Georgian Mathematical Journal
Volume 13 (2006), Number 3, 419–432

THE SECOND OBSTRUCTION FUNCTOR

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Dedicated to the 90th birthday anniversary of G. Chogoshvili

Abstract. A functor responsible for second obstruction problems is defined and investigated on the category of topological spaces. In terms of this functor we formulate and prove a classification theorem for maps, which is the reformulation of all known classification theorems for the second obstruction.

2000 Mathematics Subject Classification: 55Q25, 55S20, 55S35, 55S37, 55S45.
Key words and phrases: Obstruction, homotopy, cohomology, cochain operation, k-invariant.

1. Introduction

Homotopy problems concerned with the first obstruction are effectively solved by a cohomology functor with coefficients in the first nontrivial homotopy group of the target space. The investigation of obstruction problems concerned with the second obstruction uses the first two nontrivial homotopy groups as well the cohomology operation defined by the k-invariant of the target space. The aim of the paper is to introduce and to study the new functor called the second obstruction functor (the first one being the cohomology functor) responsible for the problems of the second obstruction.

The functor in question $OB^{p,q}_{op}(X, \pi_p, \pi_q)$ is defined in Section 3 using a pair of abelian groups (π_p, π_q) as coefficient groups and involves a unary cochain operation

$$\mathrm{op} : Z^p(-, \pi_p) \to Z^{q+1}(-, \pi_q)$$

with $\mathrm{op}(0) = 0$. More precisely, a 'cocycle' of a space X is a pair of cochains

$$(c^p.c^q),\ c^p \in C^p(X, \pi_p),\ c^q \in C^q(X, \pi_q),\ \delta c^p = 0,\ \delta c^q = \mathrm{op}(c^p),$$

Two 'cocycles' $(c^p.c^q)$ and $(\overline{c}^p.\overline{c}^q)$ are 'cohomological' if there is a pair (c^{p-1}, c^{q-1}) such that $\overline{c}^p = c^p + \delta c^{p-1}$ and $\overline{c}^q = c^q + \delta c^{q-1} + \mathrm{op}_2(c^p, c^{p-1})$. Here op_2 is the binary cochain operation $Z^p(-, \pi_p) \times C^{p-1}(-, \pi_p) \to C^q(-, \pi_q)$ defined in terms of op. The second cohomology $OB_{\mathrm{op}}(X, \pi_p, \pi_q)$ is the set of classes of 'cohomological' 'cocycles' $(c^p.c^q)$. It is a pointed set and in some cases carries an abelian group structure. The functor is topologically invariant and the homotopic maps induce the same map (Section 3).

The main application, the classification theorem in terms of the second obstruction functor, is given in Section 7. It asserts that if the first two nontrivial homotopy groups of a space B are $\pi_p(B), \pi_q(B)$ and its k-invariant is k^{q+1}, then

ISSN 1072-947X / \$8.00 /

for the q-dimensional complex X, the set $\pi(X,B)$ of homotopy classes of maps of X into B is in one-to-one correspondence with the set

$$OB_{k^{q+1}}(X,\pi_p(B),\pi_q(B)).$$

If, moreover, the other homotopy groups of B vanish, then the restriction on the dimension of X is superfluous. In particular, if B is a two stage Postnikov complex, $B=K(\pi_p,p,k^{q+1},\pi_q)$, then

$$\pi(X,K(\pi_p,p,k^{q+1},\pi_q))=OB_{k^{q+1}}(X,\pi_p(B),\pi_q(B)).$$

This theorem reformulates the classification theorems of [5], [9], [13], [12], [15], [10], [11], [14].

In the previous papers [1], [2], [3] we considered the second obstruction problems in fibrations.

2. The Needed Cochain Operations

All cochains considered here are normalized.

If $\Delta^n=(b_0b_1\cdots b_n)$ is a standard simplex, then the standard triangulation of $I\times\Delta^n$ consists of simplexes

$$(b_0b_1\cdots b_i\bar b_i\cdots\bar b_n),\quad i=0,1,2,\dots,n,$$

and its faces.

If $\sigma^n:\Delta^n\to B$ is a singular simplex, then

$$\mathrm{id}\times\sigma^n\ :I\times\Delta^n\ \to I\times B$$

is a singular $(n+1)$-chain of $I\times B$. Denote by $I\times\mathrm{Sing}(B)$ the union of all

$$\mathrm{id}\times\sigma^n\ :I\times\Delta^n\ \to I\times B,\quad \sigma^n\in\mathrm{Sing}(B).$$

Obviously, $I\times\mathrm{Sing}(B)$ is a subcomplex of $\mathrm{Sing}(I\times B\ )$. The projection

$$pr:I\times B\to B$$

defines the projection

$$pr:I\times\mathrm{Sing}(B)\ \to\mathrm{Sing}(B).$$

Lemma 1. *If $z^n\in C^n(I\times\mathrm{Sing}(B)\ ,G)$, then*

$$[\delta_{I\times B}z^n][I\times\sigma^n]=z^n(\sigma^n\times 1)-z^n(\sigma^n\times 0)-[\delta_Bc^{n-1}](\sigma^n),$$

where

$$c^{n-1}(\sigma^{n-1})=z^n[I\times\sigma^{n-1}].$$

Proof. One has

$$\begin{aligned}[\delta_{I\times B}z^n][I\times\sigma^n]&=z^n[\partial(I\times\sigma^n)]=z^n[\partial I\times\sigma^n]-z^n[I\times\partial\sigma^n]\\&=z^n[1\times\sigma^n]-z^n[0\times\sigma^n]-z^n[\textstyle\sum(-1)^i(I\times\sigma^n_i)]\\&=z^n[\sigma^n\times 1]-z^n[\sigma^n\times 0]-\textstyle\sum(-1)^iz^n[I\times\sigma^n_i]\\&=z^n[1\times\sigma^n]-z^n[0\times\sigma^n]-[\delta_B\{z^n[(I\times\sigma^{n-1})]\}][\sigma^n].\end{aligned}$$

□

For a cocycle $u^{q+1} \in Z^{q+1}(K(\pi_p, p), \pi_q)$ let

$$\mathrm{op}_{u^{q+1}} : Z^p(B, \pi_p) \to Z^{q+1}(B, \pi_q)$$

be the unary cochain operation defined by it. We will write it as a function $\mathrm{op}^{q+1}_{u^{q+1}}(z^p)$. Obviously $\mathrm{op}^{q+1}_{u^{q+1}}(0) = 0$. Vice versa, each unary cochain operation $\mathrm{op} : Z^p(B, \pi_p) \to Z^{q+1}(B, \pi_q)$ with $\mathrm{op}(0) = 0$ is represented uniquely by some normalized cocycle $u^{q+1} \in Z^{q+1}(K(\pi_p, p), \pi_q)$.

Definition 1. For a unary cochain operation $\mathrm{op} : Z^p(B, \pi_p) \to Z^{q+1}(B, \pi_q)$ define the derived binary cochain operation

$$\mathrm{op}^q_2(z^p, c^{p-1}) \in C^q(B, \pi_q), \quad z^p \in Z^p(B, \pi_p), \quad c^{p-1} \in C^{p-1}(B, \pi_p), \quad \delta z^p = 0,$$

i.e.,

$$\mathrm{op}_2 : Z^p(B, \pi_p) \times C^{p-1}(B, \pi_p) \to C^q(B, \pi_q),$$

by

$$\mathrm{op}^q_2(z^p, c^{p-1})[\sigma^q] = \mathrm{op}^{q+1}(pr^* z^p - \delta_{I\times B} c^{p-1}_0)[I \times \sigma^q], \quad \sigma^q \in \mathrm{Sing}(B),$$

where c^{p-1}_0 is the cochain c^{p-1} embedded in $0 \times B$. Since $\mathrm{op}^{q+1}(z^p)$ is a normalized cocycle, evidently,

$$\mathrm{op}^q_2(z^p, 0) = 0.$$

Proposition 1. $\delta[\mathrm{op}^q_2(z^p, c^{p-1})] = \mathrm{op}^{q+1}(z^p) - \mathrm{op}^{q+1}(z^p - \delta c^{p-1})$, *in particular,* $\delta[\mathrm{op}^q_2(0, c^{p-1})] = -\mathrm{op}^{q+1}(-\delta c^{p-1})$; *hence* $\delta[\mathrm{op}^q_2(0, c^{p-1})] = 0$ *when* $\delta c^{p-1} = 0$.

Proof. Let $z^p \in Z^p(B, \pi_p)$ and $c^{p-1} \in C^{p-1}(B, \pi_p)$; then by Lemma 1 the $(q+1)$-cocycle of $I \times \mathrm{Sing}(B)$

$$\mathrm{op}^{q+1}(pr^* z^p - \delta_{I\times B} c^{p-1}_0)$$

leads to the equality

$$\begin{aligned} 0 = {} & \mathrm{op}^{q+1}(pr^* z^p - \delta_{I\times B} c^{p-1}_0)[1 \times \sigma^{q+1}] \\ & - \mathrm{op}^{q+1}(pr^* z^p - \delta_{I\times B} c^{p-1}_0)[0 \times \sigma^{q+1}] - [\delta_B c^q](\sigma^{q+1}), \end{aligned}$$

where

$$c^q(\sigma^q) = \mathrm{op}^{q+1}(pr^* z^p - \delta_{I\times B} c^{p-1}_0)[I \times \sigma^q];$$

equivalently,

$$\delta_B\, \mathrm{op}^q_2(z^p, c^{p-1})(\sigma^{q+1}) = \mathrm{op}^{q+1}(z^p)(\sigma^{q+1}) - \mathrm{op}^{q+1}(z^p - \delta_B c^{p-1})(\sigma^{q+1}),$$

i.e.,

$$\delta_B\, \mathrm{op}^q_2(z^p, c^{p-1}) = \mathrm{op}^{q+1}(z^p) - \mathrm{op}^{q+1}(z^p - \delta_B c^{p-1}).$$ □

Definition 2. Define a derived unary operation

$$\mathrm{op}_3 : Z^{p-1}(X, \pi_p) \to Z^q(X, \pi_q)$$

as

$$\mathrm{op}^q_3(z^{p-1}) = \mathrm{op}^q_2(0, z^{p-1}).$$

One has $\mathrm{op}^q_3(0) = 0$.

Example 1. op_2 of the unary cochain operation $\mathrm{op}^{2p-i}=z^p\smile_i z^p$, $i=0,1,2,\dots$, is the binary cochain operation

$$\mathrm{op}_2^{2p-i-1}(z^p,c^{p-1})=c^{p-1}\smile_i z^p+\overline{z}^p\smile_i c^{p-1}+c^{p-1}\smile_{i-1}c^{p-1},$$

where $\overline{z}^p=z^p+\delta c^{p-1}$. In particular, $0p_3$ of the unary cochain operation $z^p\smile_i z^p$, $i=0,1,2,\dots$, is the unary cochain operation $c^{p-1}\smile_{i-1}c^{p-1},\delta c^{p-1}=0$.

Proof. Let us prove the second assertion.

$$\begin{aligned}
&\mathrm{op}_3^q(z^{p-1})(\sigma^q)=\mathrm{op}_2^q(0,z^{p-1})(\sigma^q)\\
&=\mathrm{op}^{q+1}(\delta_{I\times B}z_0^{p-1})(I\times\sigma^q)=[\delta_{I\times B}z_0^{p-1}\smile_i\delta_{I\times B}z_0^{p-1}][I\times\sigma^q]\\
&=\sum(-1)^i[\delta_{I\times B}z_0^{p-1}\smile_i\delta_{I\times B}z_0^{p-1}][(b_0b_1\cdots b_j\overline{b}_j\cdots\overline{b}_{q-1}\overline{b}_q)].
\end{aligned}$$

Steenrod's definition of $\smile_i$ [13] considers the set of $i+1$ vertices in the set $(b_0b_1b_2\cdots b_j\overline{b}_j\cdots\overline{b}_{q-1}\overline{b}_q)$. In our case, the corresponding product is 0 if j is not q and if the $(i+1)$-th vertex is not $\overline{b}_q$. In the case where $j=q$ and $(i+1)$-th vertex is $\overline{b}_q$, the product is equal to the corresponding product of z^{p-1} for $(b_0b_1\cdots b_q)$ with the set of considered vertices without the $(i+1)$-th vertex, i.e., $z^{p-1}\smile_{i-1}z^{p-1}$. □

Example 2. Let $\mathrm{op}:Z^p(B,Z)\to Z^{np}(B,Z)$ be the n-fold $\smile$-product. Then $\mathrm{op}_2:Z^p(B,Z)\times C^{p-1}(B,Z)\to C^{np}(B,Z)$ is

$$\mathrm{op}_2(z^p,c^{p-1})=\sum_1^n(-1)^{(i+1)p}\ \underbrace{\overline{z}^p\smile\cdots\smile\overline{z}^p}_{i-1}\smile c^{p-1}\smile\underbrace{z^p\smile\cdots\smile z^p}_{n-i},$$

where $\overline{z}^p=z^p-\delta c^{p-1}$.

Proof. If $\Delta^{np-1}=(b_0b_1\cdots b_{np-1})$ is the standard simplex, then

$$(-1)^i(b_0b_1\cdots b_i\overline{b_i}\cdots\overline{b}_{np-1})\tag{2.1}$$

is the standard oriented np-simplex of $I\times\Delta^{np-1}$. The value of the n-fold product of the cocycle

$$pr^*z^p-\delta_{I\times\Delta^{np-1}}c_0^{p-1}$$

(here c_0^{p-1} is c^{p-1} embedded in $0\times B$) on (2.1) is 0 unless $j=p-1,2p-1,3p-1,\dots,np-1$. If $j=ip-1$, then the value is

$$-(-1)^p(-1)^{ip-1}\,\underbrace{\overline{z}^p\smile\cdots\smile\overline{z}^p}_{i-1}\smile c^{p-1}\smile\underbrace{z^p\smile\cdots\smile z^p}_{n-i};$$

adding these elements we obtain $\mathrm{op}_2(z^p,c^{p-1})$. □

If $\Delta^q=(b_0b_1\cdots b_n)$ is a standard simplex, then the standard triangulation of $\Delta^2\times\Delta^n$ is

$$b_0b_1b_2\cdots b_i\overline{b}_i\cdots\overline{b}_j\overline{\overline{b}}_j\overline{\overline{b}}_{j+1}\cdots\overline{\overline{b}}_{n-1}\overline{\overline{b}}_n),\quad i\le j,\quad i,j=0,1,2,\dots,n.$$

If $\sigma^n:\Delta^n\to B$ is a singular simplex, then

$$\mathrm{id}\times\sigma^n\ :\Delta^2\times\Delta^n\to\Delta^2\times B$$

is a singular $(n+2)$-chain of $\Delta^2 \times B$. Denote by $\Delta^2 \times \operatorname{Sing}(B)$ the union of all

$$\operatorname{id} \times \sigma^n : \Delta^2 \times \Delta^n \to \Delta^2 \times B, \quad \sigma^n \in \operatorname{Sing}(B).$$

Obviously, $\Delta^2 \times \operatorname{Sing}(B)$ is a subcomplex of $\operatorname{Sing}(\Delta^2 \times B)$.

Lemma 2. *If* $z^{n+1} \in C^{n+1}(\Delta^2 \times \operatorname{Sing}(B), G)$, *then*

$$[\delta_{\Delta^2 \times B} z^{n+1}][\Delta^2 \times \sigma^n] = [\delta_B c^{n-1}_{(0,1,2)}][\sigma^n] + \{c^n_{(1,2)} - c^n_{(0,2)} + c^n_{(0,1)}\}[\sigma^n],$$

where

$$c^{n-1}_{(0,1,2)} = z^{n+1}[(\Delta^2 \times \sigma^{n-1})],$$
$$c^n_{(1,2)} = z^{n+1}[(1,2) \times \sigma^n],$$
$$c^n_{(0,2)} = z^{n+1}[(0,2) \times \sigma^n],$$
$$c^n_{(0,1)} = z^{n+1}[(0,1) \times \sigma^n].$$

Proof. One has

$$\begin{aligned}
&[\delta_{\Delta^2 \times B} z^{n+1}][\Delta^2 \times \sigma^n] = z^{n+1}[\partial(\Delta^2 \times \sigma^n)] \\
&= z^{n+1}[\partial\Delta^2 \times \sigma^n] + z^{n+1}[\Delta^2 \times \partial\sigma^n] \\
&= z^{n+1}[(1,2) \times \sigma^n] - z^{n+1}[(0,2) \times \sigma^n] + z^{n+1}[(0,1) \times \sigma^n] \\
&\quad + z^{n+1}\Big[\sum(-1)^i(\Delta^2 \times \sigma^n_i)\Big] \\
&= c^n_{(1,2)}[\sigma^n] - c^n_{(0,2)}[\sigma^n] + c^n_{(0,1)}[\sigma^n] + \sum(-1)^i z^{n+1}[(\Delta^2 \times \sigma^n_i)] \\
&= c^n_{(1,2)}[\sigma^n] - c^n_{(0,2)}[\sigma^n] + c^n_{(0,1)}[\sigma^n] + [\delta_B\{z^{n+1}[(\Delta^2 \times \sigma^{n-1})]\}[\sigma^n] \\
&= \{c^n_{(1,2)} - c^n_{(0,2)} + c^n_{(0,1)}\}[\sigma^n] + [\delta_B\{c^{n-1}_{(012)}][\sigma^n].
\end{aligned}$$

□

Consider $\Delta^2 \times B$ and the projection

$$pr : \Delta^2 \times \operatorname{Sing}(B) \to \operatorname{Sing}(B).$$

Define

$$\operatorname{op}_4 : Z^p(B, \pi_p) \times C^{p-1}(B, \pi_p) \times C^{p-1}(B, \pi_p) \to C^{q-1}(B, \pi_q)$$

as

$$\operatorname{op}_4^{q-1}(z^p, c^{p-1}, \overline{c}^{p-1})[\sigma^{q-1}] = \operatorname{op}^{q+1}((pr^* z^p + \delta_{\Delta^2 \times B} c^{p-1}_{01} + \delta_{\Delta^2 \times B} \overline{c}^{p-1}_0)[\Delta^2 \times \sigma^{q-1}],$$

where c^{p-1}_{01} is $pr^* c^{p-1}$ embedded in $(01) \times B$, $\overline{c}^{p-1}_0$ is $\overline{c}^{p-1}$ embedded in $0 \times B$.

Proposition 2.

$$\begin{aligned}
\delta \operatorname{op}_4^{q-1}(z^p, c^{p-1}, \overline{c}^{p-1}) &= \operatorname{op}_2^q(z^p, c^{p-1}) \\
&\quad - \operatorname{op}_2^q(z^p, c^{p-1} + \overline{c}^{p-1}) + \operatorname{op}_2^q(z^p + \delta c^{p-1}, \overline{c}^{p-1}),
\end{aligned}$$

in particular,

$$\operatorname{op}_2^q(z^p, -c^{p-1}) + \operatorname{op}_2^q(z^p + \delta c^{p-1}, c^{p-1}) = -\delta \operatorname{op}_4^{q-1}(z^p, -c^{p-1}, c^{p-1}).$$

Proof. Use Lemma 2 for the cocycle $z^{q+1} = \mathrm{op}^{q+1}(pr^* z^p - \delta_{\Delta^2\times B} c_{01}^{p-1} - \delta_{\Delta^2\times B}\overline{c}_0^{p-1})$. One has

$$0 = [\delta_B \mathrm{op}_4^{q-1}(z^p, c^{p-1}, \overline{c}^{p-1})[\sigma^q] + \mathrm{op}_2^q(z^p, c^{p-1}) \\ - \mathrm{op}_2^q(z^p, c^{p-1} + \overline{c}^{p-1}) + \mathrm{op}_2^q(z^p + \delta c^{p-1}, \overline{c}^{p-1}). \qquad \square$$

Definition 3. The restriction of operation op_4^{q-1} on

$$0 \times Z^{p-1}(B, \pi_p) \times Z^{p-1}(B, \pi_p) \subset Z^p(B, \pi_p) \times C^{p-1}(B, \pi_p) \times C^{p-1}(B, \pi_p)$$

is the operation

$$\mathrm{op}_5^{q-1} : Z^{p-1}(B, \pi_p) \times Z^{p-1}(B, \pi_p) \to C^{q-1}(B, \pi_q).$$

The following proposition is a corollary of Proposition 2.

Proposition 3.

$$\delta_B[\mathrm{op}_5^{q-1}(z^{p-1}, \overline{z}^{p-1})] = \mathrm{op}_3^q(z^{p-1}) + \mathrm{op}_3^q(\overline{z}^{p-1}) - \mathrm{op}_3^q(z^{p-1} + \overline{z}^{p-1}),$$

i.e., $\mathrm{op}_3^q(z^{p-1})$ *is an additive operation.*

Example 3. op_5 of the unary cochain operation $z^p \smile_i z^p$ is the binary cochain operation $z^{p-1} \smile_{i-1} \overline{z}^{p-1}$.

3. The Functor OB

Let X be a space, (π_p, π_q) be a pair of abelian groups and $\mathrm{op} : Z^p(-, \pi_p) \to Z^{q+1}(-, \pi_q)$ be a cochain operation. Consider pairs of normalized cochains

$$(z^p,\ z^q),$$

where $z^p \in C^p(X, \pi_p)$, $z^q \in C^q(X, \pi_q)$, such that

$$\delta z^p = 0,\ \delta z^q = \mathrm{op}(z^p).$$

Definition 4. $(z^p,\ z^q) \sim (\overline{z}^p,\ \overline{z}^q)$ if there are $c^{p-1} \in C^{p-1}(X, \pi_p)$ and $c^{q-1} \in C^{q-1}(X, \pi_q)$ such that $z^p + \delta c^{p-1} = \overline{z}^p$ and $z^q + \delta c^{q-1} + \mathrm{op}_2(\overline{z}^p, c^{p-1}) = \overline{z}^q$.

Proposition 4. $\sim$ *is an equivalence relation*

Proof. (a) $\sim$ is reflexive: $(z^p,\ z^q) \sim (z^p,\ z^q)$: $z^p + \delta_X 0 = z^p$ and $z^q = z^q + \delta_X 0 + \mathrm{op}_2(z^p, 0)$; here $\mathrm{op}_2(z^p, 0^{p-1}) = 0$ because it is a value of op on degenerated simplexes.

(b) $\sim$ is transitive: Let $z^p + \delta_X c^{p-1} = \overline{z}^p$, $z^q + \delta_X c^{q-1} + \mathrm{op}_2(\overline{z}^p, c^{p-1}) = \overline{z}^q$ and $\overline{z}^p + \delta_X u^{p-1} = \widetilde{z}^p$, $\overline{z}^q + \delta_X u^{q-1} + \mathrm{op}_2(\widetilde{z}^p, u^{p-1}) = \widetilde{z}^q$. It follows $z^p + \delta_X c^{p-1} + \delta_X u^{p-1} = \widetilde{z}^p$ and $[z^q + \delta_X c^{q-1} + \mathrm{op}_2(\overline{z}^p, c^{p-1})] + \delta_X u^{q-1} + \mathrm{op}_2(\widetilde{z}^p, u^{p-1}) = \widetilde{z}^q$. On the other hand, by Proposition 2, $\mathrm{op}_2(\widetilde{z}^p - \delta c^{p-1}, c^{p-1}) - \mathrm{op}_2(\widetilde{z}^p, c^{p-1} + u^{p-1}) + \mathrm{op}_2(\widetilde{z}^p, u^{p-1}) - \delta[\mathrm{op}_4(\widetilde{z}^p, c^{p-1}, u^{p-1})] = 0$; so we have $z^q + \delta_X[c^{q-1} - \mathrm{op}_4(\widetilde{z}^p, c^{p-1}, u^{p-1}) + u^{q-1}] + \mathrm{op}_2(\widetilde{z}^p, c^{p-1} + u^{p-1}) = \widetilde{z}^q$.

(c) $\sim$ is symmetric: If $(z^p,\ z^q) \sim (\overline{z}^p,\ \overline{z}^q)$, then $z^p + \delta_X c^{p-1} = \overline{z}^p$ and $z^q + \delta_X c^{q-1} + \mathrm{op}_2(\overline{z}^p, c^{p-1}) = \overline{z}^q$; hence $\overline{z}^p + \delta_X(-c^{p-1}) = z^p$ and $\overline{z}^q + \delta_X(-c^{q-1}) - \mathrm{op}_2(\overline{z}^p, c^{p-1}) = z^q$. Here, by Proposition 2, $-\mathrm{op}_2(\overline{z}^p, -c^{p-1}) = \mathrm{op}_2(z^p, -c^{p-1}) - \delta\,\mathrm{op}_4(z^p, -c^{p-1}, c^{p-1})$; hence $\overline{z}^q + \delta_X(-c^{q-1} - \mathrm{op}_4(z^p, -c^{p-1}, c^{p-1})) + \mathrm{op}_2(z^p, -c^{p-1}) = z^q$. $\square$

Definition 5. The set of equivalence classes of pairs $(z^p,\ z^q)$ is denoted by $OB_{k^{q+2}}(X, \pi_p, \pi_q)$. The element containing $(0,0)$ is denoted by 0.

$OB_{k^{q+2}}(X, \pi_p, \pi_q)$ is a contravariant functor on the category of topological spaces with values in the pointed sets.

Proposition 5. *If $i : X \subset Y$ is a homology isomorphism, then $OB(i)$ is a one-to-one map.*

Proof. (a) Surjectivity: Consider (z^p_X, z^q_X). In our case $Z^i(Y,G) \to Z^i(X,G)$ is epimorphfic. Then there is z^p_Y such that $z^p_Y|X = z^p_X$. We have $\delta_X z^q_X = \mathrm{op}(z^p_X)$; it follows that $\mathrm{op}(z^p_Y)$ is a coboundary, i.e., there is $\overline{z}^q_Y$ such that $\delta_Y \overline{z}^q_Y = \mathrm{op}(z^p_Y)$; then $\delta_X(z^q_X - i^*\overline{z}^q_Y) = 0$. There is $\overline{\overline{z}}^q_Y$ with $\delta\overline{\overline{z}}^q_Y = 0$ and $i^*\overline{\overline{z}}^q_Y = z^q_X - i^*\overline{z}^q_Y$, hence $z^q_X = i^*(\overline{\overline{z}}^q_Y + \overline{z}^q_Y)$. On the other hand, $\delta_Y(\overline{\overline{z}}^q_Y + \overline{z}^q_Y) = \delta_Y\overline{z}^q_Y = \mathrm{op}(z^p_Y)$; hence the pair $(z^p_Y, \overline{\overline{z}}^q_Y + \overline{z}^q_Y)$ covers (z^p_X, z^q_X)

(b) Injectivity: Let (z^p_Y, z^q_Y) and $(\overline{z}^p_Y, \overline{z}^q_Y)$ be such that $i^*(z^p_Y, z^q_Y) \sim i^*(\overline{z}^p_Y, \overline{z}^q_Y)$; then $i^*z^p_Y - i^*\overline{z}^p_Y = \delta_X c^{p-1}_X$ and $i^*z^q_Y - i^*\overline{z}^q_Y = \delta_X c^{q-1}_X + \mathrm{op}(c^{p-1}_X, i^*\overline{z}^p_Y)$. Then there is c^{p-1}_Y such that $i^*c^{p-1}_Y = c^{p-1}_X$ and $\delta_Y c^{p-1}_Y = z^p_Y - \overline{z}^p_Y$, and $i^*z^q_Y - i^*\overline{z}^q_Y = \delta_X c^{q-1}_X + \mathrm{op}_2(c^{p-1}_X, i^*\overline{z}^p_Y)$; it follows that $z^q_Y - \overline{z}^q_Y - \mathrm{op}_2(c^{p-1}_Y, \overline{z}^p_Y)$ is a coboundary. □

As easy corollaries one has the following two facts.

Theorem 1. *If $f, g : X \to Y$ are homotopic maps, then $OB(f) = OB(g)$.*

Proof. The projection $I \times X \to X$ and embeddings $h_0, h_1 : X \to I \times X$, where $h_0(x) = (0,x)$ and $h_1(x) = (1,x)$, are homology isomorphisms. Hence $OB(X) \to OB(I \times X)$ is a one-to-one map. □

Theorem 2 (Topological invariance)**.** *If L is a simplicial complex and $|L|$ is its realization, then $OB_{k^{q+2}}(|L|, \pi_p, \pi_q) \to OB_{k^{q+2}}(L, \pi_p, \pi_q)$ is a one-to-one map.*

Proposition 5 can be sharpened in form of

Theorem 3. *If $f : X \to Y$ is a homology isomorphism, then $OB(f)$ is a one-to-one map.*

Proof. In our case one has the homology isomorphisms

$$X \subset Z(f) \text{ and } Y \subset Z(f),$$

($Z(f)$ being the cylinder of the map f). On the other hand, the triangle

$$\begin{array}{ccc} X & \to & Z(f) \\ & \searrow & \uparrow \\ & & Y \end{array}$$

is commutative up to homotopy. The assertion follows from Proposition 5 and Theorem 1. □

Lemma 3. *There is a functorial exact sequence of pointed sets*

$$H^{p-1}(X,\pi_p) \overset{\mathrm{op}_3}{\to} H^q(B,\pi_q) \to OB_{k^{q+1}}(X,\pi_p,\pi_q) \to H^p(X,\pi_p) \overset{\mathrm{op}}{\to} H^{q+1}(X,\pi_q).$$

Here op_3 is a homomorphism.

Proof. Proposition 3 shows that op_3 is a homomorphism. The rest is checked easily. □

Example 4. Assume that $\pi_2 = Z$, $\pi_3 = Z$ and let $k^4 \in Z^4(K(Z,2),Z)$ be $\zeta^2 \smile \zeta^2$, where ζ^2 is the basic cocycle in $C^2(Z,2,Z)$. Then the definition of the functor $OB_{k^4}(B,\pi_2,\pi_3)$ in terms of the cocycle $k^4=\zeta^2\smile\zeta^2$ becomes as follows: one considers pairs (z^2,z^3), such that $\delta z^2=0$, $\delta z^3 = z^2\smile z^2$. The transformation is $\overline{z}^2 = z^2+\delta c^1$, $\overline{z}^3 = z^3+\delta c^2 + c^1\smile z^2 + \overline{z}^2\smile c^1$.

Example 5. Assume that $p\succeq 2$, $\pi_p = Z$, $\pi_{p+1}=Z_2$ and let $k^{p+2}\in Z^{p+2}(K(Z,p),Z_2)$ be $\zeta^p \smile_{p-2} \zeta^p$, where ζ^p is the basic cocycle in $K(Z,p)$. Then the definition of the functor $OB_{k^{p+2}}(B,\pi_p,\pi_{p+1})$ in terms of the cocycle $k^{p+2}=\zeta^p\smile_{p-2}\zeta^p$ is as follows: one considers pairs (z^p,z^{p+1}) such that $\delta z^p=0$, $\delta z^{p+1} = z^p\smile_{p-2} z^p$. The transformation is $\overline{z}^p = z^p+\delta c^{p-1}$, $\overline{z}^{p+1} = z^{p+1}+\delta c^p + \overline{z}^p\smile_{p-2} c^{p-1} + c^{p-1}\smile_{p-2} z^p + c^{p-1}\smile_{p-3} c^{p-1}$. Define the addition in $OB^{p,\,p+1}_{\zeta^p\smile_{p-2}\zeta^p}(B,Z,Z/2)$ as

$$(z^p,z^{p+1})+(z_1^p,z_1^{p+1}) = (z^p+z_1^p, z^{p+1}+z_1^{p+1}+z^p\smile_{p-1} z_1^p);$$

it is an abelian group.

Remark 1. The exact sequence of Lemma 3 for the functor

$$OB^{p,p+1}_{\zeta^p\smile_{p-2}\zeta^p}(X,Z,Z/2),$$

i.e.,

$$H^{p-1}(X,Z)\xrightarrow{Sq^2} H^{p+1}(X,Z/2Z)\to OB^{p,p+1}_{\zeta^p\smile_{p-2}\zeta^p}(X,Z,Z/2)$$
$$\to H^p(X,Z)\xrightarrow{Sq^2} H^{p+2}(X,Z/2Z)$$

is the exact sequence of abelian groups.

Example 6. Assume that $p\succeq 2$, $\pi_p=Z$, $\pi_{np}=Z$ and let $k^{np}\in Z^{np}(K(Z,p),Z)$ be an n-fold $\smile$-product of the basic cocycle ζ^p in $K(Z,p)$. Then the definition of the functor $OB^{p,np-1}_{k^{np}}(B,Z,Z)$ in terms of the cocycle $k^{np}=\zeta^p\smile_n\zeta^p$ in view of example 2 is as follows: one considers pairs (z^p,z^{np-1}) such that $\delta z^p=0$, $\delta z^{np-1}=z^p\smile z^p\smile(\times n)\smile z^p$. The transformation is $\overline{z}^p = z^p+\delta c^{p-1}$,

$$\overline{z}^{np-1} = z^{np-1}+\delta c^{np-2}+\sum_1^n (-1)^{(i+1)p}\underbrace{\overline{z}^p\smile\dots\smile\overline{z}^p}_{i-1}\smile c^{p-1}\smile\underbrace{z^p\smile\dots\smile z^p}_{n-i}.$$

4. The k-Invariant of a Space

Let Y be a topological space and $\pi_p=\pi_p(Y)$ its homotopy group. For each element $\alpha\in\pi_p$ choose a map $(\Delta_p,\partial\Delta_p)\to(Y,*)$ representing it. Then we have a map of the p-skeleton of $K(\pi_p,p)$ into the space Y which extends to the $(p+1)$-skeleton. If $\pi_i(Y)=0$, $p<i<q$, then the first obstruction for this map to be extended on the $(q+1)$-skeleton of $K(\pi_p,p)$ is a cocycle $k^{q+1}\in Z^{q+1}(K(\pi_p,p),\pi_q(Y))$. Its class is an invariant of the space Y and is called a k-invariant.

Remark 2. We assume that the map $K(\pi_p, p)^{(q)} \to Y$ is fixed.

Here are some known k-invariants (written as unary cochain operations) (see [5]);

1) S^n is the sphere, $n = 2, 3, \ldots$: the k-invariant is the operation $z^n \smile_{n-2} z^n$ (z^n is the singular main n-cocycle).

2) PC^n is the complex projective n-space: the k-invariant is the operation $\underbrace{z^2 \smile \cdots \smile z^2}_{(n+1)\text{-times}}$; here z^2 is the main 2-cocycle (see [4]).

3) A space B is aspherical in dimensions less than n: there is a pairing

$$\pi_n(B) \otimes \pi_n(B) \to \pi_{n+1}(B)$$

and the k-invariant is the operation $z^n \smile_{n-2} z^n$ (z^n is the singular main n-cocycle) (see [10], [11]).

5. The Characteristic Class of a Map

Let B be a space, $\pi_i(B) = 0$, $i < q$, $i \neq p$. Let $k^{q+1} \in Z(K(\pi_p, p), \pi_q)$ be the k-invariant of B and, as already assumed in Remark 2, the map

$$K(\pi_p, p)^{(q)} \to B$$

be fixed. Assume the map

$$f : X \to B$$

to be given. We are going to assign to it a pair

$$(z_f^p, z_f^q),$$

the second obstruction pair, as follows. Fix a point $* \in B$. Consider the cylinder $I \times \mathrm{Sing}(X)$. Consider the map on the lower base as

$$f_* : 0 \times \mathrm{Sing}(X) \to B$$

and on the p-skeleton of the upper base as the constant map

$$1 \times \mathrm{Sing}(X)^{(p)} \to * \in B.$$

Since $\pi_i(B) = 0$, $i < p$, the map

$$[1 \times \mathrm{Sing}(X)]^{(p)} \cup [0 \times \mathrm{Sing}(X)] \to B$$

extends to a map

$$[I \times \mathrm{Sing}(X)]^{(p)} \cup [0 \times \mathrm{Sing}(X)] \to B,$$

where $[I \times \mathrm{Sing}(X)]^{(p)}$ is the p-skeleton. Hence an obstruction (cellular) $(p+1)$-cocycle $\zeta^{p+1} \in Z^{p+1}(I \times X, 0 \times X)$ is defined. Let $z^p(\sigma^p) = \zeta^{p+1}(I \times \sigma^p)$. It follows that $\delta_X z^p = 0$ (z^p is called the first obstruction cocycle). Let us proceed as follows. Change the map on the upper cell $1 \times \sigma^p$ (which is a constant map) by $z^p(\sigma^p)$ (i.e., consider the map $1 \times \mathrm{Sing}(X)^{(p)} \xrightarrow{z^p} K(\pi_p, p)^{(q)} \to B$; the second map has already been fixed above) and extend it to the q-skeleton of the top of $I \times \mathrm{Sing}(X)$ as

$$1 \times \mathrm{Sing}(X)^{(q)} \xrightarrow{z^p} K(\pi_p, p)^{(q)} \to B.$$

So we have the map

$$[I \times \mathrm{Sing}(X)]^{(p)} \cup [1 \times \mathrm{Sing}(X)]^{(q)} \cup [0 \times \mathrm{Sing}(X)] \to B.$$

This map evidently extends to the $(p+1)$-skeleton and, then extends – via $\pi_i(B) = 0, p < i < q$ – to the q-skeleton of $[I \times \mathrm{Sing}(X), 0 \times \mathrm{Sing}(X)]$. One has an obstruction cocycle ζ^{q+1} on $(I \times \mathrm{Sing}(X), 0 \times \mathrm{Sing}(X))$. Let $z^q(\sigma^q) = \zeta^{q+1}(I \times \sigma^q)$. It follows that

$$(\delta_X z^q)(\sigma^{q+1}) = k^{q+1}(z^p | \sigma^{q+1}),$$

where k^{q+1} is the obstruction cocycle of the already fixed map

$$K(\pi_p, p)^{(q)} \to B$$

(i.e., the k-invariant of B), z^p is the p-cocycle defined above and $z^p|\sigma^{q+1}$ is the restriction of cocycle z^p on the singular $(q+1)$-simplex σ^{q+1}, i.e., the $(q+1)$-simplex of complex $K(\pi_p, p)$. So the pair $(z^p, z^q) \equiv (z_f^p, z_f^q)$, the second obstruction pair, is defined.

Remark 3. The procedure of constructing the obstruction pairs is such that if the pair

$$(z_f^p, z_f^q)$$

is defined on the subcomplex of $\mathrm{Sing}(X)$, then it extends (not uniquely) on the $\mathrm{Sing}(X)$.

Definition 6. Let

$$d(f) \in OB_{k^{q+1}}(X, \pi_p(B), \pi_q(B))$$

be the class containing the second obstruction pair (z_f^p, z_f^q). $d(f)$ is called *the characteristic class of the map* f.

Theorem 4. *$d(f)$ is correctly defined.*

Proof. Let us consider two embeddings

$$i_0, i_1 : \mathrm{Sing}(X) \to \mathrm{Sing}(I \times X),$$

where $i_0(\sigma) = (0, \sigma)$ and $i_1(\sigma) = (1, \sigma)$. These maps are homotopic and hence, by Theorem 1,

$$OB(i_0) = OB(i_1). \tag{5.1}$$

Let (z^p, z^q) and $(\overline{z}^p, \overline{z}^q)$ be two obstruction pairs assigned as above to the same map f. Consider the identical homotopy

$$F : I \times X \to B, \quad F(x,t) = f(x).$$

As indicated in Remark 3, the procedure of constructing the pairs is such that there exists a pair (z_F^p, z_F^q) with

$$(z_F^p, z_F^q)|(0 \times X) = (z^p, z^q)$$

and

$$(z_F^p, z_F^q)|(1 \times X) = (\overline{z}^p, \overline{z}^q).$$

It follows by (5.1) that $(z^p, z^q) \sim (\overline{z}^p, \overline{z}^q)$. $\square$

We are interested in the set of all pairs (z^p_f, z^q_f) for given f. For this, consider the homotopy

$$F : I \times X \to B$$

and a pair

$$(z^p_F, z^q_F),$$

then

Lemma 4. *$z^p_{f_1} = z^p_{f_0} + \delta_X c^{p-1}$ and $z^q_{f_1}(b_0b_1 \cdots, b_q) = z^q_{f_0}(b_0b_1 \cdots, b_q) + \delta_X c^{q-1}[(b_0b_1 \cdots, b_q)] + \sum(-1)^i[\mathrm{op}(z^p_F)][(b_0b_1 \cdots b_i\bar{b}_i \cdots \bar{b}_q)]$, where*

$$f_1(x) = F(1, x), \quad f_0(x) = F(0, x),$$

$$c^{p-1}(b_0b_1 \cdots b_{p-1}) = \sum(-1)^i z^p_F[(b_0b_1 \cdots b_i\bar{b}_i \cdots \bar{b}_{p-1})],$$

$$c^{q-1}(b_0b_1 \cdots b_{q-1}) = \sum(-1)^i z^q_F[(b_0b_1 \cdots b_i\bar{b}_i \cdots \bar{b}_{q-1})].$$

Proof. The first equality follows from Lemma 1 assuming z^n to be z^p_F. The second equality follows from the same Lemma 1 assuming z^n to be z^q_F. □

Remark 4. This lemma holds no matter what extension (z^p_F, z^q_F) is considered. Hence, to use the above formulas it is advisable to consider simple extensions.

Theorem 5. *For the given f the class $d(f)$ consists of all possible pairs (z^p_f, z^q_f).*

Proof. Consider $(z^p, z^q) \sim (\bar{z}^p_f, \bar{z}^q_f)$, i.e.,

$$z^p + \delta c^{p-1} = \bar{z}^p_f, \;\; z^q + \delta c^{q-1} + \mathrm{op}_2(\bar{z}^p_f, c^{p-1}) = \bar{z}^q_f. \tag{5.2}$$

Consider the trivial homotopy

$$F : I \times X \to B, \quad F(t, x) = f(x).$$

Define the procedure of constructing second obstruction pairs on $I \times X$ as the image of this procedure on X via the projection

$$pr : I \times X \to X.$$

Let us change the 'cross-section' on $0 \times \sigma^{p-1}$ by $-c^{p-1}(\sigma^{p-1})$ (i.e., change the map $I \times 0 \times \sigma^{p-1} \to B$ by $-c^{p-1}(\sigma^{p-1}) \in \pi_p(B)$). It is obvious that the procedure of constructing the second obstruction pair on $I \times X$ can be continued in such a way that on the upper base it remains unchanged. Then we get a pair $(\widetilde{z}^p_{I\times X}, \widetilde{z}^q_{I\times X})$ such that $\widetilde{z}^p_F = pr^*(\bar{z}^p_f) - \delta(c^{p-1}_0)$, where c^{p-1}_0 is c^{p-1} embedded in $0 \times X$,

$$\bar{z}^p_f = z^p + \delta c^{p-1}, \quad (\widetilde{z}^p_{I\times X}\widetilde{z}^q_{I\times X})|X \times 1 = (\overline{z_f}^p, \overline{z_f}^q)$$

and

$$(\widetilde{z}^p_{I\times X}\widetilde{z}^q_{I\times X})|X \times 0 = (\widetilde{z}^p_{0\times X}, \widetilde{z}^q_{0\times X}) = (z^p, \widetilde{z}^q_{0\times X}).$$

By Lemma 4,

$$\begin{aligned}\bar{z}^q_f(b_0b_1 \cdots, b_q) &= \widetilde{z}^q_{X\times 0}(b_0b_1 \cdots, b_q) + \delta_X \widetilde{c}^{q-1}[(b_0b_1 \cdots, b_q)] \\ &\quad + \sum(-1)^i[\mathrm{op}(\widetilde{z}^p_F)][(b_0b_1 \cdots b_i\bar{b}_i \cdots \bar{b}_q)].\end{aligned}$$

In view of

$$\widetilde{z}^p_F = pr^*(\overline{z}^p_f) - \delta_{I\times X}(c^{p-1}_0)$$

the above equality becomes

$$\overline{z}^q_f = \widetilde{z}^q_{0\times X} + \delta\widetilde{c}^{q-1} + \mathrm{op}_2(\overline{z}^p_f, c^{p-1}).$$

Change the cross section on $0\times\sigma^{q-1}$ by $-(\widetilde{c}^{q-1} - c^{q-1}_1)(\sigma^{p-1})$. Then

$$\widetilde{z}^q_{0\times X} + \delta\widetilde{c}^{q-1} = \widetilde{\widetilde{z}}^q_{0\times X} + \delta(-\widetilde{c}^{q-1} + c^{q-1}) + \delta\widetilde{c}^{q-1} = \widetilde{\widetilde{z}}^q_{0\times X} + \delta c^{q-1}.$$

Hence

$$\overline{z}^q_f = \widetilde{\widetilde{z}}^q_{0\times X} + \delta c^{q-1} + \mathrm{op}_2(\overline{z}^p_f, c^{p-1}).$$

It follows by 5.2 that $\widetilde{\widetilde{z}}^q_{X\times 0} = z^q$ and hence (z^p, z^q) is a obstruction pair assigned to f. □

6. The Characteristic Class of a Space

Let B be a space, $\pi_i(B) = 0$, $i < q$, $i \neq p$. Let $k^{q+1} \in Z^{q+1}(K(\pi_p, p), \pi_q)$ be the k-invariant of B.

Definition 7. We define the characteristic class of the space B

$$d(B) \in OB_{k^{q+1}}(B, \pi_p(B), \pi_q(B)),$$

as the characteristic class of the identity map

$$\mathrm{id} : B \to B,$$

i.e.,

$$d(B) = d(\mathrm{id} : B \to B) \in OB_{k^{q+1}}(B, \pi_p(B), \pi_q(B)).$$

Lemma 5. *If $f : X \to B$, then*

$$OB(f)[d(B)] = d(f).$$

Proof is obvious. □

7. The Homotopy Classification of Maps

Let B be a space, $\pi_i(B) = 0$, $i < q$, $i \neq p$, and let

$$k^{q+1} \in Z^{q+1}(K(\pi_p(B), p), \pi_q(B))$$

be the k-invariant of B. If X is a space and (z^p_X, z^q_X) is a pair for

$$OB_{K^{q+1}}(X, \pi_p(B), \pi_q(B)),$$

then a map

$$u_{(z^p_X,, z^q_X)} : X^{(q)} \to B$$

is defined in obvious way.

Lemma 6. *If X is a complex and $f : X \to B$ is a map, then the maps*

$$f, u_{(z^p_f,, z^q_f)} : X^{(q)} \to B$$

are homotopic. In particular, if $(z^p_f, z^q_f) = (0, 0)$, then f is homotopic to 0.

Proof follows immediately from the procedure of constructing of the pair (z_f^p, z_f^q). □

Theorem 6. *If B is a space, $\pi_i = 0$, $i \neq p$, $i < q$, k^{q+1} is the k-invariant of B and X is a complex of dimension q and $f, g : X \to B$, then by considering the functor $OB_{k^{q+1}}(Y, \pi_p(B), \pi_q(B))$ one has $D(f)[d(B)] = D(g)[d(B)]$ if and only if f is homotopic to g. In particular, f is null homotopic if and only if $D(f)[d(B)]=0$.*

Proof. Let $D(f)[d(B)] = D(g)[d(B)]$ and $(z_B, z_B) \in d(B)$, then by Theorem 5 we have $(z_f, z_f) = (z_g, z_g)$; from Lemma 6 it follows that f and g are homotopic. □

Theorem 7 (Steenrod's classification theorem)**.** *If X is a complex of dimension q and B is as in Theorem 6, then*

$$OB_{k^{q+1}}(X, \pi_p(B), \pi_q(B))$$

is in one-to-one correspondence with $\pi(X, B)$, the set of homotopy classes of maps of X into B.

Proof. Follows from preceding theorem and Lemma 6. □

8. Hopf Invariant

Let X and B be simplicial complexes. Then by Theorem 4 we can use a simplicial version of OB and the above theorem can be formulated for simplicial maps. In particular,

Theorem 8. *Let $f : S^{n+1} \to S^n$ be a simplicial map in some subdivisions of the spheres. Let z^n be the main simplicial cocycle of S^n and let $c^{n-1} \in C^{n-1}(S^{n+1}, Z)$ be such that $\delta c^{n-1} = f^* z^n$, then f is an essential map if and only if the cocycle of S^{n+1}, $c^{n-1} \smile_{n-3} c^{n-1} + c^{n-1} \smile_{n-2} f^* z^n$, is not a coboundary.*

Proof. The k-invariant of S^n is $z^n \smile_{n-2} z^n$. Hence one must use the functor of Example 4 if $n = 2$ and that of Example 5 if $n > 2$. It follows that $(z^n, 0) \in d(S^n)$ (simplicial $(n+1)$-cochains are 0). Transforming $f^*(z^n, 0) = (f^* z^n, 0)$ using the pair $(-c^{n-1}, 0)$, $\delta c^{n-1} = f^* z^n$, we obtain the pair $(0, c^{n-1} \smile_{n-3} c^{n-1} + c^{n-1} \smile_{n-2} f^* z^n)$. □

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(Received 20.11.2004; revised 28.06.2006)

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