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A LOCAL SYSTEM OF SPACES AND A BIGRADED MODEL OF FIBRATION

NODAR BERIKASHVILI

Dedicated to the memory of G. Lomadze

Abstract. On the singular complex of a space, a local system of acyclic spaces is constructed leading, for Serre fibrations, to a bigraded differential model for the chain complex of the total space.

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1. Introduction

In independent works [4] and [3] the following bigraded model of the chain complex of a fibration $F \to E \xrightarrow{\pi} B$ is given: the bigraded generators of the bigraded model $C_{**}(E) = \{C_{pq}(E,G)\}$ are continuous maps $\sigma^{pq} : \Delta^p \times \Delta^q \to E$ such that the composition $\pi\sigma^{pq} : \Delta^p \times \Delta^q \to B$ does not depend on the second argument. The differentials $d^{'}$ and $d^{''}$ are defined in an obvious way via the differentials of standard simplexes Δ^p and Δ^q.

In [1] another bigraded model in the case B is a polyhedron, $B = |K|$, is considered as a bicomplex $C_{**}(E) = C_p(B, C_q(\pi^{-1}[\mathrm{st}(\sigma)], G)$, where $\mathrm{st}(\sigma)$ denotes the star of a simplex $\sigma \in K$. The proof, in the spirit of Zeeman's dihomology [6], is an easy one. The general case of a fibration with arbitrary B is reduced then to that of polyhedron B by a standard but tedious technique.

Our aim is to construct the bigraded model of E by an analog technique for the general case of B. If $\mathrm{Sing}(B)$ is the singular complex of the space B, then in general it is not any longer an acyclic complex $\mathrm{st}(\sigma^m)$, $\sigma^m \in \mathrm{Sing}(X)$ (for example, $\mathrm{st}(\sigma^0)$ is homologically isomorphic to $\mathrm{Sing}(B)$ when B is a connected space). In Section 2 we construct such a system of spaces. In Section 3 we construct the bigraded model of a Serre fibration. In Sections 4 and 5 we equip the cohomology version of a bigraded model with cochain operations $\smile -$ and $\smile_i -$.

The main result of this paper was announced in [2]

2. A Local System of Spaces on the Singular Complex

Let X be a space and $\mathrm{Sing}(X)$ be its singular complex. If $\sigma^m \in \mathrm{Sing}(X)$, i.e.,

$$\sigma^m : \Delta^m \to X,$$

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and if $(i_1, i_2, \dots, i_k)$ is a subset of the set $(0, 1, 2, \dots, m)$, then its complementary subset, i.e., the subset

$$(\cdots \widehat{i_1} \cdots \widehat{i_2}, \dots, \widehat{i_k} \cdots) \subset (0, 1, 2, \dots, m)$$

is a $(m-k)$-dimensional face of $\Delta^m = (0, 1, 2, 3, \dots, m)$. Let denote it by $\Delta^m_{i_1 i_2 \cdots i_k}$. The restriction of σ^m on $\Delta^m_{i_1 i_2 \cdots i_k}$ let us call by the $(i_1, i_2, \dots, i_k)$-th face of σ^m and denote it by $\sigma^m_{i_1, i_2, \dots, i_k}$. One has $\dim(\sigma^m_{i_1, i_2, \dots, i_k}) = m - k$.

Let us fix σ^n and define a simplicial complex $K_1(\sigma^n)$ as follows. Consider all pairs $f = (\sigma^m, 0 \preceq i_1 < i_2 < \cdots < i_k \preceq m)$ such that $\sigma^m_{i_1, i_2, \dots, i_k} = \sigma^n$; for such a pair consider a copy of the standard m-simplex and denote it by Δ^m_f; let $K_1(\sigma^n) = \cup_{m,f} \Delta^m_f$ (a disjoint union). Each simplex Δ^m_f, $f = (\sigma^m, 0 \preceq i_1 < i_2 < \cdots < i_k \preceq m)$, contains a standard n-simplex Δ^n as the face $[\Delta^m_f]_{i_1, i_2, \dots, i_k}$. Let us identify all them in $K_1(\sigma^n)$ as one Δ^n. The resulting factor space $K_2(\sigma^n)$ is still a simplicial complex. Obviously, $K_2(\sigma^n)$ is a cone with the first vertex of the simplex Δ^n at the top. Hence there is a standard contraction of $K_2(\sigma^n)$ to the first vertex of Δ^n.

If for a pair $f = (\sigma^m, 0 \preceq i_1 < i_2 < \cdots < i_k \preceq m)$, $(j_1 < j_2 < \cdots < j_p)$ is a subset of $(i_1 < i_2 < \cdots < i_k)$ we define a new pair $f_{j_1, j_2, \dots, j_p}$, the $(j_1, j_2, \dots, j_p)$-th face of the pair f, as the pair $[\sigma^m_{j_1, j_2, \dots, j_p}, 0 \preceq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-p} \preceq m - p]$, where $\sigma^m_{j_1, j_2, \dots, j_p}$ is the $(j_1, j_2, \dots, j_p)$-th face of σ^m and the numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-p}$ correspond in an obvious way to the elements of $(i_1, i_2, \dots, i_k) \backslash (j_1, j_2, \dots, j_p)$. In the complex $K_2(\sigma^n)$ the pairs f and $f_{j_1, j_2, \dots, j_p}$ are independently given as simplexes. Let us jet identify the pair $f_{j_1, j_2, \dots, j_p}$ with the $(j_1 j_2 \cdots j_p)$-th face of the pair f. So we get a factor space of $K_2(\sigma^n)$, the space $K(\sigma^n)$. Of course, still $\Delta^n \subset K(\sigma^n)$ and $K(\sigma^n)$ is still contractible to 0, the first vertex of Δ^n.

$K(\sigma)$ is not any longer a simplicial complex because two different faces of a simplex Δ^m_f can be identified in $K(\sigma)$. But $K(\sigma)$ is obviously a simplicial set without degeneracy operators.

Below $K(\sigma)$ denotes a simplicial set or its realization.

Definition 1. If σ^n is a face of σ^p, $\sigma^n < \sigma^p$, then one defines a map

$$\epsilon_{\sigma^n \sigma^p} : K(\sigma^p) \to K(\sigma^n)$$

as the one induced by the obvious inclusion

$$K_2(\sigma^p) \subset K_2(\sigma^n).$$

Definition 2. For each $\Delta^m_f = (\sigma^m,\ 0 \preceq i_1 < i_2 < \cdots < i_k \preceq m)$ consider the map

$$\sigma^m : \Delta^m_f \to X.$$

These maps define a map $K_2(\sigma^n) \to X$. Obviously, this map factors as

$$K_2(\sigma^n) \to K(\sigma^n) \to X.$$

Let the second map be denoted by ϵ_{σ^n},

$$\epsilon_{\sigma^n} : K(\sigma^n) \to X.$$

So, on the singular complex $\mathrm{Sing}(X)$, we have constructed a local system of simplicial sets $\{K(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\}$.

Lemma 1. *The system* $\{K(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\}$ *is a local system of simplicial sets on* $\mathrm{Sing}(X)$*, i.e.,*

(a) $K(\sigma)$, $\sigma \in \mathrm{Sing}\, X$, *is a simplicial set without degeneracy operators;*
(b) *if* $\sigma^n < \sigma^p$, *then the map* $\epsilon_{\sigma^n\sigma^p}$ *is a map of simplicial sets.*
(c) *if* $\sigma^n < \sigma^p < \sigma^q$, *then the diagram*

$$\begin{tikzcd} K(\sigma^q) \arrow[r] \arrow[dr] & K(\sigma^p) \arrow[d] \\ & K(\sigma^n) \end{tikzcd}$$

is commutative;
(d) *if* $\sigma^n < \sigma^p$, *then the diagram*

$$\begin{tikzcd} K(\sigma^p) \arrow[r] \arrow[dr] & K(\sigma^n) \arrow[d] \\ & X \end{tikzcd}$$

is commutative.

Proof. The commutativity of the above diagrams is obvious from the definitions of the maps involved. □

The local system $\{K(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\}$ is a covariant functor on the category of topological spaces: if $f : X \to Y$ is a map, then for a simplex $\sigma^n \in \mathrm{Sing}(X)$ we define in an obvious way the map

$$K(f) : K(\sigma^n) \to K(f(\sigma^n))$$

and the diagrams

$$\begin{tikzcd} K(\sigma^p) \arrow[r] \arrow[d] & K(\sigma^n) \arrow[d] \\ K(f(\sigma^p)) \arrow[r] & K(f(\sigma^n)) \end{tikzcd},$$

$\sigma^n < \sigma^p$, and

$$\begin{tikzcd} K(\sigma^n) \arrow[r] \arrow[d] & K(f(\sigma^n)) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

are commutative.

Let X be a polyhedron, $X = |L|$, where L is a simplicial complex. Consider L as an ordered simplicial complex. Then L is a subcomplex of $\mathrm{Sing}(X)$ and $L \subset \mathrm{Sing}(X)$ is a homology isomorphism. For $\sigma \in L$ consider $\mathrm{st}(\sigma) \subset L$. Then $\{\mathrm{st}(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\}$ is a local system of simplicial contractible complexes on L.

Obviously, one has the map $\mathrm{st}(\sigma) \to K(\sigma)$, i.e., there is a map of local systems on L

$$\{\mathrm{st}(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\} \to \{K(\sigma), \epsilon_{\sigma\sigma}, \epsilon_\sigma\}.$$

3. A Bigraded Chain Model for a Serre Fibration

Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration and consider the local system of acyclic spaces

$$K(\sigma), \quad \sigma \in \mathrm{Sing}(B),$$

on the singular complex of B, which is discussed in Section 2. For each $\sigma \in \mathrm{Sing}(B)$ the map

$$\epsilon_\sigma : K(\sigma) \to B$$

induces a fibration

$$F \to E(\sigma) \xrightarrow{\pi} K(\sigma).$$

In this way we have a local system of spaces

$$E(\sigma), \quad \sigma \in \mathrm{Sing}(B),$$

on the singular complex of B. If G is a coefficient group, then

$$C_q(E(\sigma), G), \quad \sigma \in \mathrm{Sing}(B),$$

form a local system of abelian groups on B and we obtain a bigraded abelian group

$$U_{*,*} = C_*(B, C_*(E(\sigma), G))$$

that carries two differentials

$$d' : C_p(B, C_q(E(\sigma), G)) \to C_{p-1}(B, C_q(E(\sigma), G))$$

(the differential of B) and

$$d'' : C_p(B, C_q(E(\sigma), G)) \to C_p(B, C_{q-1}(E(\sigma), G))$$

(the differential of $E(\sigma)$). Thus we have a bigraded complex

$$U_{*,*} = \{C_*(B, C_*(E(\sigma), G)), d^{'}, d^{''}\}. \tag{3.1}$$

Lemma 2. *If the base B of a Serre fibration $F \to E \xrightarrow{\pi} B$ is contractible, then $F \to E$ is a homology isomorphism.*

Proof. The first proof (using the Whitehead theorem) is trivial: from an exact sequence of homotopy groups of a fibration it follows that $\pi_*(F) \to \pi_*(E)$ is an isomorphism; from the Whitehead theorem it follows that $F \to E$ is a homology isomorphism.

The second proof is an elementary one and consists in constructing (inductively on q) a chain map

$$f : C_q(E) \to C_q(\pi^{-1}(*))$$

and a chain homotopy

$$D_q : C_q(E) \to C_{q+1}(E))$$

with

$$f|\pi^{-1}(0)) = id : C_*(\pi^{-1}(*)) \to C_*(\pi^{-1}(*))$$

and

$$\partial D + D\partial = id - f. \qquad \square$$

By Lemma 2 the first spectral sequence of the bigraded complex (3.1) can be written as follows

$$E^2_{pq} = H_p(B, H_q(F)) \underset{p}{\Longrightarrow} H_{p+q}(U_{**}) \tag{3.2}$$

(where $H_q(F)$ is a local system on B).

The map $\epsilon_\sigma : K(\sigma) \to B$ induces a chain map

$$C_*(E(\sigma), G) \to C_*(E, G).$$

Hence there is a map

$$f_0 : C_0(B, C_*(E(\sigma), G) \to C_*(E, G)$$

with $f_0 d'' = \partial f_0$. It is easy to check that the composition

$$C_1(B, C_*(E(\sigma), G) \to C_0(B, C_*(E(\sigma), G) \to C_*(E, G)$$

is a zero homomorphism. Hence if one defines

$$f_p = 0 : C_p(B, C_*(E(\sigma), G) \to C_*(E, G), \quad p > 0,$$

then one has a chain map

$$f : U = C_*(B, C_*(E(\sigma), G) \to C_*(E, G).$$

Theorem 1. *The chain map*

$$f : C_*(B, C_*(E(\sigma), G)) \to C_*(E, G))$$

is a homology isomorphism.

The proof will be given in Section 6.

Theorem 1 has (by 3.2) the following corollary.

Theorem 2 (Serre)**.** *If $F \to E \xrightarrow{\pi} B$ is a Serre fibration, then there is a spectral sequence*

$$E^2_{pq} = H_p(B, H_q(F, G)) \underset{p}{\Longrightarrow} H_{p+q}(E, G).$$

4. The Multiplicative Structure in the Bigraded Model

Let

$$C_*(B, C_*(E(\sigma)))$$

be the bigraded chain model of a Serre fibration $F \to E \xrightarrow{\pi} B$. Then

$$\{C^*(B, C^*(E(\sigma), G)), \delta', \delta''\}$$

is a model of cochain complex of E and one has a cochain map

$$C^*(E, G) \to C^*(B, C^*(E(\sigma), G)) \tag{4.1}$$

which is defined using a standard map

$$C^q(E, G) \to C^0(B, C^q(E(\sigma), G)).$$

Let $G = \Lambda$ be a commutative ring. Then the bicomplex

$$C^*(B, C^*(E(\sigma), \Lambda))$$

carries a multiplicative structure defined as follows: if $x^{p,q} \in C^p(B, C^q(E(\sigma), \Lambda))$, $y^{r,s} \in C^r(B, C^s(E(\sigma), \Lambda))$, then let

$$x^{p,q} y^{r,s} = (-1)^{qr} x^{p,q} \smile y^{r,s},$$

where on the right is a $\smile$-product in the singular complex of B of the p-cochain $x^{p,q}$ and r-cochain $y^{r,s\cdot}$, and the coefficients, q-cochains and s-cochains are paired by the $\smile$-product in the space $E(\sigma^{p+r})$. The product is subject of the standard relation

$$d(xy) = dxy + (-1)^{p+q} x dy.$$

One sees that map (4.1) is multiplicative. So we have

Theorem 3. *Map* (4.1) *induces an isomorphism*

$$H^*(E, \Lambda) \to H^*(C^*(B, C^*(E(\sigma), \Lambda)))$$

of cohomology algebras.

5. Steenrod's $\smile_i$-Product in a Serre Fibration

Consider a graded algebra (A, δ) (where A is a Z_2-module) with Steenrod's $\smile_i$-products, i.e., one has

$$\delta(a^p \smile_i b^q) = \delta a^p \smile_i b^q + a^p \smile_i \delta b^q + a^p \smile_{i-1} b^q + b^q \smile_{i-1} a^p, \tag{5.1}$$

and $\smile_0$ is associative. If (B, δ) is another such algebra, then the tensor product

$$A \otimes B$$

carries $\smile_i$-products defined as

$$\sum_{j=0}^{i} \smile_j \otimes \smile_{i-j} T^j, \tag{5.2}$$

where $T(a^p \otimes b^r) = (b^p \otimes a^r)$. It is easy to check that Steenrod's coboundary formula (5.1) holds. Formula (5.2) can be rewritten in the detailed form

$$\begin{aligned}(a^p \otimes b^r) \smile_i (a^q \otimes b^s) &= (a^p \smile_0 a^q) \otimes (b^r \smile_i b^s) + (a^p \smile_1 a^q) \otimes (b^s \smile_{i-1} b^r) \\ &+ (a^p \smile_2 a^q) \otimes (b^r \smile_{i-2} b^s) + (a^p \smile_3 a^q) \otimes (b^s \smile_{i-3} b^q) + \cdots .\end{aligned} \tag{5.3}$$

Let

$$C^*(B, C^*(E(\sigma), G))$$

be the bigraded cochain model of a Serre fibration $F \to E \xrightarrow{\pi} B$.

Let $G = Z_2$. Then by (5.3) the bicomplex

$$C^*(B, C^*(E(\sigma), Z_2))$$

carries Steenrod's $\smile_i$ $-$products defined as follows: if $x^{p,q} \in C^p(B, C^q(E(\sigma), Z_2))$, $y^{r,s} \in C^r(B, C^s(E(\sigma), Z_2))$, then let $x^{p,q} \smile_i y^{r,s}$ be defined by

$$x^{p,q} \smile_i y^{r,s} = x^{p,q} \smile_0^i y^{r,s} + x^{p,q} \overline{\smile}_1^{i-1} y^{r,s} + x^{p,q} \smile_2^{i-2} y^{r,s} + x^{p,q} \overline{\smile}_3^{i-3} y^{r,s} + \cdots ,$$

where on the right the expression

$$x^{p,q} \smile^k_l y^{r,s}$$

means a $\smile_l$-product in the singular complex of B of the p-cochain $x^{p,q}$ and the r-cochain $y^{r,s\cdot}$ and the coefficients, q-cochains and s-cochains are paired by the $\smile_k$-product in the space $E(\sigma^{p+r-i})$; on the other hand, the expression

$$x^{p,q} \overline{\smile}^k_l y^{r,s}$$

means a $\smile_l$-product in the singular complex of B of the p-cochain $x^{p,q}$ and the r-cochain $y^{r,s\cdot}$ and the coefficients, s-cochains and q-cochains are paired by the $\smile_k$-product in the space $E(\sigma^{p+r-i})$.

The Steenrod coboundary formula (5.1) obviously holds.

Proposition 1. *The cochain map*

$$C^*(E, Z_2) \to C^*(B, C^*(E(\sigma), Z_2)) \tag{5.4}$$

preserves $\smile_i$ -product.

Proof. The map

$$C^q(E, Z_2) \to C^0(B, C^q(E(\sigma), Z_2))$$

obviously preserves $\smile_i$-product. □

Theorem 4. *Map* 4.1 *induces an isomorphism of cohomology algebras*

$$H^*(E, Z_2) \to H^*(C^*(B, C^*(E(\sigma), Z_2)))$$

as modules over the Steenrod algebra.

6. The Proof of Theorem 1

6.1. **The bigraded model of a covering.** Let E be a space and $U = \{U_\alpha\}$ be its open covering, i.e., U_α is an open set of E and $\cup U_\alpha = E$. The well-known tool in the singular homology theory is

Proposition 2 (see [5])**.** *If $S_U(E)$ is the union of subcomplexes* $\mathrm{Sing}(U_\alpha)$*, then $S_U(E) \subset \mathrm{Sing}(E)$ is a homology isomorphism.*

Let $N(U)$ be the nerve of the covering U. In the spirit of [6], consider, in the product

$$N(U) \times S_U(E),$$

the subcomplex $V(U)$ of all pairs (σ, τ), $\tau \in \mathrm{Sing}\,|\sigma|$, where $|\sigma| = U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$.

Denote $V_\tau = \{\sigma | (\sigma, \tau) \in V\} \subset N(U)$ and $V_\sigma = \{\tau | (\sigma, \tau) \in V\} \subset S_U(E)$.

Obviously, V_τ is a simplex and hence it is acyclic.

If G is an abelian group, then the chain complex $C(V, G)$ is bigraded

$$V_{**} = C_{**}(V, G). \tag{6.1}$$

The system $\{V_\sigma\}$ is a local system of spaces on $N(U)$. So we can consider the above bicomplex as

$$C_*(N(U), C_*(V_\sigma, G)). \tag{6.2}$$

As in Section 3 we have a chain map

$$C_*(S_U(E), G) \longleftarrow C_*(N(U), C_*(V_\sigma, G)).$$

Lemma 3. *The above map is a homology isomorphism*

Proof. The second spectral sequence of bicomplex (6.2) is the same as that of bicomplex (6.1). Being first term of the spectral sequence

$$E^1_{p,q} = C_p(S_U(E), H_q(V_\varsigma, G)$$

and being V_τ acyclic, one has

$$E^1_{p,q} = 0, \quad p > 0,$$
$$E^1_{p,q} = C_q(S_U(E), G), \quad p = 0.$$

It follows that the first term of the spectral sequence is equal to the chain complex $C_*(S_U(E), G)$. □

The first spectral sequence of the bigraded complex 6.2 is

$$E^2_{pq} = H_p(N(U), H_q(|\sigma|, G)) \underset{p}{\Rightarrow} H_{p+q}[C_*(N(U), C_*(V_\sigma, G)].$$

Hence, by Lemma 3 and Proposition 2, we get

$$E^2_{pq} = H_p(N(U), H_q(|\sigma|, G)) \underset{p}{\Rightarrow} H_{p+q}(E, G).$$

Remark 1. This is a version for the singular homology theory of Leray spectral sequence of a covering.

6.2. **The bigraded model of a fibration with the base a simplicial complex.** If the base of a fibration $F \to E \xrightarrow{\pi} B$ is a polyhedron, $B = |L|$, then consider the open covering of B, $U = \{\widetilde{st}(a)\}$, where $\widetilde{st}(a)$ is the union of all open simplexes of L having a as a vertex. The well-known fact is

Proposition 3. *$N(U)$ is isomorphic to L.*

Consider the open covering of E: $\pi^{-1}U = \{\pi^{-1}[\widetilde{st}(a)]\}$. Obviously,

$$N(\pi^{-1}(U)) = N(U).$$

Hence Lemma 3 and Proposition 2 give a homology isomorphism

$$C_*(E, G) \leftarrow C_*(L, C_*(\pi^{-1}[\widetilde{st}(\sigma)], G)). \tag{6.3}$$

The map

$$\widetilde{st}(\sigma) \to \mathrm{st}(\sigma)]$$

induces an isomorphism

$$H_q(\pi^{-1}[\widetilde{st}(\sigma)], G) \to H_q(\pi^{-1}[\mathrm{st}(\sigma)], G).$$

Hence one has a homology isomorphism

$$C_*(L, C_*(\pi^{-1}[\widetilde{st}(\sigma)], G)) \to C_*(L, C_*(\pi^{-1}[\mathrm{st}(\sigma)], G)),$$

i e., by (6.3) we have

Proposition 4. *For $F \to E \xrightarrow{\pi} |L|$ one has the bigraded complex*

$$C_*(L, C_*(\pi^{-1}[\mathrm{st}(\sigma), G)))$$

and a homology isomorphism

$$C_*(E, G) \leftarrow C_*(L, C_*(\pi^{-1}[\mathrm{st}(\sigma), G))).$$

On the other hand, the map

$$C_*(L, C_*(\pi^{-1}[\mathrm{st}(\sigma)], G)) \to C_*(|L|, C_*(E(\sigma), G))$$

is a homology isomorphism.

6.3. **Proof of Theorem 1.** If B is a space, then there are a simplicial complex L and $f : |L| \to B$, inducing an isomorphism of homotopy groups. Let for a Serre fibration $F \to E \xrightarrow{\pi} B$, $F \to E' \xrightarrow{\pi} |L|$ be the induced fibration. Then $C_*(E', G) \to C_*(E, G)$ is a homology isomorphism.

Proof of Theorem 1. One has a diagram

$$\begin{tikzcd}
C_*(E,G) & C_*(E',G) \arrow[l] & C_*(L, C_*(\pi^{-1}[\mathrm{st}(\sigma)], G)) \arrow[l] \arrow[d] \\
& C_*(B, C_*(E[K(\sigma)], G))) \arrow[ul, "s"'] & C_*(|L|, C_*(E(K(\sigma)], G)))) \arrow[l]
\end{tikzcd}.$$

It is easy to check that the diagram is commutative. The above reasoning shows that the maps of the diagram except s are homology isomorphisms. Hence s, too, is a homology isomorphism. $\square$

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Author's address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: berika@rmi.acnet.ge
URL: `http://www.rmi.acnet.ge/~berika`