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# TRANSFERRED CHERN CLASSES IN MORAVA $K$-THEORY

MALKHAZ BAKURADZE AND STEWART PRIDDY

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ABSTRACT. Let $\eta$ be a complex $n$-plane bundle over the total space of a cyclic covering of prime index $p$. We show that for $k \in \{1, 2, ..., np\} \setminus \{p, 2p, ..., np\}$ the $k$-th Chern class of the transferred bundle differs from a certain transferred class $\omega_k$ of $\eta$ by a polynomial in the Chern classes $c_p, ..., c_{np}$ of the transferred bundle. The polynomials are defined by the formal group law and certain equalities in $K(s)^*B(Z/p \times U(n))$.

## 1. STATEMENTS

Let $p$ be a prime, and let $\pi$ be the cyclic group of order $p$. For a given action of $\pi$ on a space $X$ consider the regular covering

$$\rho : E\pi \times X \to E\pi \times_\pi X.$$

Let us write for short $X_{h\pi} := E\pi \times_\pi X$. For the permutation action of $\pi$ on $X = Y^p$ we have $Y^p_{h\pi} = E\pi \times_\pi Y^p$. Let $t$ be a generator of $\pi$ and let $N_\pi = 1 + t + ... + t^{p-1}$ be the trace map. For an $n$-plane bundle $\eta^n$, the corresponding classifying map

$$f : X \to BU(n)$$

induces the classifying map $(f, tf, ..., t^{p-1}f)$ for the bundle $N_\pi\eta^n$ and thereby a map of orbit spaces

$$f_{\eta^n} : X_{h\pi} \to BU(n)^p_{h\pi}.$$

For the covering

$$\rho_\pi : E\pi \times BU(n)^p \to BU(n)^p_{h\pi}$$

and the universal $n$-plane bundle $\xi^n \to BU(n)$ consider the Atiyah transfer bundle [2]

$$\xi^n_\pi \to BU(n)^p_{h\pi},$$

i.e., the $np$-plane bundle

$$\xi^n_\pi = E\pi \times_\pi (\xi^n)^{\times p}.$$

Then the map $f_{\eta^n}$ classifies the Atiyah transfer bundle for $\eta^n$ and $\rho$. So by naturality of transfer [1] we can consider $\rho_\pi$ as the universal example.

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Let $K(s)^*$, $s \geq 1$, be the $s$th Morava $K$-theory at $p$. We recall that by the Künneth isomorphism, $K(s)^*(BU(n)^p) = F \oplus T$ as a $\pi$ module, where $F$ is free and $T$ is trivial.

**Definition 1.1.** Let $\omega_k^{(n)} \in F$ be defined modulo $\ker N_\pi = \mathrm{Im}(1-t)$ by

$$N_\pi(\omega_k^{(n)}) = c_k(N_\pi(\xi^n)),$$

where the $c_k$ are Chern classes, $k \in \{1, ..., np\} \setminus \{p, 2p, ..., np\}$.

By naturality of the transfer, $Tr_\pi^* = Tr_\pi^* t$, where

$$Tr_\pi^* : K(s)^*(E\pi \times BU(n)^p) \to K(s)^*(BU(n)^p_{h\pi});$$

hence $Tr_\pi^*(\omega_k^{(n)})$ is well defined.

We write $\omega_k(\eta^n)$ for the pullback by the map $f_{\eta^n}$ of orbit spaces defined above.

Recall that $K(s)^*(B\pi) = F_p[v_s, v_s^{-1}][z]/(z^{p^s})$, where $z = c_1(\theta)$ is the Chern class of the canonical complex line bundle over $B\pi$.

**Lemma 1.2.** *We can define a polynomial in $n+1$ variables,*

$$A_k^{(n)}(z^{p-1}, Z_1, ..., Z_n) \in K(s)^*[z, Z_1, ..., Z_n],$$

*uniquely by the equation in $K(s)^*B(\pi \times U(n))$:*

$$C_k - v_s z^{p^s-1} p^{-1} \sum_{\substack{i_1+2i_2+...+ni_n=k \\ i_0+i_1+...+i_n=p}} \binom{p}{i_0, i_1, ..., i_n} c_1^{i_1} ... c_n^{i_n} = A_k^{(n)}(z^{p-1}, C_p, ..., C_{pn})$$

*where $C_i = c_i(\xi^n \oplus \theta \otimes \xi^n \oplus ... \oplus \theta^{p-1} \otimes \xi^n)$, $c_j = c_j(\xi^n)$ are Chern classes, and $k \in \{1, ..., np\} \setminus \{p, ..., np\}$.*

For example, in $K(s)^*(B\pi \times BU(1))$ one has

$$C_1 = v_s \left( z^{p^s-1} c_1 + \sum_{i=1}^{s-1} z^{p^s - p^i} C_p^{p^{i-1}} \right);$$

thus

$$A_1^{(1)}(z^{p-1}, Z_1) = v_s \sum_{i=1}^{s-1} z^{p^s-p^i} Z_1^{p^{i-1}}.$$

Then using the polynomials $A_k^{(n)}$ we evaluate the transferred classes $\omega_k(\eta^n)$ for regular coverings:

**Theorem 1.3.** *Let $\rho : X \to X/\pi$ be the regular cyclic covering of prime index $p$ defined by a free action of $\pi$ on $X$, and let $Tr^* = Tr_\rho^*$ be the associated transfer homomorphism. Let $\eta^n \to X$ be a complex $n$-plane bundle, $\eta_\pi^n \to X/\pi$ the $pn$-plane bundle defined by Atyiah transfer, and $\psi \to X/\pi$ be the complex line bundle associated with $\rho$. Then*

$$c_k(\eta_\pi^n) - Tr^*(\omega_k(\eta^n)) = A_k^{(n)}(c_1^{p-1}(\psi), c_p(\eta_\pi^n), ..., c_{pn}(\eta_\pi^n)),$$

*where $k \in \{1, ..., np\} \setminus \{p, ..., np\}$.*

**Example 1.4.** $\omega_1 = c_1(\eta^n)$; hence if $k = 1$ we have

$$c_1(\eta_\pi^n) - Tr^*(c_1(\eta^n)) = A_1^{(n)}(c_1^{p-1}(\psi), c_p(\eta_\pi^n), ..., c_{pn}(\eta_\pi^n)).$$

If, in addition, $n = 1$, then by the above example we have for the line bundle $\eta \to X$ and the transferred Chern class,

$$c_1(\eta_\pi) - Tr^*(c_1(\eta)) = v_s \sum_{i=1}^{s-1} z^{p^s - p^i} c_p(\eta_\pi)^{p^{i-1}}.$$

## 2. Proofs

For the proof of Lemma 1.2 we need the following property of formal group laws (FGL) observed in [3].

**Lemma 2.1.** *For the formal group law in Morava $K$-theory $K(s)^*$, $s > 1$, we have mod $x^{p^{2(s-1)}}$ (or modulo $y^{p^{2(s-1)}}$)*

$$F(x,y) \equiv x + y - v_s \sum_{0<j<p} p^{-1}\binom{p}{j}(x^{p^{s-1}})^j (y^{p^{s-1}})^{p-j}.$$

*Proof.* As D. Ravenel explained, this result can be derived from the recursive formula for the FGL given in [5, 4.3.8]: For the FGL in Morava $K$-theory it reads

$$F(x,y) = F(x+y, v_s w_1(x,y)^{p^{s-1}}, v_s^{e_2} w_2(x,y)^{p^{2(s-1)}}, ...)$$

where $w_i$ is a certain homogeneous polynomial of degree $p^i$ defined by [5, 4.3.5] and $e_i = (p^{is} - 1)/(p^s - 1)$. In particular, $w_0 = x + y$ (the first term in the formula above),

$$w_1 = - \sum_{0<j<p} p^{-1}\binom{p}{j} x^j y^{p-j},$$

and $w_i \notin (x^p, y^p)$.

For $s > 1$ we can reduce modulo the ideal $v_s^{e_2}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ and get

$$\begin{aligned} F(x,y) &\equiv F(x+y, v_s w_1(x,y)^{p^{s-1}}) \\ &= F(x+y+v_s w_1(x,y)^{p^{s-1}}, v_s w_1(x+y, v_s w_1(x,y)^{p^{s-1}})^{p^{s-1}}, \dots) \\ &\equiv F(x+y+v_s w_1(x,y)^{p^{s-1}}, v_s w_1(x^{p^{s-1}} + y^{p^{s-1}}, v_s^{p^{s-1}} w_1(x,y)^{p^{2(s-1)}})), \end{aligned}$$

and modulo $v_s^{1+p^{s-1}}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ we have $F(x,y) \equiv x + y + v_s w_1(x,y)^{p^{s-1}}$. $\square$

**Proof of Lemma 1.2.** We begin by considering the case $n = 1$. Let $\sigma_i$ be the $i$-th symmetric functions in $p$ variables. Then

$$C_i = \sigma_i(x, F(x,z), ..., F(x, (p-1)z)),$$

where $x = c_1(\xi^1)$, $z = c_1(\theta)$ are Chern classes in $K(s)^*B(\pi \times U(1)) = K(s)^*[[z,x]]/(z^{p^s})$. Consider the equation

$$C_k = - \sum_{0 \le i \le p^s} \lambda_{ki} C_p^i + p^{-1}\binom{p}{k} x^k z^{p^s - 1};\ \ 1 \le k \le p-1.$$

We want to prove that such $\lambda_{ki}$ exist uniquely as elements in $K(s)^*[[z]]/(z^{p^s})$ and to compute these elements as polynomials in $z^{p-1}$. Then $A_k^{(1)}(z^{p-1}, C_p) = \sum_{0 \le i \le p^s} \lambda_{ki} C_p^i$.

$C_k$ is the Chern class of the bundle $\xi \otimes (1 + \theta + \theta^2 + ... + \theta^{p-1})$ and can be written as a series in the Chern classes of $\xi$, that is $x$, and the Chern classes of $1 + \theta + \theta^2 + ... + \theta^{p-1}$. But the Chern classes of the latter bundle are elementary symmetric functions in $z, 2z, ..., (p-1)z$, the Chern classes of $\theta, \theta^2, ..., \theta^{p-1}$, all of which vanish except for the $(p-1)$-th class, which is $-z^{p-1}$. Hence we can write the classes $C_k$ as series in $x$ and $z^{p-1}$. Lemma 2.1 enables us to write $C_k$ as explicit polynomials in $x$ and $z^{p-1}$. Now noting that $C_p = x^p \bmod z^{p-1}$ we obtain from the above equation a system of linear equations in variables $\lambda_{kj}$ by equating the coefficients at $x^i$, $i \geq 0$. Vanishing also implies that $\lambda_{k0} = 0$, $k = 1, ..., p-2$ and $\lambda_{p-10} = c_{p-1}(1 + \theta + ... + \theta^{p-1}) = -z^{p-1}$.

Then equating the coefficients at $x^p$,...,$x^{p^{s+1}}$ in the above equation after rewriting it in terms of $x$ and $z^{p-1}$ as above, we have a system of $p^s$ linear equations in $p^s$ variables $\lambda_{ki}$, $i = 1, ..., p^s$ . The determinant of this system is invertible since the diagonal coefficients are invertible and all other coefficients lie in the (nilpotent) augmentation ideal. Thus the elements $\lambda_{ki}$ are uniquely defined.

Of course, equating the coefficients at $x^i$ for $i \neq p, 2p, ..., p^{s+1}$ will produce other equations in $\lambda_{kj}$, $j = 1, ..., p^s$. But these equations are derived from the old equations above. These additional equations make the matrix upper triangular. This defines $A_k^{(1)}$.

In the general $n$ case we proceed analogously, noting that $C_{ip} = c_i^p \bmod z^{p-1}$, $i = 1, ..., n$. Our additional claim again is that the $A_k^{(n)}$ are polynomials, that is, only a finite number of elements $\lambda_{k,i_1,...,i_n}$ are nontrivial. Here we need the splitting principle and Lemma 2.1 to express explicitly the elements $C_i$, $i = 1, ..., np$ in terms of polynomials in $z^{p-1}$ and $c_1, ..., c_n$. For $k \in \{1, ..., np\} \setminus \{p, ..., np\}$, let

$$A_k^{(n)}(z^{p-1}, C_p, ..., C_{pn}) = \sum_{0 \leq i_1, ..., i_n \leq p^s} \lambda_{k,i_1,...,i_n} C_p^{i_1} ... C_{np}^{i_n}.$$

We define $\lambda_{k,0...0} = c_k(n + n\theta + ... + n\theta^{p-1})$ again by looking at reductions to $K(s)^*(B\pi)$. The other $n(p^s+1)-1$ elements $\lambda_{k,i_1,...,i_n}$ can be defined as the solution of a system of $n(p^s+1)-1$ linear equations with an invertible determinant. This system is obtained from the equation after using Lemma 2.1 to rewrite it in terms of $z^{p-1}$ and $c_1, ..., c_n$ and equating coefficients at $c_1^{pi_1} ... c_n^{pi_n}$. The solution defines $\lambda_{k,i_1,...,i_n}$, $0 \leq i_j \leq p^s$. Again the additional equations in these elements arise from the coefficients of other monomials and are not new. The desired polynomials are thus uniquely defined. □

*Proof of Theorem* 1.3. Consider the homotopy orbit space $BU(n)^p_{h\pi} = E\pi \times_\pi BU(n)^p$ as the universal example. The diagonal map $BU(n) \to BU(n)^p$ induces the inclusion

$$i : B\pi \times BU(n) \to E\pi \times_\pi BU(n)^p.$$

We use a result from [4] (Proposition 4.2) which implies that since $BU(n)$ is a unitary-like space (i.e., $K(s)^*BU(n)$ has no nilpotent elements) the map $(i \vee \rho_\pi)^*$ is a monomorphism. Since $\rho_\pi^* Tr^* = N_\pi$, the difference

$$c_k(\eta_\pi^n) - Tr^*(\omega_k^{(n)})$$

belongs to $\ker \rho_\pi^*$ and hence is detected by $i^*$. The result now follows from Lemma 1.2. □

Note we can replace the cyclic group by the symmetric group $\Sigma_p$ and use the polynomials $A_k^{(n)}$ to evaluate the disparity or "gap" between Chern class $c_k(\xi^n_{\Sigma_p})$ and $\operatorname{Im} Tr^*_{\Sigma_p}$, for $\rho_{\Sigma_p} : E\Sigma_p \times U(n)^p \to BU(n)^p_{h\Sigma_p}$.

Namely, the Euler characteristic of the coset space $\Sigma_p \wr U(n)/\pi \wr U(n) = (p-1)!$ is prime to $p$. Hence the inclusion $\rho_{\pi,\Sigma_p} : \pi \to \Sigma_p$ induces a monomorphism

$$K(s)^*(B(\Sigma_p \wr U(n))) \to K(s)^*(B(\pi \wr U(n))).$$

Hence Hunton's result above holds for $BU(n)^p_{h\Sigma_p}$.

Now let $\varsigma_k^{(n)} \in F$ be defined modulo $\ker N_{\Sigma_p}$ by $N_{\Sigma_p}(\varsigma_k^{(n)}) = c_k(N_\pi(\xi^n))$. Again $Tr^*_{\Sigma_p}(\varsigma_k^{(n)})$ is well-defined: as above, $\ker N_\pi = \mathrm{Im}(1-t^*)$ and therefore

$$a \in F \cap \ker N_{\Sigma_p} \Rightarrow \sum_{g\in\Sigma_p/\pi} g^*a \in \mathrm{Im}(1-t^*) \Rightarrow Tr^*_{\Sigma_p}(\sum_{g\in\Sigma_p/\pi} g^*a) = 0$$
$$\Rightarrow (p-1)! Tr^*_{\Sigma_p}(a) = 0 \Rightarrow Tr^*_{\Sigma_p}(a) = 0.$$

Then for the $np$-plane bundle over $BU(n)^p_{h\Sigma_p}$,

$$\xi^n_{\Sigma_p} = E\Sigma_p \times_{\Sigma_p} (\xi^n)^{\times p},$$

the difference

$$c_k(\xi^n_{\Sigma_p}) - Tr^*_{\Sigma_p}(\varsigma_k^{(n)})$$

belongs to $\ker \rho^*_{\Sigma_p}$ and hence is detected by the polynomials

$$A_k^{(n)}(y, c_p(\xi^n_{\Sigma_p}), ..., c_{np}(\xi^n_{\Sigma_p})),$$

where

$$y \in K(s)^*(B(\Sigma_p)) = K(s)^*[[y]]/(y^{m_s}),$$

$|y| = 2(p-1)$, and $m_s = [(p^s - 1/(p-1))] + 1$.

Thus we have:

**Theorem 2.2.**

$$c_k(\xi^n_{\Sigma_p}) - Tr^*_{\Sigma_p}(\varsigma_k^{(n)}) = A_k^{(n)}(y, c_p(\xi^n_{\Sigma_p}), ..., c_np(\xi^n_{\Sigma_p})),$$

*for* $k \in \{1, ..., np\} \setminus \{p, ..., np\}$.

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We would like to thank Neil Strickland for suggesting an elegant proof of Lemma 1.2 completely within the context of subgroups of formal groups. This material is available from N.P.Strickland@sheffield.ac.uk. We have decided not to include it since it uses rather different techniques and is no shorter than our proof for which we now offer some motivation. The lemma says that, modulo image of the transfer

$$Tr^* : K(s)^*B(U(n)) \to K(s)^*B(\pi \times U(n))$$

for the covering $\pi \times 1 : E\pi \times BU(n) \to B\pi \times BU(n)$, the class $C_k$ can be written as a polynomial in $z^{p-1}$ and $C_p, ..., C_{np}$, that this expression vanishes when restricted to $K(s)^*BU(n)$, and that the indeterminacy is obtained by applying the transfer to $p^{-1}C_k$ after restricting to $K(s)^*(BU(n))$. This uses Fröbenius reciprocity and the fact that $Tr^*(1) = [p](z)/z = v_s z^{p^s-1}$ in Morava $K$-theory.

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Razmadze Mathematical Institute, Tbilisi 380093, Republic of Georgia
*Current address*: Max-Planck-Institut Für Mathematik, Bonn, Germany
*E-mail address*: bakuradz@mpim-bonn.mpg.de

Department of Mathematics, Northwestern University, Evanston, Illinois 60208
*E-mail address*: priddy@math.northwestern.edu