathfrak{T}$ be a finite irreflexive tree with the root r, and let n be the depth of $\mathfrak{T}$. We build a tree $\mathfrak{T}^*$ by adjoining a new root to $\mathfrak{T}$; that is, if $\mathfrak{T} = (T, R)$, then $\mathfrak{T}^* = (T^*, R^*)$, where $T^* = T \sqcup \{*\}$ and $R^* = R \cup \{(*, t) : t \in T\}$. Then $\mathfrak{T}^*$ is of depth $n + 1$. By Bezhanishvili & Morandi (2010, lemma 3.4), each finite tree of depth $n+1$ is a d-morphic image of $\omega^{n+1} + 1$. Therefore, there exists an onto d-morphism $f : \omega^{n+1} + 1 \twoheadrightarrow T^*$. It follows from the proof of Bezhanishvili & Morandi (2010, lemma 3.4) that $f^{-1}(*) = \{\omega^{n+1}\}$ and $f^{-1}(r) = \{\omega^n \cdot k \ : \ 0 < k < \omega\}$. Thus, $f^{-1}(T) = \omega^{n+1}$. Let g be the restriction of f to ω^{n+1}. Then g is clearly an onto d-morphism from the limit ordinal ω^{n+1} onto the initial tree $\mathfrak{T}$ with $g^{-1}(r) = \{\omega^n \cdot k \ : \ 0 < k < \omega\}$. □

Let X be a completely regular space. We recall that a *compactification* of X is a compact Hausdorff space Y such that X is homeomorphic to a dense subspace of Y. Without loss of generality we identify X with the dense subspace of Y which is homeomorphic to X. Let $Y^* = Y - X$. As usual, we call Y^* the *remainder* of Y.

Let X be a topological space and Y a subspace of X. We recall that Y is a *retract* of X if there is a continuous onto map $f : X \to Y$ such that $f(y) = y$ for each $y \in Y$. In this case we call f a *retraction*; if in addition the f inverse image of each compact subset of Y is compact in X, then we call f a *compact retraction*.

LEMMA 4.3. *Let X be a noncompact locally compact zero-dimensional Hausdorff space, S a noncompact locally compact subspace of X, $f : X \to S$ a compact retraction, and Y a zero-dimensional compactification of S. Then there is a zero-dimensional compactification Z of X such that Z^* is homeomorphic to Y^* and $Z^* \subseteq \mathrm{cl}(S)$.*

Proof. Since X is locally compact and zero dimensional, there is a basis $\mathcal{B}_X$ of compact clopen subsets of X. As S is a subspace of a zero-dimensional Hausdorff space, S is also zero-dimensional Hausdorff. Therefore, S is a noncompact locally compact zero-dimensional Hausdorff space, and so it is an open subset of Y. Let $\mathrm{Cp}(Y)$ denote the basis of all clopen subsets of Y. We set Z to be the disjoint union of X and Y^*, and define a topology on Z by letting $\mathcal{B}_Z = \mathcal{B}_X \cup \mathcal{B}_Y$ be the basis for the topology, where

$$\mathcal{B}_Y = \{(U \cap Y^*) \cup f^{-1}(U \cap S) : U \in \mathrm{Cp}(Y)\}.$$

To see that $\mathcal{B}_Z$ is a basis, it is obvious that the union of the elements of $\mathcal{B}_Z$ is Z. We show that $\mathcal{B}_Z$ is closed under finite intersections. That $\mathcal{B}_X$ and $\mathcal{B}_Y$ are closed under finite intersections is obvious. On the other hand, if $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$, then $U \cap V = U \cap f^{-1}(W \cap S)$ for some $W \in \mathrm{Cp}(Y)$. We clearly have that $W \cap S$ is clopen in S, and so $f^{-1}(W \cap S)$ is clopen in X. Therefore, $U \cap f^{-1}(W \cap S)$ is a clopen subset of U, which is compact. Thus, $U \cap V$ is a compact clopen of X, and so belongs to $\mathcal{B}_X$.

Note that if we extend $f : X \to S$ to $\overline{f} : Z \to Y$ by setting $\overline{f}(x) = x$ for each $x \in Y^*$, then the topology on Z given by the basis $\mathcal{B}_Z$ is the least topology on Z containing all the open subsets of X and making $\overline{f}$ continuous.

We show that Z is Hausdorff. Let $x, y \in Z$ with $x \neq y$. If $x, y \in X$, then since $\mathcal{B}_X$ is a basis of X and X is Hausdorff, there exist disjoint $U, V \in \mathcal{B}_X$ separating x and y. If $x, y \in Y^*$, then as Y is Hausdorff, there exist disjoint $U, V \in \mathrm{Cp}(Y)$ separating x and y. But then $(U \cap Y^*) \cup f^{-1}(U \cap S)$ and $(V \cap Y^*) \cup f^{-1}(V \cap S)$ are disjoint elements of $\mathcal{B}_Y$ separating x and y. Finally, let $x \in X$ and $y \in Y^*$. Then there exists a compact clopen U of X containing x. Therefore, $f(U)$ is a compact subset of S, and hence a compact subset of Y. Thus, $Y - f(U)$ is an open subset of Y containing y. Since Y is zero dimensional, there exists a clopen subset V of Y such that $y \in V \subseteq Y - f(U)$. But then $(V \cap Y^*) \cup f^{-1}(V \cap S)$ is an open subset of Z containing y and disjoint from U. Thus, Z is Hausdorff.

Next we show that Z is compact. Let $\mathcal{U} \subseteq \mathcal{B}_Z$ be a cover of Z. We let $\mathcal{U}_Y = \{U \in \mathrm{Cp}(Y) : (U \cap Y^*) \cup f^{-1}(U \cap S) \in \mathcal{U}\}$. Then $\mathcal{U}_Y$ is an open cover of Y^*. Since Y^* is compact, there exist $U_1, \dots, U_n \in \mathcal{U}_Y$ such that $Y^* \subseteq U_1 \cup \dots \cup U_n$. Let $F = Y - (U_1 \cup \dots \cup U_n)$. Then $F \subseteq S$ is a compact subset of Y and as S is an open subset of Y, F is also compact in S. Since f is a compact retraction, $f^{-1}(F)$ is a compact subset of X, and so a compact subset of Z. Therefore, there exist $V_1, \dots, V_m \in \mathcal{U}$ such that $f^{-1}(F) \subseteq V_1 \cup \dots \cup V_m$. Thus, $Z = V_1 \cup \dots \cup V_m \cup [(U_1 \cap Y^*) \cup f^{-1}(U_1 \cap S)] \cup \dots \cup [(U_n \cap Y^*) \cup f^{-1}(U_n \cap S)]$, and so Z is compact.

To see that Z is zero-dimensional, let $U \in \mathcal{B}_X$. Then U is a compact clopen subset of X. Since X is an open subset of Z, U is compact open in Z; and as Z is compact Hausdorff, U is a clopen subset of Z. Now let $U \in \mathcal{B}_Y$. Then $U = (V \cap Y^*) \cup f^{-1}(V \cap S)$ for some $V \in \mathrm{Cp}(Y)$. Therefore, $Z - U = Z - [(V \cap Y^*) \cup f^{-1}(V \cap S)] = [Z - (V \cap Y^*)] \cap [Z - f^{-1}(V \cap S)] = [X \cup (Y^* \cap (Y - V))] \cap [Y^* \cup f^{-1}((Y - V) \cap S)] = f^{-1}((Y - V) \cap S) \cup (Y^* \cap (Y - V))$; and as $Y - V \in \mathrm{Cp}(Y)$, we obtain that $Z - U \in \mathcal{B}_Y$. Thus, each element of $\mathcal{B}_Z$ is a clopen subset of Z, and so Z is zero-dimensional.

We show that $Z^* \subseteq \mathrm{cl}(S)$. Let $x \in Z^*$ and $U \in \mathcal{B}_Z$ be a neighborhood of x. Then there exists $V \in \mathrm{Cp}(Y)$ such that $U = (V \cap Y^*) \cup f^{-1}(V \cap S)$. Since $\mathrm{cl}_Y(S) = Y$, we have $V \cap S \neq \emptyset$. Therefore, $f^{-1}(V \cap S) \neq \emptyset$, and as f is a retraction, we obtain $S \cap f^{-1}(V \cap S) \neq \emptyset$. Thus, $S \cap U \neq \emptyset$, and so $Z^* \subseteq \mathrm{cl}(S)$.

It follows that Z is a compact Hausdorff zero-dimensional space such that $Z^* \subseteq \mathrm{cl}(S)$. Therefore, $\mathrm{cl}(X) = Z$, and so Z is a zero-dimensional compactification of X. Finally, that the identity map from Z^* to Y^* is continuous follows from the definition of $\mathcal{B}_Y$. Thus, there is a continuous bijection between compact Hausdorff spaces Z^* and Y^*, which means that Z^* and Y^* are homeomorphic. $\square$

LEMMA 4.4. *For a limit ordinal ω^{n+1} and a closed subset X of the Cantor space* **C**, *there exists a compactification Z of ω^{n+1} such that:*

1. *Z is a Stone space;*
2. *The remainder Z^* is homeomorphic to X; and*
3. *$Z^* \subseteq \mathrm{cl}(\{\omega^n \cdot k : 0 < k < \omega\})$.*

Proof. Let X be a closed subset of **C**. Then X is a compact Hausdorff metrizable space. Therefore, by Terasawa (1997, theorem 1), there is a compactification Y of ω such that the remainder Y^* is homeomorphic to X. In fact, the proof of Terasawa (1997, theorem 1) implies that Y can be chosen to be zero-dimensional.[1] Now consider the following partition

[1] Indeed, since X is a subspace of **C**, it is zero-dimensional; therefore, in the proof of Terasawa (1997, theorem 1) we can choose the basis $\{U_n : n < \omega\}$ to consist of clopen subsets of X, which makes each element $H_{n,m}$ of the basis of Y clopen.

of ω^{n+1} into ω-many pairwise disjoint clopen intervals:

$$[0, \omega^n], (\omega^n, \omega^n \cdot 2], \ldots, (\omega^n \cdot (k-1), \omega^n \cdot k], \ldots.$$

Using this partition, we define a retraction $f : \omega^{n+1} \twoheadrightarrow \{\omega^n \cdot k : k < \omega\}$ by sending all the points in $(\omega^n \cdot (k-1), \omega^n \cdot k] \subseteq \omega^{n+1}$ to $\omega^n \cdot k$. It is easy to see that f is an onto continuous map and that $\{\omega^n \cdot k : k < \omega\} \subseteq \omega^{n+1}$ with its subspace topology is homeomorphic to ω. Since compact subsets of ω are finite, it follows that the f inverse image of a compact subset of $\{\omega^n \cdot k : k < \omega\}$ is a finite union of intervals of the form $(\omega^n \cdot (k-1), \omega^n \cdot k]$. Since each of these is compact, so is their finite union. Consequently, f is a compact retraction.

Now ω^{n+1} is a noncompact locally compact zero-dimensional Hausdorff space, $\{\omega^n \cdot k : k < \omega\}$ is a noncompact locally compact subspace of ω^{n+1}, $f : \omega^{n+1} \twoheadrightarrow \{\omega^n \cdot k : k < \omega\}$ is a compact retraction, and Y is a zero-dimensional compactification of $\{\omega^n \cdot k : k < \omega\}$ such that Y^* is homeomorphic to X. Thus, we are in a position to apply Lemma 4.3, by which we obtain a zero-dimensional compactification Z of ω^{n+1} such that Z^* is homeomorphic to X and $Z^* \subseteq \text{cl}(\{\omega^n \cdot k : 0 < k < \omega\})$. □

§5. Main results. In this section we prove our main results, that the modal logic of Stone spaces is **K4**, and that the modal logic of weakly scattered Stone spaces is **K4G**. As a corollary, we obtain that the modal logic of compact Hausdorff spaces is also **K4** and that the modal logic of weakly scattered compact Hausdorff spaces is **K4G**.

The key observation in establishing our main results is that each finite quasitree $\mathfrak{F} = (W, R)$ is a d-morphic image of an appropriately chosen Stone space. Our strategy will be as follows:

1. Represent $\mathfrak{F}$ as the disjoint union of two finite frames $\mathfrak{D}$ and $\mathfrak{T}$ in such a way that:
   - $\mathfrak{D}$ is a top-reflexive quasitree, hence a **K4D**-frame;
   - $\mathfrak{T}$ is the disjoint union of irreflexive trees $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$, hence a **GL**-frame.
2. Use Lemma 3.1 to build a d-morphism f from the Cantor space **C** onto $\mathfrak{D}$.
3. Use Lemma 4.2 to build d-morphisms g_i from limit ordinals ω^{k_i+1} onto the trees $\mathfrak{T}_i$.
4. Combine **C** and $\omega^{k_1+1}, \ldots, \omega^{k_n+1}$ to obtain a Stone space X.
5. Combine f and $g_1, \ldots, g_n$ to obtain a d-morphism from X onto $\mathfrak{F}$.

For Step (1) we employ a method reminiscent of the Cantor–Bendixson theorem which represents each space X as the disjoint union of an open subspace U and a closed subspace F so that U is scattered and F is dense-in-itself.

LEMMA 5.1. *Let $\mathfrak{F} = (W, R)$ be a finite quasitree. Then there exist finite (possibly empty) frames $\mathfrak{D} = (D, R_D)$ and $\mathfrak{T} = (T, R_T)$ such that:*

(i) $W = D \cup T$, $D \cap T = \emptyset$, R_D is the restriction of R to D, and R_T is the restriction of R to T;
(ii) $\mathfrak{D}$ is a top-reflexive quasitree; and
(iii) $\mathfrak{T}$ is the disjoint union of irreflexive trees $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$.

Proof. We first build D by applying repeatedly the operator R^{-1} to W until we reach the (largest) fixpoint. More precisely, let $D_0 = W$ and $D_{i+1} = R^{-1}(D_i)$. Clearly $D_{i+1} \subseteq D_i$.

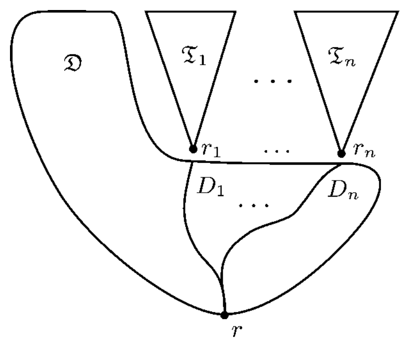


Fig. 1. Bisecting a quasitree into top-reflexive and irreflexive parts.

Since W is finite, at some stage $k < \omega$ we obtain $D_{k+1} = D_k$. Set $D = D_k$ and $\mathfrak{D} = (D, R_D)$. It follows from the construction that $\mathfrak{D}$ is a top-reflexive quasitree. Now set $T = W - D$ and $\mathfrak{T} = (T, R_T)$. Clearly $W = D \cup T$ and $D \cap T = \emptyset$. Moreover, $\mathfrak{T}$ consists of irreflexive points, and so there are no nondegenerate clusters in $\mathfrak{T}$. Let $r_1, \ldots, r_n$ be the minimal points of T. We set $T_i = R(r_i) \cup \{r_i\}$ and $\mathfrak{T}_i = (T_i, R_{T_i})$. Since $\mathfrak{F}$ is a quasitree, it is obvious that each $\mathfrak{T}_i$ is an irreflexive tree, that $T = \bigcup_{i=1}^n T_i$, and that the trees $\mathfrak{T}_i$ are disjoint. Therefore, $\mathfrak{T}$ is the disjoint union of irreflexive trees $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$. $\square$

The construction described in Lemma 5.1 is shown in Figure 1, where r denotes a root of $\mathfrak{F}$. For each $i \leq n$ let $D_i = R^{-1}(r_i)$. It is clear that $D_i \subseteq D$ and that for each $u, v \in D_i$ such that $u \neq v$ we have uRv or vRu. It is also clear that $R^{-1}(D) = D$ and that $D = \bigcup_{i=1}^n D_i$ iff each quasimaximal point of $\mathfrak{F}$ is irreflexive (in which case it is a maximal point of $\mathfrak{F}$).

For Step (2) we use Lemma 3.1 to obtain a d-morphism $f : \mathbf{C} \twoheadrightarrow \mathfrak{D}$. For each $i \leq n$ let $C_i = f^{-1}(D_i)$. It is readily seen that each C_i is a closed subspace of $\mathbf{C}$, hence a Stone space.

For Step (3) note that by Lemma 4.2, for each $i \leq n$, there exists a limit ordinal ω^{k_i+1} and a d-morphism $g_i : \omega^{k_i+1} \twoheadrightarrow T_i$.

Next we concentrate on Step (4). Fix $i \leq n$ and consider the limit ordinal ω^{k_i+1} and the Stone space $C_i \subseteq \mathbf{C}$. By Lemma 4.4, there exists a Stone space Y_i such that Y_i is a compactification of ω^{k_i+1} and the remainder Y_i^* is homeomorphic to C_i. We look at Y_i as the disjoint union of ω^{k_i+1} and Y_i^*, where ω^{k_i+1} is an open dense subspace of Y_i, Y_i^* is a closed subspace of Y_i homeomorphic to C_i, and $Y_i^* \subseteq \mathrm{cl}(\{\omega^{k_i} \cdot k : k < \omega\})$. The space Y_i is shown in Figure 2.

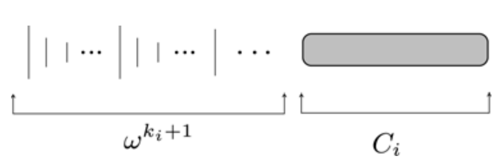


Fig. 2. The compactification of ω^{k_i+1} with the remainder C_i.

To build X we employ the following lemma:

LEMMA 5.2. *Let X, Y, Z be Stone spaces and $i : Z \hookrightarrow X$, $j : Z \hookrightarrow Y$ continuous injections. Then there exists a Stone space Z' and closed subspaces X', Y' of Z' such that X' is homeomorphic to X, Y' is homeomorphic to Y, and $Z' = X' \cup Y'$.*

Proof. This follows easily from the well-known fact that the category **Stone** of Stone spaces and continuous maps is closed under pushouts. In fact, Z' is the pushout of the diagram $X \hookleftarrow Z \hookrightarrow Y$ in the category **Stone**. More precisely, Z' is the factor space of the topological sum $X \oplus Y$ by the equivalence relation $\{(i(z), j(z)) : z \in Z\}$. □

We denote the pushout of the diagram $X \hookleftarrow Z \hookrightarrow Y$ by $X \oplus_Z Y$ and point out that since we are working with compact Hausdorff spaces, continuous injections are in fact topological (homeomorphic) embeddings. We consider an example which will be the starting point in the construction of the space X to follow.

Suppose we are given an ordinal ω^{k_1+1} and its compactification Y_1 such that Y_1^* is homeomorphic to a closed subspace C_1 of $\mathbf{C}$. Then using Lemma 5.2 we can identify the copies of C_1 present in both $\mathbf{C}$ and Y_1 to obtain the space $X_2 = \mathbf{C} \oplus_{C_1} Y_1$ such that:

(a) X_2 is a Stone space based on the disjoint union of ω^{k_1+1} and $\mathbf{C}$,
(b) ω^{k_1+1} is homeomorphic to an open subspace of X_2,
(c) $\mathbf{C}$ is homeomorphic to a closed subspace of X_2, and
(d) $Y_1^* \subseteq \mathrm{cl}(\{\omega^{k_1} \cdot k : k < \omega\})$.

This situation is depicted in Figure 3 below.

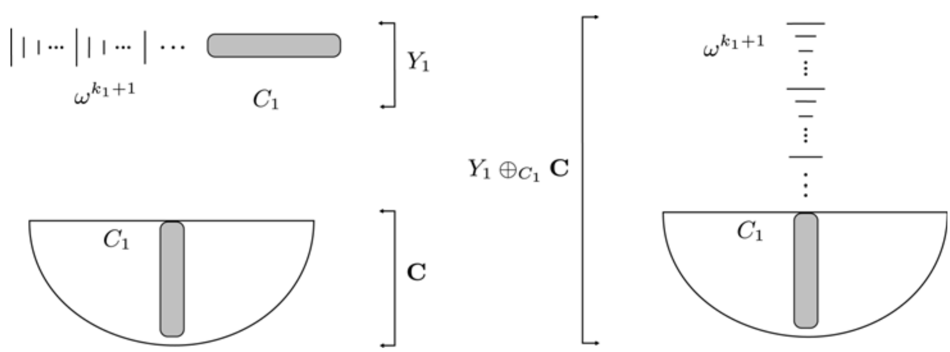


Fig. 3. Glueing of Y_1 and $\mathbf{C}$ along the shared closed subspace C_1.

Since $\mathbf{C}$ is a closed subspace of X_2, and C_2 is a closed subspace of $\mathbf{C}$, it follows that C_2 is (homeomorphic to) a closed subspace of X_2. This enables us to iterate the procedure and now adjoin ω^{k_2+1} to X_2 along C_2. A formal definition of the procedure is obtained by putting:

$$\begin{aligned}
&X_1 = \mathbf{C};\\
&X_2 = X_1 \oplus_{Y_1^*} Y_1;\\
&\vdots\\
&X_{n+1} = X_n \oplus_{Y_n^*} Y_n.
\end{aligned}$$

By identifying each Y_i^* with C_i, we can write each X_{i+1} as $X_{i+1} = X_i \oplus_{C_i} Y_i$. Finally, we set $X = X_{n+1}$. We clearly have that X is a Stone space, which concludes our Step (4). Pictorially X can be represented as in Figure 4.

We point out that since the constructed X is a metrizable Stone space, it is in fact homeomorphic to a closed subspace of $\mathbf{C}$ (see, e.g., Engelking, 1977, theorem 6.2.16). Moreover, since the topological sum of $\omega^{k_1+1}, \ldots, \omega^{k_n+1}$ is homeomorphic to a limit ordinal ω^{k+1}, we can think of the scattered part of X as a limit ordinal.

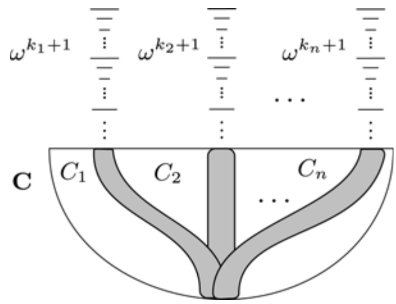


Fig. 4. The Stone space X.

For our final Step (5), we need to construct an onto d-morphism $h : X \to \mathfrak{F}$. For this we observe that X is the disjoint union of $\mathbf{C}$ and $\omega^{k_1+1}, \ldots, \omega^{k_n+1}$. Now let $x \in X$. We set

$$h(x) = \begin{cases} f(x), & x \in \mathbf{C} \\ g_i(x), & x \in \omega^{k_i+1}. \end{cases}$$

That h is a well defined onto map is obvious. It is left to be shown that h is a d-morphism. We first show that the restriction of h to each Y_i, which we denote by h_i, is a d-morphism. Let $\mathfrak{T}_i^+$ denote the range of h_i, which is a subframe of $\mathfrak{F}$ based on the set $T_i \cup D_i$. Let also f_i denote the restriction of f to C_i.

LEMMA 5.3. *The map $h_i : Y_i \to \mathfrak{T}_i^+$ is a d-morphism.*

Proof. To see that h_i is continuous, let U be an upset of $\mathfrak{T}_i^+$. If $U \subseteq T_i$, then $h_i^{-1}(U) = g_i^{-1}(U)$, which is open in Y_i since g_i is continuous and ω^{k_i+1} is an open subset of Y_i. If $U \cap D_i \neq \emptyset$, then $U = (U \cap D_i) \cup T_i$ and $h_i^{-1}(U) = f_i^{-1}(U) \cup \omega^{k_i+1}$, which is open in Y_i because $f_i^{-1}(U)$ is open in Y_i^* and ω^{k_i+1} is open and dense in Y_i.

To see that h_i is open, let U be an open subset of Y_i. If $U \subseteq \omega^{k_i+1}$, then $h_i(U) = g_i(U)$, which is an upset of $\mathfrak{T}_i$ since g_i is open. Therefore, $h_i(U)$ is also an upset of $\mathfrak{T}_i^+$. Suppose now that $U \cap Y_i^* \neq \emptyset$. Then $f(U) = g_i(U \cap \omega^{k_i+1}) \cup f_i(U \cap Y_i^*)$. By Lemma 4.4, $\omega^{k_i} \cdot k \in U \cap \omega^{k_i+1}$ for some $k < \omega$; and by Lemma 4.2, $g_i(\omega^{k_i} \cdot k) = r_i$. Since g_i is open, $g_i(U \cap \omega^{k_i+1}) = T_i$. Thus, as f_i is open, $T_i \cup f_i(U \cap Y_i^*)$ is an upset of $\mathfrak{T}_i^+$.

That h_i is r-dense is obvious because there are no reflexive points in T_i and f_i is r-dense. Similarly, as both f_i and g_i are i-discrete, it is easy to see that h_i is i-discrete. Consequently, $h_i : Y_i \to \mathfrak{T}_i^+$ is a d-morphism. $\square$

Now we show that h is a d-morphism. Since $\mathfrak{F}$ is finite, by Bezhanishvili *et al.* (2005, corollary 2.8), it is sufficient to show that $d(h^{-1}(w)) = h^{-1}(R^{-1}(w))$ for each $w \in W$, where d denotes the derived set operator of X.

LEMMA 5.4. *For each $w \in W$ we have $d(h^{-1}(w)) = h^{-1}(R^{-1}(w))$.*

Proof. First we recall that if Y is a closed subspace of X and $A \subseteq Y$, then $d_X(A) = d_Y(A)$. Now let $w \in W$. If $w \in D$, then $R^{-1}(w) \subseteq D$. Therefore, $h^{-1}(w) = f^{-1}(w)$ and $h^{-1}(R^{-1}(w)) = f^{-1}(R^{-1}(w))$. Since f is a d-morphism, we have:

$$d_X(h^{-1}(w)) = d_{\mathbf{C}}(f^{-1}(w)) = f^{-1}(R^{-1}(w)) = h^{-1}(R^{-1}(w)).$$

Next suppose that $w \in T_i$ for some $i \leq n$. Then $h^{-1}(w) = h_i^{-1}(w)$ and $h^{-1}(R^{-1}(w)) = h_i^{-1}(R_{T_i^+}^{-1}(w))$. By Lemma 5.3, h_i is a d-morphism. Now as Y_i is a closed subspace of X, we have:

$$d_X(h^{-1}(w)) = d_{Y_i}(h_i^{-1}(w)) = h_i^{-1}(R_{T_i^+}^{-1}(w)) = h^{-1}(R^{-1}(w)).$$

Thus, $d(h^{-1}(w)) = h^{-1}(R^{-1}(w))$. □

As a result, we obtain that for each quasitree $\mathfrak{F}$, there exists a (metrizable) Stone space X such that $\mathfrak{F}$ is a d-morphic image of X. In addition, if $\mathfrak{F}$ is top-irreflexive, then X is weakly scattered. Indeed, since h is open, continuous, and i-discrete, the h inverse image of the set of maximal points of $\mathfrak{F}$ is precisely the set of isolated points of X, which is dense in X because the set of maximal points of $\mathfrak{F}$ is dense in the Alexandroff topology of $\mathfrak{F}$. This immediately leads to our main theorem.

THEOREM 5.5. **K4** *is the modal logic of Stone spaces and* **K4G** *is the modal logic of weakly scattered Stone spaces.*

Proof. That **K4** is sound with respect to the class of Stone spaces follows from Proposition 2.2(1) because each Stone space is Hausdorff, hence a T_D-space. To prove completeness, let **K4** $\nvdash \varphi$. By Theorem 4.1(1), **K4** is complete with respect to finite quasitrees. Therefore, there exists a finite quasitree $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By our construction, there exists a Stone space X and an onto d-morphism $f : X \twoheadrightarrow \mathfrak{F}$. By Bezhanishvili *et al.* (2005, corollary 2.9), onto d-morphisms preserve validity of formulas. Therefore, $X \not\models \varphi$, and so **K4** is sound and complete with respect to the class of Stone spaces.

That **K4G** is sound with respect to the class of weakly scattered Stone spaces follows from Proposition 2.2(1) and an easily verifiable fact that if X is weakly scattered, then $X \models$ **G**. To prove completeness, let **K4G** $\nvdash \varphi$. By Theorem 4.1(4), **K4G** is complete with respect to finite top-irreflexive quasitrees. Therefore, there exists a finite top-irreflexive quasitree $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By our construction, there exists a weakly scattered Stone space X and an onto d-morphism $f : X \twoheadrightarrow \mathfrak{F}$. Now since onto d-morphisms preserve validity of formulas, we obtain $X \not\models \varphi$, and so **K4G** is sound and complete with respect to the class of weakly scattered Stone spaces. □

Since the class of Stone spaces is contained in the class of compact Hausdorff spaces and the class of weakly scattered Stone spaces is contained in the class of weakly scattered compact Hausdorff spaces, the following is an immediate corollary to Theorem 5.5:

COROLLARY 5.6. **K4** *is the modal logic of compact Hausdorff spaces and* **K4G** *is the modal logic of weakly scattered compact Hausdorff spaces.*

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