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ON THE OBSTRUCTION FUNCTOR

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Abstract. An obstruction functor in terms of Postnikov towers is introduced and studied.

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1. Introduction

Let $\pi_* = \{\pi_1, \pi_2, \dots, \pi_i, \dots\}$ be a graded abelian group. According to [1] the obstruction functor $DO(-, \pi_*)$ is a contravariant functor from the category Top of topological spaces B into the category of sets with distinguished subset and element $* \in \overline{DO}(B, \pi_*) \subset DO(B, \pi_*)$, with the following properties.

Property 1.1. If f is homotopic to g then $DO(f) = DO(g)$.

Property 1.2. For any Serre fibration $F \to E \to B$ with suitable assumption on the fiber, there is defined a functorial (with respect to induced fibrations) element $do(E) \in DO(B, \pi_*)$, where $\pi_* = \pi_*(F)$ is the sequence of homotopy groups of the fiber, and E has a cross section if and only if $do(E) \in \overline{DO}(B, \pi_*)$.

Property 1.3. If $\pi_i = 0$ for all $i \neq n$ then DO is the singular cohomology group $H^{n+1}(-, \pi_n)$, $\overline{DO} = 0$ and $do(E)$ is the classical first obstruction class.

Property 1.4. Functor DO is constructed in terms of cochains of space B and groups π_i. There is a reasonable criterion to define whether two cochain representations give the same element of $DO(B, \pi_*)$ or not.

The construction of the obstruction functor in first nontrivial case when $\pi_k = 0,\ k \neq p,\ k \neq q$ is given (in terms of twisted tensor product) and investigated in [1].

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Here we construct the obstruction functor in general case in terms of Moore-Postnikov towers with the Property 1.4 somewhat weakened: DO is constructed in terms of cochains of complexes of tower not only of B.

In sections 2-4 we construct the functor DO and obstruction element $do(E)$ and establish their general properties (Properties 1,1-1,3). In section 5 we give a criterion of equivalence of two towers (Property 1.4).

Main results of this paper (partially in slightly different form) was announced in [2].

2. Preliminaries

Below, throughout this paper, all simplicial sets and topological spaces will be connected and arcwise connected respectively.

Let N be a simplicial set and π be an abelian group. Below we use only normalized cochains $c^n \in C^n(N,\pi)$. Let $Z^n(N,\pi)$ be the group of $n-$dimensional cocycles.

Recall the definition of Postnikov construction $P = P(N,\pi,z^n)$, $z^n \in Z^n(N,\pi)$ (see for details [3]). Let $E(\pi,n)$ be the complex whose q-simplexes are cochains $u \in C^n(\Delta[q],\pi)$ and face and degeneracy operators ∂_i and s_i are defined by

$$\partial_i u = e_i^{\#} u,\ s_i u = d_i^{\#} u;\ u \in C^n(\Delta_q,\pi),\ 0 \le i \le q,$$

where $\Delta[q]$ is the standard simplicial q-simplex, $e_i : \Delta[q-1] \to \Delta[q]$ and $d_i : \Delta[q+1] \to \Delta[q]$ are the standard maps.

The Eilenberg-MacLane complex $K(\pi,n)$ is defined as the subcomplex of $E(\pi,n)$ whose simplexes $u \in C^n(\Delta[q],\pi)$ are cocycles $u \in Z^n(\Delta[q];\pi)$. For a simplicial set N the following bijections are well know:

$$C^n(N,\pi) = \mathrm{Hom}(N, E(\pi,n));\ Z^n(N,\pi) = \mathrm{Hom}(N, K(\pi,n)).$$

If $c^n \in C^n(N,\pi)$ then the simplicial map $\hat{c}^n : N \to E(\pi,n)$ is defined as follows. For a given simplex $\sigma^q \in N$ let $t_{\sigma^q} : \Delta[q] \to N$ be the standard map for σ^q. Let

$$\hat{c}^n(\sigma^q) = t_{\sigma^q}^{\#}(c^n).$$

One verifies that $\hat{c}^n$ is a simplicial map. Observe also that for t_σ we have $t_{f(\sigma)} = f \circ t_\sigma$, where f is a simplicial map.

If $c^n = z^n \in Z^n(N,\pi)$ then $\hat{z}^n(N) \subset K(\pi,n)$ and hence there is a simplicial map $\hat{z}^n : N \to K(\pi,n)$. We use also the standard map

$$\delta : E(\pi,n-1) \to K(\pi,n).$$

The complex $E(\pi,n)$ is acyclic and the homotopy groups of realization $\mid K(\pi,n) \mid$ are zero except $\pi_n(\mid K(\pi,n) \mid) = \pi$.

Definition 1. The Postnikov construction $P = P(N,\pi,z^n)$, $z^n \in Z^n(N,\pi)$, is defined as the subcomplex of $N \times E(\pi,n-1)$ consisting of all simplexes (s,u), $s \in N$, $u \in E(\pi,n-1)$, such that $t_s^{\#}(z^n) = \delta u$. The projection $N \times E(\pi, n-1) \to N$ defines the projection $P(N, \pi, z^n) \xrightarrow{p} N$. There is a standard $(n-1)$-cochain $c^{n-1} \in C^{n-1}(P, \pi)$, in the sequel called P-cochain, defined by $c^{n-1}((\sigma^{n-1}, g)) = g$, for $\sigma^{n-1} \in N$ and $g \in \pi$, where g on the left side is looked up as an $(n-1)$-cochain of $\Delta[n-1]$.

Obviously the P-cochain is functorial and $p^{\#}(z^n) = \delta c^{n-1}$. Note that, for a simplicial map $f : N \to M$ and $z^n \in Z^n(M, \pi)$, we have the induced map

$$P(f) : P(N, \pi, f^{\#}(z^n)) \to P(M, \pi, z^n)$$

of P-constructions, with $P(f)(s, u) = (f(s), u)$.

Lemma 1. *If $f : N \to M$ induces an isomorphism of homology groups then $P(f) : P(N, \pi, f^{\#}(z^n)) \to P(M, \pi, z^n)$ induces an isomorphism of homology groups as well, where $z^n \in Z^n(M, \pi)$.*

Proof. Spectral sequence arguments. □

Let $F \to E \to\mid N \mid$ be a Serre fibration over the Milnor realization of simplicial set N and s^{n-1} be a cross section over the $(n-1)$-skeleton of $\mid N \mid$. Then it is defined an obstruction cocycle $z(s^{n-1}) \in Z^n(N, \pi_{n-1}(F))$. If s_1^{n-1} is another cross section over the $(n-1)$-skeleton which coincides with s^{n-1} on the $(n-2)$-skeleton then it is defined a difference cochain $d(s^{n-1}, s_1^{n-1}) = c^{n-1} \in C^{n-1}(N, \pi_{n-1}(F))$ and $z(s^{n-1}) - z(s_1^{n-1}) = \delta c^{n-1}$. For a cochain $c^{n-1} \in C^{n-1}(N, \pi_{n-1}(F))$ there exists a cross section s_1^{n-1} coinciding with s^{n-1} on the $(n-2)$-skeleton such that $d(s^{n-1}, s_1^{n-1}) = c^{n-1}$. We have the following almost evident

Lemma 2. *Let $F \to E \to\mid N \mid$ be a Serre fibration over the Milnor realization of a simplicial set N and let s^{n-1} be a cross section over the $(n-1)$-skeleton of $\mid N \mid$. Consider the Postnikov construction*

$$P = P(N, \pi_{n-1}(F), z(s^{n-1}))$$

for obstruction cocycle $z(s^{n-1}) \in Z^n(N, \pi)$. Consider the fibration over $\mid P \mid$ induced from given fibration by the projection $\mid P \mid\to\mid N \mid$. Consider the induced cross section over $(n-1)$-skeleton of $\mid P \mid$ defined by $s^{n-1} :\ pr^{\#}(s^{n-1})$. Consider the cross section s_1^{n-1}, which coincides with the latter cross section over $(n-2)$-skeleton such that (perturb the cross section $pr^{\#}(s^{n-1})$ by P-cochain c^{n-1})

$$z(pr^{\#}(s^{n-1})) - z(s_1^{n-1}) = \delta c^{n-1},$$

then the cross section s_1^{n-1} extends over the n-skeleton of $\mid P \mid$.

Definition 2. In the conditions of preceding Lemma 2 every extension of the cross section s_1^{n-1} over n-skeleton of $\mid P \mid$ we will call P-extension of the cross section s^{n-1}.

3. Definition of Functor $DO(-,\pi_*)$

Let N be a simplicial set and $\pi_* = \{\pi_1, \pi_2, \pi_3, \dots\}$ be a sequence of abelian groups.

Definition 3. A tower $t(N; z^2, z^3, z^4, \dots)$ over N is a sequence of simplicial sets and their projections

$$N = K_0 \leftarrow K_1 \leftarrow K_2 \leftarrow \cdots$$

together with a sequence of cocycles $z^2,\ z^3,\ z^4,\ \dots,\ z^{i+2} \in Z^{i+2}(K_i, \pi_{i+1})$, $i \geq 0$, such that

$$K_{i+1} = P(K_i, \pi_{i+1}, z^{i+2}).$$

Let $f : M \to N$ be a simplicial map and t be a tower over N. Then, in obvious way, we define the induced tower $\bar{t} = f^{\#}(t)$ over M by taking $\bar{z}^{i+2} = f_i^{\#}(z^{i+2})$, $i \geq 0$, where $f_0 = f$ and $f_{i+1} = P(f_i)$. Hence one has a map $T(f) : T(N, \pi_*) \to T(M, \pi_*)$. Thus the set of all towers $T(N, \pi_*)$ defines a contravariant functor $T(-, \pi_*)$ from the category of simplicial sets to the category of sets.

Definition 4. A tower $t(N; z^2, z^3, \dots)$ *is said to be zero on* N *if* $z^2 = 0$ (then $N \subset K_1$ by standard manner), $z^3 \mid N = 0$ (then $N \subset K_2$), $z^4 \mid N = 0$, (then $N \subset K_3$), and so on. Denote by $\tilde{T}(N, \pi_*)$ the set of all towers zero on N. $\tilde{T}$ is also a contravariant functor.

Lemma 3. *If a chain map $f : C \to C_1$ is an epimorphism and induces an epimorphism of homology, then $f\,|_{Z(C)}: Z(C) \to Z(C_1)$ is an epimorphism as well* (*here $Z(C)$ is the group of cycles*).

Proof. Is trivial. □

Corollary 1. *For a pair $i : N_1 \subset N$, if i induces an isomorphism of the integral homology groups, then*

$$T(i) : T(N, \pi_*) \to T(N_1, \pi_*) \ \textit{and}\ \tilde{T}(i) : \tilde{T}(N, \pi_*) \to \tilde{T}(N_1, \pi_*)$$

are surjective.

Proof. Let $t_1(N_1, z_1^2, z_1^3, \dots) \in T(N_1, \pi_*)$. By Lemma 3 there is a cocycle $z^2 \in Z^2(N, \pi_1)$ with $z^2 \mid N_1 = z_1^2$. Hence $P(N_1, \pi_1, z_1^2) \subset P(N, \pi_1, z^2)$ and by Lemma 1 they have isomorphic homology groups. By Lemma 3 there is a cocycle $z^3 \in P(N, \pi_1, z^2)$ with $z^3 \mid P(N_1, \pi_1, z_1^2) = z_1^3$. So one inductively constructs the tower $t(N, \pi_*)$ such that $T(i)(t) = t_1$. Now let t_1 be a tower zero on N_1. Then we can construct t as follows. Since $z_1^2 = 0$ we can select $\overline{z}^2 = 0$. For $z^3 \in P(N, \pi_1, \overline{z}^2)$ with $z^3 \mid P(N_1, \pi_1, z_1^2) = z_1^3$ let $\overline{z}^3 = z^3 - p^{\#}(z^3 \mid N)$, where $p : P \to N$ is the projection. Then $\overline{z}^3 \mid P(N_1, \pi_1, z_1^2) = z_1^3$ too and $\overline{z}^3 \mid N = 0$. And so on. This completes the proof. □

Definition 5. Two towers, $t(N; z^2, z^3, \dots)$ and $t_1(N; z_1^2, z_1^3, \dots)$, are equivalent (notation $t \sim t_1$), if there is a tower $t_2(N \times I; z_2^2, z_2^3, \dots)$, where $I = \Delta[1]$ is the unit interval, such that

$$t(N; z^2, z^3, \dots) = t_2(N \times I; z_2^2, z_2^3, \dots) \mid N \times 0,$$
$$t_1(N; z_1^2, z_1^3, \dots) = t_2(N \times I; z_2^2, z_2^3, \dots) \mid N \times 1.$$

Theorem 1. $\sim$ *is an equivalence relation.*

Proof. Let t, t_1 and t_2 be towers over a simplicial set N and let $t \sim t_1$, $t_1 \sim t_2$. Then one has two towers $\overline{t}$ and $\widetilde{t}$ over $N \times I$ such that

$$t = \overline{t} \mid N \times 0, \quad t_1 = \overline{t} \mid N \times 1, \; t_1 = \widetilde{t} \mid N \times 0, \quad t_2 = \widetilde{t} \mid N \times 1.$$

Consider the product $N \times \Delta[2]$. Then t and t_1 define the sum tower on the $N \times (01) \bigcup N \times (12)$. By the Corollary 1, this tower extends from $N \times (01) \bigcup N \times (12)$ on $N \times \Delta[2]$. The restriction of obtained tower on $N \times (02)$ provides the transitivity $t \sim t_2$. Reflexivity $t \sim t$ we obtain considering the tower $t \times I = T(pr)(t)$, where $pr : N \times I \to N$ is the projection. Symmetricity: consider Sing([01]) and two imbeddings

$$\begin{aligned} i_0, i_1 &: N \times I \to N \times \mathrm{Sing}([01]) \\ i_0(\sigma \times 0) &= \sigma \times 0, \;\; i_0(\sigma \times 1) = \sigma \times 1, \\ i_1(\sigma \times 0) &= \sigma \times 1, \;\; i_1(\sigma \times 1) = \sigma \times 0 \end{aligned}$$

from $t \sim t_1$ follows by Corollary 1 there is an extension of the tower on $i_0(N \times I)$ to a tower on $N \times \mathrm{Sing}([01])$. Its restriction on the subcomplex $i_1(N \times I)$ provides the equivalence $t_1 \sim t$. □

Now we can define the obstruction functor.

Definition 6. For a simplicial set N, let

$$DO(N, \pi_*) = T(N, \pi_*)/ \sim .$$

Definition 7. $\overline{DO}(N, \pi_*)$ is a subset of elements of $DO(N, \pi_*)$ containing at least one tower zero on N. Besides, by condition $z^i = 0$ for all $i \geq 2$, we define distinguished element $* \in \overline{DO}(N, \pi_*)$.

Theorem 2. *If* $f : N_1 \subset N$ *induce an isomorphism of homology then*

$DO(f) : DO(N, \pi_*) \to DO(N_1, \pi_*)$ *and* $\overline{DO}(f) : \overline{DO}(N, \pi_*) \to \overline{DO}(N_1, \pi_*)$

are $1-1$ *maps.*

Proof. Surjectivity of $DO(i)$ and $\overline{DO}(i)$. In virtue of Corollary 1 $T(i)$ *and* $\tilde{T}(i)$ are surjective. Hence $DO(i)$ *and* $\overline{DO}(i)$ are surjective as well. Injectivity of DO(i): Let t and $\bar{t}$ be towers over N such that

$$t \mid N_1 \sim \bar{t} \mid N_1.$$

Hence there exists a tower t_I over $N_1 \times I$ which provides this equivalence. Consider $N \times I$ and it's three subcomplexes $N \times 0$, $N \times 1$, $N_1 \times I$. The union of these subcomplexes U has the same homology as $N \times I$. Consider the tower $\widetilde{t}$ on U given by

$$\tilde{t} \mid N \times 0 = t, \quad \tilde{t} \mid N \times 1 = \bar{t}, \ \tilde{t} \mid N_1 \times I = t_I.$$

By virtue of Corollary 1 of Lemma 3, there is an extension of $\widetilde{t}$ on the $N \times I$. It gives the equivalence $t \sim \bar{t}$. This completes the proof of injectivity of $DO(i)$. Injectivity for $\overline{DO}(i)$ follows from injectivity for $DO(i)$. □

Definition 8. Let for topological space B

$$\begin{aligned} DO(B, \pi_*) &= DO(\mathrm{Sing}(B), \pi_*) \\ \overline{DO}(B, \pi_*) &= \overline{DO}(\mathrm{Sing}(B), \pi_*) \end{aligned}$$

Lemma 4. *The inclusion*

$$N \subset \mathrm{Sing}(|N|)$$

induce the equalities

$$\begin{aligned} DO(N, \pi_*) &= DO(\mathrm{Sing}(|N|), \pi_*) \\ \overline{DO}(N, \pi_*) &= \overline{DO}(\mathrm{Sing}(|N|), \pi_*) \end{aligned}$$

Proof. Follows from Theorem 2. □

Lemma 5. *If B is a topological space and i_0 and i_1 are imbeddings as upper and lower base of $B \times I$ then $DO(i_0) = DO(i_1)$.*

Proof. From Theorem 2 follows that $DO(i_0)$ and $DO(i_1)$ are $1-1$ maps. Hence for the projection

$$pr : B \times I \to B$$

$DO(pr)$ is $1-1$ map. It follows $DO(i_0) = DO(i_1)$. □

As a Corollary we have

Theorem 3. *If a map of topological spaces f is homotopic to g then $DO(f) = DO(g)$ and $\overline{DO}(f) = \overline{DO}(g)$.*

Jet we can prove

Theorem 4. *If a map of topological spaces*

$$f : B_1 \to B$$

induce an isomorphism of homology then

$DO(f) : DO(B, \pi_*) \to DO(B_1, \pi_*)$ *and* $\overline{DO}(f) : \overline{DO}(B, \pi_*) \to \overline{DO}(B_1, \pi_*)$

are $1-1$ maps.

Proof. Let $C(f)$ be the cylinder of the map f. One has a homotopy commutative triangle

$$\begin{array}{ccccc} & & C(f) & & \\ & \overset{k}{\nearrow} & & \overset{q}{\nwarrow} & \\ B_1 & & \overset{f}{\longrightarrow} & & B \end{array}$$

with k and q being the standard imbeddings. By virtue of above results one has the commutative triangle

$$\begin{array}{ccccc} & & DO(C(f)) & & \\ & \overset{DO(k)}{\swarrow} & & \overset{DO(q)}{\searrow} & \\ DO(B) & & \overset{DO(f)}{\longrightarrow} & & DO(B_1) \end{array}$$

with $DO(k)$ and $DO(q)$ $1-1$ maps in virtue of Theorem 2. Then it follows that $DO(f)$ is $1-1$ map too. Analogously one has that $\overline{DO}(f)$ is $1-1$ map. □

4. Definition of Obstruction Element $do(E)$

Let $F \to E \to B$ be a Serre fibration. Here we assign to this fibration an element

$$do(E) \in DO(B, \pi_*(F))$$

with the following properties:

(i) $do(E)$ is functorial.

(ii) the fibration E has a cross section if and only if $do(E) \in \overline{DO}(B, \pi_*(F))$.

Consider first the case $B =\mid N \mid$. Let $K_0 = N$. Consider a cross section s^1 on the 1-skeleton of $\mid N \mid = \mid K_0 \mid$. Let $z^2 \in Z^2(K_0, \pi_1(F))$ be its obstruction cocycle, $z^2 = z(s^1)$. Then we obtain $K_1 = P(K_0, \pi_1(F), z^2)$ and a fibration $K_1 \to K_0$. Let s^2 be a cross section over 2-skeleton of $\mid K_1 \mid$, which is the P-extension of s^1 in the sense of Definition 2 . Let $z^3 = z(s^2) \in Z^3(K_1, \pi_2(F))$ and $K_2 = P(K_1, \pi_2(F), z^3)$. Let s^3 be a cross section over 3-skeleton of $\mid K_2 \mid$, i.e. the P-extension of s_1, and $z^4 = z(s^3) \in Z^4(K_2, \pi_3(F))$, etc. Proceeding inductively we construct the needed tower.

In general case let $S(B)$ be the singular complex of B, $\omega : \mid S(B) \mid \to B$ be the standard map and let $\overline{E} \to \mid S(B) \mid$ be a fibration induced by ω. Then, by above way, we construct a tower for E using the fibration $\overline{E}$.

Definition 9. The constructed tower we call a P-tower of a fibration E or a geometric tower.

Lemma 6. *If t is a P-tower of a fibration over B and f is a map $B_1 \to B$ then $T(f)t$ is a P-tower of the induced fibration over B_1*

Proof. is an easy checking. □

Lemma 7. *If $F \to E \to B$ is a fibration, $B_1 \subset B$ and t is a P-tower of the restricted fibration over B_1 then t extends to a P-tower over B.*

Proof. is an easy one. □

Definition 10. We define $do(E) \in DO(B, \pi_*(F))$ as the class of any P-tower (over Sing(B)) of the fibration E.

Theorem 5. *The class $do(E) \in DO(B, \pi_*(F))$ is uniquely defined.*

Proof. Obviously it is enough to consider the case $F \to E \to\mid N \mid$. Let

$$t = (K_0,\ z^2;\ K_1,\ z^3;\ K_2,\ z^4; \dots)$$

and

$$\bar{t} = (K_0,\ \bar{z}^2;\ \bar{K}_1,\ \bar{z}_3;\ \bar{K}_2,\ \bar{z}^4; \dots)$$

be two P-towers of E. Consider the projection

$$pr :\mid N \mid \times I \to\mid N \mid$$

and let $E \times I$ be the fibration induced by pr. By Lemma 7 there is a P-tower for this fibration whose restriction on $N \times 0$ is t and on $N \times 1$ is $\overline{t}$. □

Theorem 6. *$do(E)$ is functorial.*

Proof. Follows from Lemma 6. □

Theorem 7. *Any tower from $do(E)$ is a P-tower of E*

Proof. For a fibration $E \to\mid N \mid$ (the general case is trivial after this one) let us consider a tower t over $N \times I$ such that its restriction on $N \times 0$ (we denote it by t_N) is geometric, i.e. it is a P-tower of E. It is enough to show that t is a P-tower of induced fibration. There exists a cochain $c^1 = c^1_{N \times I}$ such that

$$z^2_{N \times I} - pr^{\#}(z^2_N) = \delta c^1_{N \times I},\ c^1 \in C^1(N \times I, \pi_1).$$

Indeed, the left side is zero over $N \times 0$ and $H^*(N \times I, N \times O) = 0$. Besides, $pr^{\#}(z^2_N)$ is geometric since z^2_N is geometric. By classical fact $pr^{\#}(z^2_N) + \delta c^1$ is geometric as well (theorem about difference cochain). Hence $z^2_{N \times I}$ is geometric. Knowing $z^2_{K^1_{N \times 0}}$ to be geometric, consider one of geometric $\bar{z}^3_{K^1_{N \times I}}$ cocycles extending geometric $z^3_{K^1_{N \times 0}}$. By $H^*(K^1_{N \times I}, K^1_{N \times 0}) = 0$ there is $c^2 = c^2_{K^1_{N \times I}}$ such that

$$z^3_{K^1_{N \times I}} - pr^{\#}(z^3_{K^1_N}) = \delta c^2_{K^1_{N \times I}},\ c^2 \in C^1(K^1_{N \times I}, \pi_2).$$

it follows $z^3_{K^1_{N \times I}}$ is geometric. The same proof is valid for $z^4_{K^2_{N \times I}} z^5_{K^3_{N \times I}} z^6_{K^4_{N \times I}}$ and so on. □

Theorem 8. *For a fibration $F \to E \to\mid N \mid$ one has $do(E) \in \overline{DO}(\mid N \mid,$ $\pi_*(F))$ if and only if there exists a cross section for E.*

Proof. Sufficiency: Let $do(E) \in \overline{DO}(N, \pi_*(F))$. This means that there is a tower in $do(E)$ which is zero over N. By preceding theorem this tower is geometric. From this it is not hard to deduce that the procedure of constructing of geometric sequence gives a cross section over the whole $\mid N \mid$.Necessity: Let $s : \mid N \mid \to E$ be a cross section. Construct the P-tower by using this cross section as follows. Let s_1 be the restriction of cross section s on 1-skeleton of $\mid N \mid = \mid K_0 \mid$. The cocycle $z^2 = z(s_1)$, as obstruction cocycle of s_1 which is extendable over 2-skeleton of $\mid K_0 \mid$, is zero, $z^2 = 0$ and hence $N \subset K_1$. In constructing z^3, having the freedom of extension, extend, perturbed by P-cochain c^1, cross section $pr(s_1)$ on 2-simplexes of $\mid N \mid \subset \mid K_1 \mid$ by s (see Lemma 2 and take in consideration that $c^1 \mid N = 0$). We became $z^3 = 0$ over N and so on. □

About the property 1.3. If the graded group π_* have only one nontrivial component, say π_n, and $F \to E \to \mid N \mid$ is a Serre fibration with $\pi_*(F) = \pi_n$, then towers from $T(N, \pi_*)$ reduce to fibration $K_n = P(N, \pi_n, z^{n+1})$. Then we have: $T(N, \pi_n) = Z^{n+1}(N, \pi_n)$, there is only one tower $P(N, \pi_n, 0) = N \times K(\pi_n, n)$ zero on N and $P(N, \pi_n, z^{n+1}) \sim P(N, \pi_n, \overline{z}^{n+1})$ if and only if $[z^{n+1}] = [\overline{z}^{n+1}]$. Hence $DO(N, \pi_n) = H^{n+1}(N, \pi_n)$ and $do(E)$ is the classical first obstruction class of E.

5. Criterion of Equivalence of Towers

Our aim in this section is to formulate a criterion of equivalence of two towers (Theorem 10 below). A tool for this is a notion of maps of towers which leads actually to an alternative definition of functor $DO(B, \pi_*)$ as a set of towers on B modulo isomorphism of towers. Below we denote by the same symbols maps and induced homomorphisms. We denote cochains and corresponding maps by the same symbols as well: for example $C^n(N, \pi) = \mathrm{Hom}(N, E(\pi, n))$.

5.1. **Maps of P-constructions.**

Definition 11. A map of P-constructions is a couple of maps

$$(f, F) = (f : K \to L;\ F : P(K, \pi, z^n) \to P(L, \pi, \bar{z}^n))$$

such that $fp = pF$ and F is a $K(\pi, n-1)$-map: $F(k, c + z) = F(k, c) \circ z$, where the operation $\circ$ is the standard action of Eilenberg-MacLane complex on the total complex P.

Lemma 8. *Let $P(K, \pi, z^n)$ and $P(L, \pi, \bar{z}^n)$ be P-constructions and let $f : K \to L$ be a map. Then there exists a map of P-constructions*

$$(f, F) = (f : K \to L;\ F : P(K, \pi, z^n) \to P(L, \pi, \bar{z}^n))$$

if and only if there exists a cochain $a^{n-1} : K \to E(\pi, n-1)$ such that

$$\delta a^{n-1} = \bar{z}^n \circ f - z^n.$$

One has $F = F_{a^{n-1}}$, where $F_{a^{n-1}}(k,c) = (f(k), c + a^{n-1}(k))$. The cochain a^{n-1} is uniquely determined by the map (f,F).

Proof. Easy to show that for a given a^{n-1} with $\delta a^{n-1} = \bar{z}^n \circ f - z^n$ the map $F_{a^{n-1}}$ satisfies the needed conditions. Suppose now that a map of P-constructions (f,F) is given. We have to construct a cochain a^{n-1} satisfying the conditions $\delta a^{n-1} = \bar{z}^n \circ f - z^n$ and $F = F_{a^{n-1}}$. Since $fp = pF$, one has $F(k,c) = (f(k), \varphi(k,c))$, where

$$\varphi : P(K,\pi,z^n) \to E(\pi, n-1).$$

Let us introduce the map

$$\psi : P(K,\pi,z^n) \to E(\pi, n-1)$$

given by $\psi(k,c) = \varphi(k,c) - c$. This map does not depend on the second argument $c \in E(\pi, n-1)$. Indeed, suppose $(k,c),(k,c^{`}) \in P(K,\pi,z^n)$. Thus $\delta c = z^n(k) = \delta c^{`}$, i.e. $c - c^{`} \in K(\pi, n-1)$. Then

$$\begin{aligned}\psi(k,c^{`}) = \psi(k, c + c^{`} - c) = \varphi(k, c + c^{`} - c) - c^{`} = \\ = \varphi(k,c) + (c^{`} - c) - c^{`} = \varphi(k,c) - c = \psi(k,c).\end{aligned}$$

This fact implies that there exists the unique cochain $a^{n-1} : K \to E(\pi, n-1)$ such that $\psi = a^{n-1} \circ p$. It remains to show that $F = F_{a^{n-1}}$ and $\delta a^{n-1} = \bar{z}^n \circ f - z^n$. Indeed

$$\begin{aligned}F(k,c) = (f(k), \varphi(k,c)) = (f(k), c + \psi(k,c)) = \\ = (f(k), c + (a^{n-1} \circ p)(k,c)) = (f(k), c + a^{n-1}(k)) = F_{a^{n-1}}(k,c).\end{aligned}$$

Now look at δa^{n-1}. Since

$$F(k,c) = (f(k), c + a^{n-1}(k)) \in P(L,\pi,\bar{z}^n),$$

we have $\delta(c + a^{n-1}(k)) = \bar{z}^n(f(k))$. Thus $z^n(k) + \delta(a^{n-1}(k)) = \bar{z}^n(f(k))$. □

Note that for $P(f)$ in Lemma 1 we have $P(f) = F_{a^{n-1}}$ with $a^{n-1} = 0$.

Proposition 1. *Let $(f, F_{a^{n-1}})$ and $(g, F_{b^{n-1}})$ be maps of P-constructions*

$$P(K,\pi,z^n) \overset{(f,F_{a^{n-1}})}{\to} P(L,\pi,\bar{z}^n) \overset{(g,F_{b^{n-1}})}{\to} P(S,\pi,\tilde{z}^n),$$

then the composition again is a map $(gf, F_{c^{n-1}})$ of P-constructions, where

$$c^{n-1} = a^{n-1} + b^{n-1} \circ f.$$

Proof. We have

$$\begin{aligned}(F_{b^{n-1}} \circ F_{a^{n-1}})(k,c) = F_{b^{n-1}}(f(k), c + a^{n-1}(k)) = \\ = (g(f(k)), c + a^{n-1}(k) + b^{n-1}(f(k))) = (g(f(k)), c + c^{n-1}(k)) = F_{c^{n-1}}(k,c).\end{aligned}$$ □

Proposition 2. *If (f,F) is a morphism of P-constructions and f is an isomorphism [monomorphism], then F is an isomorphism [monomorphism] too.*

Proof. By Lemma 8 $F = F_{a^{n-1}}$ for some a^{n-1}. Let

$$(f(k_1), c_1 + a^{n-1}(k_1)) = (f(k_2), c_2 + a^{n-1}(k_2)).$$

Then $k_1 = k_2$, $c_1 + a^{n-1}(k_1)) = c_2 + a^{n-1}(k_2)$ and hence $c_1 = c_2$. Suppose now that f is an isomorphism. Then the opposite map will be $F_{b^{n-1}}$, where $b^{n-1} = -a^{n-1} \circ f^{-1}$. Indeed, to the composition $F_{b^{n-1}} \circ F_{a^{n-1}}$ corresponds the zero cochain:

$$c^{n-1} = a^{n-1} + b^{n-1} \circ f = a^{n-1} - a^{n-1} \circ f^{-1} \circ f = 0.$$

Thus $F_{b^{n-1}} \circ F_{a^{n-1}} = F_{c^{n-1}} = F_0$. Since $f^{-1} \circ f = id$ one has $F_0 = id$. □

Let $P(K, \pi, z^n)$ be a P-construction and L be a simplicial set. Then the total complex $P(K \times L, \pi, \bar{z}^n)$, where

$$\bar{z}^n = z^n \circ pr : K \times L \to K \to K(\pi, n),$$

we can identify with $P(K, \pi, z^n) \times L$

Let $I = \Delta[1]$ and let, for a simplicial set M, $i_\varepsilon : M \to M \times I$, $\varepsilon = 0, 1$, be the standard imbeddings. Obviously the standard maps $i_\epsilon = (i_\epsilon, i_\epsilon) : P \to P \times I$, where $\epsilon = 0, 1$, are maps of P-constructions. By this, in obvious way, we introduce the notion of homotopy of maps of P-constructions.

Definition 12. *Two maps of* P-constructions

$$(f, F), (f^{`}, F^{`}) : P(K, \pi, z^n) \to P(L, \pi, \bar{z}^n)$$

we call homotopic, if there exists a map of P-constructions

$$(h, H) : P(K, \pi, z^n) \times I \to P(L, \pi, \bar{z}^n)$$

such that $hi_0 = f,\ hi_1 = f^{`},\ Hi_0 = F,\ Hi_1 = F^{`}$.

Proposition 3. *Two maps of P-constructions $(f, F_{a^{n-1}})$ and $(f^{`}, F_{a^{`n-1}})$ are homotopic if and only if $f \sim f^{`}$ by some homotopy $h : K \times I \to L$ for which there exists a cochain $b^{n-1} : K \times I \to E(\pi, n-1)$ such that*

$$\delta b^{n-1} = \bar{z}^n \circ h - z^n \circ pr, \quad b^{n-1} \circ i_0 = a^{n-1},\ b^{n-1} \circ i_1 = a^{`n-1}.$$

Proof. Suppose there exists a map (h, H) with suitable properties. Then by Lemma 8, there exists a cochain $b^{n-1} : K \times I \to E(\pi, n-1)$ such that $\delta b^{n-1} = \bar{z}^n \circ h - z^n \circ pr$. Let us show that $b^{n-1} \circ i_0 = a^{n-1}$. Consider the composition (hi_0, Hi_0) which coincides with the map (f, F). To this map corresponds the cochain a^{n-1}, to i_0- the zero cochain, and to (h, H) corresponds the cochain b^{n-1}. Then by Proposition 1 we have$a^{n-1} = 0 + b^{n-1} \circ i_0$. In a similar way we have $b^{n-1} \circ i_1 = a^{`n-1}$. Suppose now that a homotopy $h$ and a cochain $b^{n-1}$, satisfying the suitable conditions, are given. Then by Lemma 8 there exists a map of $P$-constructions $(h, H)$. It remains to show that $Hi_0 = F$, $Hi_1 = F^{`}$. Again, consider the composition $(h, H) \circ (i_0, i_0) = (f, Hi_0)$. Then by Proposition 1 to this map corresponds the cochain $0 + b^{n-1} \circ i_0 = a^{n-1}$. Thus by Lemma 8 we get $H \circ i_0 = F$. In similar way $H \circ i_1 = F^{`}$. □

5.2. **Map of towers.** In definition of towers we restrict ourselves to towers of height n i. e. instead of Definition 3 consider the notion of n-towers:

Definition 13. An n-tower $t^{(n)}(N; z^2, \dots, z^{n+1})$ over N with coefficients in $\pi_* = (\pi_1, \dots, \pi_n)$ is a sequence of simplicial sets and standard projections $N = K_0 \overset{p}{\leftarrow} K_1 \overset{p}{\leftarrow} K_2 \overset{p}{\leftarrow} \cdots \overset{p}{\leftarrow} K_n$, where $K_{i+1} = P(K_i, \pi_{i+1}, z^{n+1})$, $z^{i+2} \in Z^{i+2}(K_i, \pi_{i+1})$, $0 \le i \le n$.

The set $T^{(n)}(N, \pi_*)$ of n-towers defines a contravariant functor and there is an equivalence relation on $T^{(n)}(N, \pi_*)$ similar to that of in Definition 5. The obvious changes in Definitions 4, 5, 6,7, 8 leads to functors $DO^{(n)}(B, \pi_*)$, $\overline{DO}^{(n)}(B, \pi_*)$.

Definition 14. A map of n-towers $t^{(n)} \to \bar{t}^{(n)}$is defined as a sequence of simplicial maps $f_i : K_i \to \overline{K}_i$ where $i = 0, 1, 2, \dots, n$ such that each (f_i, f_{i+1}) is a map of P-constructions.

It follows from Lemma 8 that a map $t^{(n)} \to \bar{t}^{(n)}$ exists if and only if there exists a sequence of cochains $a = (a^1, a^2, \dots, a^n)$, $a^i \in C^i(K_{i-1}, \pi_i)$, such that $f^{\#}_{i-2}(\bar{z}^i) - z^i = \delta a^{i-1}$, where $f_{i-2} = F_{a^{i-2}}$, $2 \le i \le n+1$.

Proposition 4. *Two towers $t^{(n)}, \bar{t}^{(n)} \in T^n(N, \pi_*)$ are equivalent if and only if there exists an isomorphism of towers $\{f_i\} : t^{(n)} \to \bar{t}^{(n)}$, with $f_0 = id_N$.*

Proof. Let

$$t^{(n)} = t^{(n)}(N; z^2, z^3, \dots),\ \bar{t}^{(n)} = \bar{t}^{(n)}(N; \bar{z}^2, \bar{z}^3, \dots),$$
$$\widetilde{t}^{(n)} = \widetilde{t}^{(n)}(N \times I; \widetilde{z}^2, \widetilde{z}^3, \dots),$$
$$T^{(n)}(i_0)(\widetilde{t}^{(n)}) = t^{(n)},\ T^{(n)}(i_1)(\widetilde{t}^{(n)}\ ) = \bar{t}^{(n)},$$

where, for an arbitrary M, $i_\epsilon : M \to M \times I$, $\epsilon = 0, 1$ are the standard inclusions. Below maps and corresponding homomorphisms for cochains we denote by same symbols. Let, now,

$$\varphi = \{\varphi_i\} : t^{(n)} \to \widetilde{t}^{(n)} \quad \text{and} \quad \bar{\varphi} = \{\bar{\varphi}_i\} : \bar{t}^{(n)} \to \widetilde{t}^{(n)}$$

be maps of n-towers, corresponding to $\varphi_0 = i_0$, $\bar{\varphi}_0 = i_1$. For $F_{a^{n-1}}$ we below use more explicit notation $F_{(f, a^{n-1})}$. Let $f_0 = id_N : K_0 = N \to \overline{K}_0 = N$. Then, $\varphi_0 \sim \bar{\varphi}_0 \circ f_0$ by homotopy

$$h_0 = id : K_0 \times I = N \times I \to \widetilde{K}_0 = N \times I.$$

Since

$$i_0(h_0(\widetilde{z}^2) - pr(z^2)) = \varphi_0(\widetilde{z}^2) - z^2 = z^2 - z^2 = 0,$$

there exists a cochain $\bar{b}^1 \in C^1(K_0 \times I, \pi_1)$ such that$h_0(\widetilde{z}^2) - pr(z^2) = \delta \bar{b}^1$. Let

$$b^1 = \bar{b}^1 - pr(i_0(\bar{b}^1)),$$

where, for an arbitrary M, $pr : M \times I \to M$ is the standard projection. Then, since $\delta i_0(\bar{b}^1) = i_0(\delta \bar{b}^1) = 0$, we have

$$i_0(b^1) = 0,\ h_0(\widetilde{z}^2) - pr(z^2) = \delta b^1.$$

Let $a^1 = i_1(b^1)$. Then

$$\delta a^1 = i_1(\delta b^1) = i_1(h_0(\widetilde{z}^2)) - i_1(pr(z^2)) = (\overline{\varphi}_0 \circ f_0)(\widetilde{z}^2) - z^2 = f_0(\overline{z}^2) - z^2.$$

Now we consider the map $f_1 = F_{(f_0,a^1)}$ and homotopy

$$h_1 = F_{(h_0,b^1)} : K_1 \times I \to \widetilde{K}_1$$

from Lemma 8 and proposition 3. Since $i_0(b^1) = 0$ we have $h_1 \circ i_0 = F_{(\varphi_0,0)} = \varphi_1$ and

$$h_1 \circ i_1 = F_{(\tilde{\varphi}_0 \circ f_0, i_1(b^1))} = F_{(\overline{\varphi}_0,0)} \circ F_{(f_0,a^1)} = \overline{\varphi}_1 \circ f_1.$$

So, by h_1, we have $\varphi_1 \sim \overline{\varphi}_1 \circ f_1$ and so on. Thus we can construct a morphism

$$\{f_i\} : t^{(n)} \to \overline{t}^{(n)} \text{ with } f_0 = id_N.$$

Now let $\{f_i\} : t^{(n)} \to \overline{t}^{(n)}$ be a morphism with $f_0 = id_N$. Let $(a^1, a^2, \dots)$ be a corresponding sequence: $f_i^{\#}(\overline{z}^{i+2}) - z^{i+2} = \delta a^{i+1}$. For an inclusion $f : M \to M_1$ and $c^i \in C^i(M)$ define $\underline{c}^i = \underline{c}^i(f) \in C^i(M_1)$ as follows. If $\tau^i = f(\sigma^i)$, then $\underline{c}^i(\tau^i) = c^i(\sigma^i)$ and $\underline{c}^i(\tau^i) = 0$ otherwise. Let $\varphi_0 = i_0$, $\overline{\varphi}_0 = i_1$, $\widetilde{K}_0 = N \times I$ and$\widetilde{z}^2 = pr(z^2) + \delta \underline{a}^1$, where $\underline{a}^1 = \underline{a}^1\ (\overline{\varphi}_0 f_0)$. Then

$$i_0(\widetilde{z}^2) = i_0(pr(z^2)) + \delta i_0(\underline{a}^1) = z^2 + 0 = z^2,$$
$$i_1(\widetilde{z}^2) = i_1(pr(z^2)) + \delta i_1(\underline{a}^1) = z^2 + \delta a^1 = \overline{z}^2.$$

Let

$$\widetilde{K}_1 = P(\widetilde{K}_0, \pi_1, \widetilde{z}^2),\ \varphi_1 = F_{(i_0,0)} : K_1 \to \widetilde{K}_1,\ \overline{\varphi}_1 = F_{(i_1,0)} : \overline{K}_1 \to \widetilde{K}_1,$$
$$pr_1 = F_{(pr,-\underline{a}^1)} : \widetilde{K}_1 \to K_1,\ \widetilde{z}^3 = pr_1(z^3) + \delta \underline{a}^2,$$

where $\underline{a}^2 = \underline{a}^2(\overline{\varphi}_1 f_1) = \underline{f_1^{-1}(a^2)}(\overline{\varphi}_1)$. It is clear (see Proposition 2) that φ_1 and $\overline{\varphi}_1$ are inclusions and, moreover, $Im\varphi_1 \cap Im\overline{\varphi}_1 = \emptyset$. Then, using Proposition 1 we have

$$\varphi_1(\widetilde{z}^3) = \varphi_1(pr_1(z^3)) + \delta\varphi_1(\underline{a}^2) = (F_{(pr,-\underline{a}^1)} \circ F_{(i_0,0)})(z^3) + 0 =$$
$$= F_{(id,0+i_0(-\underline{a}^1))}(z^3) = F_{(id,0)}(z^3) = id_{K_1}(z^3) = z^3$$

and

$$\overline{\varphi}_1(\widetilde{z}^3) = \overline{\varphi}_1(pr_1(z^3)) + \delta\overline{\varphi}_1(\underline{a}^2) = (F_{(pr,-\underline{a}^1)} \circ F_{(i_1,0)})(z^3) + \delta f_1^{-1}(a^2) =$$
$$= F_{(id,0+i_1(-\underline{a}^1))}(z^3) + f_1^{-1}(\delta a^2) = F_{(id,-a^1)}(z^3) + f_1^{-1}(\delta a^2) =$$
$$= f_1^{-1}(z^3) + f_1^{-1}(\delta a^2) = f_1^{-1}(z^3 + \delta a^2) = \overline{z}^3.$$

Let $\widetilde{K}_2 = P(\widetilde{K}_1, \pi_2, \widetilde{z}^3)$ and so on we construct a desired tower $\widetilde{t}^{(n)}$. □

Remark 1. Let $t = t(N; z^2, z^3, \dots)$ be a tower and $c^p \in C^p(K_{p-1}, \pi_p)$, $p \geq 1$. Consider a new tower $\bar{t} = \bar{t}(N; \bar{z}^2, \bar{z}^3, \dots)$ and a map (isomorphism) of towers $f = \{f_i\} : \bar{t} \to t$, where $\bar{z}^i = z^i$ for $i \leq p$, $f_i = id$ for $i \leq p-1$, $\bar{z}^{p+1} = z^{p+1} - \delta c^p$, $f_p = F_{(id,c^p)}$ (see Lemma 2) and the rest of the $\{\bar{K}^i\}$ and $\{f_i\}$ are induced from t (beginning with f_p). Then, by Proposition 4 we have $\bar{t} \sim t$ and this is a particular case of perturbation of a representative tower of an element of $DO(N, \pi_*)$.

An obvious corollary is

Theorem 9. *$DO^{(n)}(B, \pi_*)$ is equal to the set $T^{(n)}(N, \pi_*)$ of n-towers modulo isomorphism of towers.*

We shall use the following notion of homotopy of n-towers. Two morphisms of towers $f, g : t^{(n)} \to \overline{t}^{(n)}$ we call homotopic if there exists a morphism (homotopy) of towers $F : t^{(n)} \times I \to \overline{t}^{(n)}$ such that $Fi_0 = f$ and $Fi_1 = g$, where

$$i_\epsilon : t^{(n)} = T^{(n)}(i_\epsilon)(t^{(n)} \times I) \to t^{(n)} \times I,\ \epsilon = 0, 1.$$

Note that we have $f \sim f (F = pr)$ and this notion is compatible with compositions. This follows from the diagram

$$M \times I \overset{id \times \Delta}{\to} M \times I \times I \overset{F \times id}{\to} K \times I \overset{\Phi}{\to} L,$$

where $F : M \times I \to K$ and $\Phi : K \times I \to L$ are some homotopies and Δ is the diagonal map.

From Proposition 3 we have the following

Corollary 2. *Two maps of towers $f = \{a^i\}$ and $f = \{\overline{a}^i\}$ are homotopic if and only if $f_0 \sim f_0$ by some homotopy F_0 for which there exists a sequence of cochains*

$$b = (b^1, \cdots, b^i, \cdots, b^n), b^i \in C^i(K_{i-1} \times I, \pi_i), 1 \leq i \leq n,$$

which satisfies the following conditions (i) $i_0(b^i) = a^i, i_1(b^i) = \overline{a}^i$, (ii) $\delta b^{i-1} = F_{b^{i-2}}(\overline{z}^i) - pr(z^i)$, where $F_{b^0} = F_0$ and $F_{b^i} : K_i \times I \to K_i$ are maps given by

$$F_{b^i}((k, c) \times \xi) = F_{b^{i-1}}((k, c) \times \xi), c + t_{k \times \xi} b^i), 1 \leq i \leq n.$$

5.3. **Criterion of equivalence of towers.** For a simplicial set N, let us fix a tower $t \in T(N, \pi_*)$ and let $t^{(n)} \in T^{(n)}(N, \pi_*)$ be the restrictions of t up to n-stage. Denote by $A^{(n)} = A_{t^{(n)}}$ the set of automorphisms of the n-tower $t^{(n)}$ with $f_0 = id$:

$$\{f_0 = id, f_1, \dots, f_n\} : t^{(n)} \to t^{(n)}.$$

The composition turns $A^{(n)}$ into a group. According to Lemma 8 each element $\{f_k\} \in A^{(n)}$ can be interpreted as a sequence of cochains

$$a^{(n)} = \{a^k\},\ a^k \in C^k(K_{k-1}, \pi_k),\ k = 1, 2, \dots, n;\ \delta a^{k-1} = f_{a^{k-2}}(z^k) - z^k.$$

Remark that if $a^i = 0$ for all i with $1 \le i \le p \le n-1$, then $f_i = id$ for all $0 \le i \le p$ and $\delta a^{p+1} = z^{p+2} - z^{p+2} = 0$.

By Proposition 1 to the composition $\{f_{\overline{a}^k}\} \circ \{f_{a^k}\} : t^{(n)} \to t^{(n)}$ corresponds the sequence $\{a^k + f_{a^{k-1}}(\overline{a}^k)\}$, thus the operation of the group $A^{(n)}$ in terms of cochains is given by

$$\overline{a}^{(n)} * a^{(n)} = \{a^k + f_{a^{k-1}}(\overline{a}^k)\}.$$

Then it follows from Proposition 1 and above formula for the group operation of $A^{(n)}$ that there are the restriction epimorphisms $i_\varepsilon : A_{t^{(n)} \times I} \to A^{(n)},\ \varepsilon = 0, 1$, given by $i_\varepsilon(\{b^i\}) = \{i_\varepsilon^{\#}(b^i)\}$. For example, if $a^{(n)} \in A_{t^{(n)}}$, we have $i_\varepsilon(pr\{a^{(n)}\}) = a^{(n)}$, where $pr(\{a^i\}) = \{pr^{\#}(a^i)\} \in A_{t^{(n)} \times I}$. Let

$$A^0_{t^{(n)} \times I} = \mathrm{Ker} i_0 = \{b^{(n)} \mid b^{(n)} = (b^1, \dots, b^n) \in A_{t^{(n)} \times I},\ i_0(b^i) = 0\}.$$

We introduce also the restriction homomorphisms $\overline{q}_n : A^{(n+1)} \to A^{(n)}$ given by

$$\overline{q}_n(a^1, \dots, a^n, a^{n+1}) = (a^1, \dots, a^n),\ n \ge 1.$$

Define now $B^{(n)} = Im(i_1 \mid A^0_{t^{(n)} \times I})$. Then it follows from above that $B^{(n)} \subset A^{(n)}$ is a normal subgroup of $A^{(n)}$ and we can define the factorgroup

$$G^{(n)} = A^{(n)} / B^{(n)}.$$

As above for $A^{(n)}$, the subgroup $B^{(n)}$ and consequently $G^{(n)}$ also have a description in terms of cochains, which we now give.

Proposition 5. *A sequence of cochains* $a^{(n)} = (a^1, a^2, \cdots, a^n) \in A^{(n)} = A_{t^{(n)}}$ *belongs to subgroup* $B^{(n)}$ *if and only if there is exists a sequence of cochains*

$$b^{(n)} = (b^1, b^2, \cdots, b^n) \in C(K_{n-1} \times I, \pi_i),\ 1 \le i \le n$$

which satisfies the following conditions(i) $i_0(b^i) = a^i, i_1(b^i) = \overline{a}^i$, *(ii)* $\delta b^{i-1} = F_{b^{i-2}}(\overline{z}^i) - pr(z^i)$, *where* $F_{b^0} = F_0$ *and* $F_{b^i} : K_i \times I \to K_i$ *are maps given by*

$$F_{b^i}((k, c) \times \xi) = F_{b^{i-1}}((k, c) \times \xi), c + t_{k \times \xi} b^i),\ 1 \le i \le n.$$

Proof. Let $i_1(b^{(n)}) = a^{(n)}$, where $b^{(n)} \in A^0_{t^{(n)} \times I}$. Applying Lemma 8 for $pr \circ b^{(n)}$ one can show that conditions of the proposition hold for $\{b^i\}$. Suppose now that conditions (i) and (ii) hold. Then

$$\delta b^1 = F_0(z^2) - pr(z^2) = \overline{F}_0(pr(z^2)) - pr(z^2),$$

where $\overline{F}_0 = id : N \times I \to N \times I$. Let $\overline{F}_{b^1} = F_{(\overline{F}_0, b^1)}$.Then, by Proposition 1 $pr \circ \overline{F}_{b^1} = F_{b^1}$ and we have

$$\delta b^2 = F_{b^1}(z^3) - pr(z^3) = (pr \circ \overline{F}_{b^1})(z^3) - pr(z^3) = \overline{F}_{b^1}(pr(z^3)) - pr(z^3).$$

And so on we show that $b^{(n)} = \{b^i\} \in A^0_{t^{(n)} \times I}$. Finally, it is clear that $i_1(b^{(n)}) = a^{(n)}$. □

There is the following immediate corollary of Proposition 5 and Corollary 2.

Corollary 3. *An element $a \in A^{(n)}$ belongs to subgroup $B^{(n)}$ if and only if $1 \sim a$ by some homotopy F for which $F_0 = pr$, where $1 = id \in A^{(n)}$.*

Now we are going to describe some special elements of the subgroup $B^{(n)}$.

Lemma 9. *For a fixed tower t it is possible to define a functorial map*

$$C^p(K_p, \pi_{p+1}) \to B^{(n)}, \ p \le n-1,$$

which assigns to c^p a collection $a_{c^p} = \{a^k_{c^p}\}, k = 1, 2, \cdots, n$ with

$$a^k_{c^p} = 0 \text{ for } k \le p \text{ and } a^{p+1}_{c^p} = \delta c^p.$$

Proof. Let (0) and (1) be two 0-simplices of $I = \Delta[1]$. We denote by same symbols all corresponding degenerate simplices as well. Let $c^p \in C^p(K_p, \pi_{p+1})$ and $I^q = I \times I \times \cdots \times I$ (q factors). For $q \ge 0$ define the cochain $c^p_q \in C^p(K_p \times I^q, \pi_{p+1})$ as follows. For $\xi = (1) \times \cdots \times (1)$ let $c^p_q(\tau \times \xi) = c^p(\tau)$ and $c^p_q(\tau \times \xi) = 0$ otherwise. In particular $c^p_0 = c^p$. For an arbitrary L, we denote by

$$\widetilde{i}_\epsilon : L \times I^q \to L \times I^{q+1}, \ \epsilon = 0, 1,$$

all maps given by $\widetilde{i}_\epsilon(\tau \times \xi_1 \times \cdots \times \xi_q) = \tau \times \xi_1 \times \cdots \times (\epsilon) \times \cdots \times \xi_q$. Then we have

$$\widetilde{i}_1(c^p_{q+1}) = c^p_q \text{ and } \widetilde{i}_0(c^p_{q+1}) = 0.$$

Let, besides, $F : M \times I \to L$ be a homotopy. Below we will use the standard cochain homotopy

$$d_F : C^*(L, G) \to C^{*-1}(M, G)$$

given by

$$d(c^n)(m^{n-1}) = c^n(\Sigma_{i=0}^{n-1}(-1)^i F(s_i m^{n-1} \times s_{n-1} \ldots s_{i+1} s_{i-1} \ldots s_0 \xi^1)),$$

where $c^n \in C^n(L, G)$, $m^{n-1} \in M$, $\xi^1 = (0,1) \in \Delta[1]$ and G is an abelian group. Let, now, $F_q : K_{p+1} \times I^q \to K_{p+1}$ be a map given by

$$F_q((\sigma, c) \times \xi) = (\sigma, c + t_{\sigma \times \xi}(\delta c^p_q)),$$

where $q \ge 0$ and $\xi \in I^q$. Then we have $F_0 = F_{(id, \delta c^p)}$ and, for $q \ge 1$,

$$F_q \circ \widetilde{i}_0 = pr^{(q-1)} \text{ and } F_q \circ \widetilde{i}_1 = F_{q-1},$$

where, for an arbitrary L, $pr^{(n)} : L \times I^n \to L$ is the projection and $pr^{(0)} = id$. For $q \geq 1$ we consider F_q as a homotopy by last coordinate. Then F_q defines a functorial (for induced maps of P-constructions) cochain homotopy

$$d_{F_q} : C^*(K_{p+1}, \pi_{p+2}) \to C^{*-1}(K_{p+1} \times I^{q-1}, \pi_{p+2}).$$

Inspection shows that for $q \geq 2$ we have

$$\widetilde{i}_1^{\#} \circ d_{F_q} = d_{F_{q-1}} \text{ and } \widetilde{i}_0^{\#} \circ d_{F_q} = 0.$$

Consider now $K_{p+2} = P(K_{p+1}, \pi_{p+2}, z^{p+3})$ and the maps

$$\Phi_q : K_{p+2} \times I^q \to K_{p+2}, q \geq 0,$$

given by

$$\Phi_q((\sigma, c) \times \xi) = (F_q(\sigma \times \xi), c + t_{\sigma\times\xi}(d_{F_{q+1}}(z^{p+3}))).$$

Then it follows from above that Φ_q is well defined, $\Phi_0 = F_{(F_0, d_{F_1}(z^{p+3}))}$ and

$$\Phi_q \circ \widetilde{i}_1 = \Phi_{q-1},\ \Phi_q \circ \widetilde{i}_0 = pr^{(q-1)},\ q \geq 1;$$

$$\widetilde{i}_1^{\#} \circ d_{\Phi_q} = d_{\Phi_{q-1}},\ \widetilde{i}_0^{\#} \circ d_{\Phi_q} = 0,\ q \geq 2.$$

Then, by and analogously to Φ_q, we define $\Theta_q : K_{p+3} \times I^q \to K_{p+3}$ and so on. Define now a map $F : t^{(n)} \times I \to t^{(n)}$ by taking $F = (pr, \dots, pr, F_1, \Phi_1, \Theta_1, \dots)$. Then it follows from above that $F \circ i_0 = id$ and $F \circ i_1 = a_{c^p}$, where

$$a_{c^p} = (0, \dots, 0, \delta c^p, d_{F_1}(z^{p+3}), d_{\Phi_1}(z^{p+4}), d_{\Theta_1}(z^{p+5}), \dots) \in A_{t^{(n)}}.$$

Finally, it follows from Corollary 3 that $a_{c^p} \in B^{(n)}$ and we define a functorial map $C^p(K_p, \pi_{p+1}) \to B^{(n)}$ by $c^p \mapsto a_{c^p}$. □

Let $\overline{q}_k : A^{(k+1)} \to A^{(k)}$ be the above introduced restriction homomorphisms. For fixed n, consider a sequence of groups and epimorphisms

$$A_{n,n} \overset{q_{n-1}}{\to} A_{n,n-1} \to \cdots \to A_{n,2} \overset{q_1}{\to} A_{n,1},$$

where $A_{n,n} = A^{(n)}$ and $A_{n,k} = Im(\overline{q}_k \circ \cdots \circ \overline{q}_{n-2} \circ \overline{q}_{n-1})$, $q_k = \overline{q}_k \mid A_{n,k+1}$, $1 \leq k \leq n-1$. Besides, let $B_{n,n} = B^{(n)}$ and $B_{n,k} = q_k(B_{n,k+1})$, $1 \leq k \leq n-1$. Then the group $B_{n,k}$ is a normal subgroup of $A_{n,k}$, $1 \leq k \leq n$.

Now we can consider the factorgroups $G_{n,k} = A_{n,k}/B_{n,k}$, $1 \leq k \leq n$, and the epimorphisms $\beta_{k-1} : G_{n,k} \to G_{n,k-1}$, $2 \leq k \leq n$. induced by epimorphisms q_{k-1} .

The next proposition gives a partial information for groups $G_{n,k}$ and, in particular, for group $G^{(n)} = A^{(n)}/B^{(n)} = G_{n,n}$ too.

Proposition 6. *$G_{n,1}$ is a factorgroup of a subgroup of $H^1(N, \pi_*)$ and there are exact sequences of groups*

$$0 \leftarrow G_{n,k-1} \overset{\beta_{k-1}}{\leftarrow} G_{n,k} \overset{\alpha_k}{\leftarrow} H_{n,k}, 2 \leq k \leq n,$$

where $H_{n,k}$ is a subgroup of $H^k(K_{k-1}, \pi_k)$ and $a_k([z^k]) = [(0, \cdots, 0, z^k)]$

Proof. By definition $A_{n,1}$ is a subgroup of $Z^1(N,\pi_1)$. By Lemma 9 we have $B^1(N,\pi_1) \subset B_{n,1}$. Consequently $G_{n,1}$ is a factorgroup of a subgroup of $H^1(N,\pi_1)$. Let, now, $g^{(k)} = [a^{(k)}] \in Ker\beta_{k-1}$, where $a^{(k)} \in A_{n,k}$. Consider the element $b^{(k-1)} = (q_{k-1}(a^{(k)}))^{-1} \in B_{n,k-1}$. Let $q_{k-1}(b^{(k)}) = b^{(k-1)}$, where $b^{(k)} \in B_{n,k}$. Then $q_{k-1}(a^{(k)} * b^{(k)}) = 1$ and $a^{(k)} * b^{(k)} = (0,\dots,0,z^k)$, where $z^k \in Z^k(K_{k-1},\pi_k)$. Consequently $g^{(k)} = [(0,\dots,0,z^k)]$. Let

$$\overline{Z}^k = \{z^k \mid z^k \in Z^k(K_{k-1},\pi_k),\ [(0,\dots,0,z^k)] \in G_{n,k}\}.$$

According to the formula of group operation in $A^{(n)}$, $\overline{Z}^k$ is a subgroup of $Z^k(K_{k-1},\pi_k)$. By Lemma 9 we have $B^k(K_{k-1},\pi_k) \subset \overline{Z}^k$. Consider the factorgroup

$$H_{n,k} = \overline{Z}^k / B^k(K_{k-1},\pi_k).$$

Obviously $H_{n,k}$ is a subgroup of $H^k(K_{k-1},\pi_k)$. Define now a homomorphism $\alpha_k : H_{n,k} \to G_{n,k}$ by $\alpha_k([z^k]) = [(0,\dots,0,z^k)]$, where $[z^k] \in H_{n,k}$. This homomorphism is well defined. Indeed, if $[z_1^k]$, $[z_2^k] \in H_{n,k}$ and $z_1^k - z_2^k = \delta c^{k-1}$, then, by Lemma 9 we have

$$(0,\dots,0,z_2^k)^{-1} * (0,\dots,0,z_1^k) = (0,\dots,0,\delta c^{k-1}) \in B_{n,k}.$$

Obviously we have $Im\alpha_k = Ker\beta_{k-1}$ and this completes the proof. □

The constructed group $G^{(n)}$ we use to formulate the criterion of equivalence of towers. Consider a natural action

$$H^{n+2}(K_n,\pi_{n+1}) \times G^{(n)} \to H^{n+2}(K_n,\pi_{n+1})$$

given by $h \circ g = f_n^*(h)$, here $[\{f_i\}] = g$. The action is well defined: if $\left[\{f_i^{'}\}\right] = g$, then f_n is homotopic to $f_n^{'}$, thus they induce the same homomorphisms of cohomology. Obviously we have $h \circ 1 = h$ and $h \circ (g_2 g_1) = (h \circ g_2) \circ g_1$.

Theorem 10. *Let t and $\overline{t}$ be two elements of $T(N,\pi_*)$ such that $z^k = z^k$ for $k \le n+1$, i.e. the restrictions $t^{(n)}$ and $\overline{t}^{(n)}$ are equal and let $G^{(n)}$ be the group corresponding to $t^{(n)}$. Then the restrictions $t^{(n+1)}$ and $\overline{t}^{(n+1)}$ are equivalent in $T^{(n+1)}(N,\pi_*)$ if and only if there exists an element $g \in G^{(n)}$ such that $[\overline{z}^{n+2}] = [z^{n+2}] \circ g$.*

Proof. Let $t^{(n+1)}$ and $\overline{t}^{(n+1)}$ be equivalent in $T^{(n+1)}(N,\pi_*)$, i.e. there exists a morphism

$$\{f_0 = id, f_1, \dots, f_n, f_{n+1}\} : t^{(n+1)} \to \overline{t}^{(n+1)},$$

and let $(a^1,\dots,a^n,a^{n+1})$ be the corresponding sequence of cochains. Then

$$(a^1,\dots,a^n) \in A^{(n)} \text{ and } \delta a^{n+1} = f_n^{\#}(\overline{z}^{n+2}) - z^{n+2}.$$

Thus $\left[z^{n+2}\right] \circ g^{-1} = \left[\overline{z}^{n+2}\right]$, where $g = \left[(a^1, \ldots, a^n)\right]$. Suppose now that there exists $g \in G^{(n)}$ such that $\left[z^{n+2}\right] \circ g = \left[\overline{z}^{n+2}\right]$. Let $g^{-1} = \left[(a^1, \ldots, a^n)\right]$. The sequence $(a^1, \ldots, a^n)$ defines a morphism

$$\{f_0 = id, f_1, \ldots, f_n\} : t^{(n)} \to \overline{t}^{(n)}$$

and there exists a cochain a^{n+1} such that $\delta a^{n+1} = f_n^{\#}(\overline{z}^{n+2}) - z^{n+2}$. Then the sequence $(a^1, \ldots, a^n, a^{n+1})$ defines a morphism

$$\{f_0 = id, f_1, \ldots, f_n, f_{n+1}\} : t^{(n+1)} \to \overline{t}^{(n+1)}.$$

□

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