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Unilateral contact problems with a friction

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Abstract

The boundary contact problem for a micropolar homogeneous elastic hemitropic medium with a friction is investigated. Here, on a part of the elastic medium surface with a friction, instead of a normal component of force stress there is prescribed the normal component of the displacement vector. We give their mathematical formulation of the Problem in the form of spatial variational inequalities. We consider two cases, the so-called coercive case (when elastic medium is fixed along some part of the boundary) and semi-coercive case (the boundary is not fixed). Based on our variational inequality approach, we prove the existence and uniqueness theorems and show that solutions continuously depend on the data of the original problem. In the semi-coercive case, the necessary condition of solvability of the corresponding contact problem is written out explicitly. This condition under certain restrictions is sufficient, as well.

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1. Introduction

In the present paper we investigate the one-sided contact problem for a homogeneous hemitropic elastic medium with a friction. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed by means of the displacement and rotation vectors.

In their works [1] and [2], E. and F. Cosserats created and presented the model of solid medium in which every material point has six degrees of freedom, three of which are defined by displacement components and the other three by the components of rotation (for the history of the model of elasticity see [3–9] and references therein).

A micropolar medium, not isotropic with respect to the inversion, is called a hemitropic or noncentrosymmetric medium.

Improved mathematical models describing hemitropic properties of elastic materials have been obtained and considered in [10] and [11]. The main equations of that model are interconnected and generate a matrix second

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order differential operator of dimension 6×6 . Particular problems for solid media of hemitropic theory of elasticity have been considered in [12,13,8] and [9]. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity with the use of the potential method for smooth and nonsmooth Lipschitz domains were studied in [12], the one-sided contact problems of statics of the hemitropic theory of elasticity free from friction were investigated in [14–18], and the contact problems of statics and dynamics with a friction were considered in [19–29]. Analogous one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [30–35] and the references therein).

In the present work, we present the basic equations of statics of the theory of elasticity for homogeneous hemitropic media in a vector–matrix form, introduce the generalized stress operator and quadratic form of potential energy. Then we describe mathematical model of boundary conditions which show the contact between a hemitropic medium and a solid body with regard for the friction effect. We will consider the case, where some part of the elastic medium boundary is fixed mechanically. The problem is reduced equivalently to the variational inequality, the question on the existence and uniqueness of a weak solution of the initial problem is treated, and a continuous Lipschitz dependence of the solution on the data of the problem is investigated. Further, we will investigate more complicated cases, where friction is considered on the whole medium boundary. In such cases, the corresponding mathematical problem is, in general, unsolvable. The necessary conditions of solvability are established and the sufficient conditions for the existence of a solution are formulated explicitly.

2. Basic equations and Green’s formulas

2.1. Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a C^∞ -smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. The domain Ω is assumed to be filled with a homogeneous hemitropic material.

The basic equilibrium equations in the hemitropic theory of elasticity written in components of the displacement and rotation vectors are of the form

$$\begin{aligned} (\mu + \alpha) \Delta u(x) + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} u(x) + (\kappa + \nu) \Delta \omega(x) \\ + (\delta + \kappa - \nu) \operatorname{grad} \operatorname{div} \omega(x) + 2\alpha \operatorname{curl} \omega(x) + \rho F(x) = 0, \\ (\kappa + \nu) \Delta u(x) + (\delta + \kappa - \nu) \operatorname{grad} \operatorname{div} u(x) + 2\alpha \operatorname{curl} u(x) + (\gamma + \varepsilon) \Delta \omega(x) \\ + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \omega(x) + 4\nu \operatorname{curl} \omega(x) - 4\alpha \omega(x) + \rho \Psi(x) = 0, \end{aligned} \tag{2.1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the vector of rotation, $F = (F_1, F_2, F_3)^\top$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3)^\top$ are the mass force and mass moment calculated per unit of mass, ρ is density of the elastic medium, and $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \kappa$ and ε are elastic constants (see [11,13]). Here and in what follows, the symbol $(\cdot)^\top$ denotes transposition.

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$\begin{aligned} L(\partial) &= \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6 \times 6}, \\ L^{(1)}(\partial) &:= (\mu + \alpha) \Delta I_3 + (\lambda + \mu - \alpha) Q(\partial), \\ L^{(2)}(\partial) &= L^{(3)}(\partial) := (\kappa + \nu) \Delta I_3 + (\delta + \kappa - \nu) Q(\partial) + 2\alpha R(\partial), \\ L^{(4)}(\partial) &:= [(\gamma + \varepsilon) \Delta - 4\alpha] I_3 + (\beta + \gamma - \varepsilon) Q(\partial) + 4\nu R(\partial), \end{aligned}$$

where I_k is the unit $k \times k$ -matrix and

$$Q(\partial) = [\partial_k \partial_j]_{3 \times 3}, \quad R(\partial) = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

The system of Eqs. (2.1) can be rewritten in the matrix form

$$L(\partial)U(x) + \mathcal{G}(x) = 0, \quad x \in \Omega,$$

where $U = (u, \omega)^\top$ and $\mathcal{G} = (\rho F, \rho \Psi)^\top$.

By $T(\partial, n)$ we denote the generalized stress operator of dimension 6×6 (see [13]):

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) \end{bmatrix}, \quad T^{(j)} = [T_{pq}^{(j)}]_{3 \times 3}, \quad j = \overline{1, 4},$$

where

$$\begin{aligned} T_{pq}^{(1)}(\partial, n) &:= (\mu + \alpha)\delta_{pq}\partial_n + (\mu - \alpha)n_q\partial_p + \lambda n_p\partial_q, \\ T_{pq}^{(2)}(\partial, n) &:= (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha \sum_{k=1}^3 \varepsilon_{pqk}n_k, \\ T_{pq}^{(3)}(\partial, n) &:= (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q, \\ T_{pq}^{(4)}(\partial, n) &:= (\gamma + \varepsilon)\delta_{pq}\partial_n + (\gamma - \varepsilon)n_q\partial_p + \delta n_p\partial_q - 2\nu \sum_{k=1}^3 \varepsilon_{pqk}n_k. \end{aligned}$$

Here, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward (with respect to Ω) unit normal vector at the point $x \in S$, and $\partial_n = \partial/\partial n$ is the normal derivative in the direction of the vector n . The six-component generalized stress vector has the form

$$T(\partial, n)U = (\mathcal{T}U, \mathcal{M}U)^\top,$$

where $\mathcal{T}U := T^{(1)}u + T^{(2)}\omega$ is the force stress vector and $\mathcal{M}U := T^{(3)}u + T^{(4)}\omega$ is the moment stress vector.

2.2. Green's formulas

For the real-valued vector functions $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ of the class $[C^2(\overline{\Omega})]^6$ the following Green's formula [13]

$$\int_{\Omega} [L(\partial)U \cdot U' + E(U, U')] dx = \int_S \{T(\partial, n)U\}^+ \cdot \{U'\}^+ dS, \tag{2.2}$$

is valid, where $\{\cdot\}^+$ denotes the trace operator on S from Ω , and $E(\cdot, \cdot)$ is a bilinear form defined by the equality

$$\begin{aligned} E(U, U') = E(U', U) = \sum_{p,q=1}^3 \left\{ (\mu + \alpha)u'_{pq}u_{pq} + (\mu - \alpha)u'_{pq}u_{qp} \right. \\ \left. + (\chi + \nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + (\chi - \nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + (\gamma + \varepsilon)\omega'_{pq}\omega_{pq} \right. \\ \left. + (\gamma - \varepsilon)\omega'_{pq}\omega_{qp} + \delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta \omega'_{pp}\omega_{qq} \right\}, \end{aligned}$$

where u_{pq} and ω_{pq} are the so-called tensors of deformation and torsion-bending for hemitropic media,

$$u_{pq} = u_{pq}(U) = \partial_p u_q - \sum_{k=1}^3 \varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \omega_{pq}(U) = \partial_p \omega_q, \quad p, q = 1, 2, 3. \tag{2.3}$$

Here and in the sequel, by $a \cdot b$ we denote the scalar product of two vectors $a, b \in \mathbb{R}^m : a \cdot b = \sum_{j=1}^m a_j b_j$.

Under certain assumptions on elastic constants (see [11,17,24]), specific energy of deformation $E(U, U)$ is a positive definite quadratic form with respect to $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $C_0 > 0$, depending only on the elastic constants, such that

$$E(U, U) \geq C_0 \sum_{p,q=1}^3 [u_{pq}^2 + \omega_{pq}^2].$$

The following assertion describes the null space of the energy quadratic form $E(U, U)$ (see [13]).

Lemma 2.1. Let $U = (u, \omega)^\top \in [C^1(\bar{\Omega})]^6$ and $E(U, U) = 0$ in Ω . Then

$$u(x) = [a \times x] + b, \quad \omega(x) = a, \quad x \in \Omega,$$

where a and b are arbitrary three-dimensional constant vectors and $[\cdot \times \cdot]$ denotes the cross product of two vectors.

Vectors of the type $([a \times x] + b, a)$ are called generalized rigid vectors. We observe that a generalized rigid displacement vector vanishes, i.e., $a = b = 0$ if it is zero at a single point.

Throughout the paper, $L_p(\Omega)$ ($1 \leq p \leq \infty$), $L_2(\Omega) = H^0(\Omega)$ and $H^s(\Omega) = H_2^s(\Omega)$, $s \in \mathbb{R}$, denote, respectively, the Lebesgue and Bessel potential spaces (see e.g., [36,37]). The corresponding norms we denote by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. By $\mathcal{D}(\Omega)$ we denote the class of $C^\infty(\Omega)$ functions with support in the domain Ω . If M is an open proper part of the manifold $\partial\Omega$, i.e., $M \subset \partial\Omega$, $M \neq \partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ on M ,

$$H^s(M) := \{r_M \varphi : \varphi \in H^s(\partial\Omega)\},$$

where r_M stands for the restriction operator on the set M . Further, let

$$\tilde{H}^s(M) := \{\varphi \in H^s(\partial\Omega) : \text{supp } \varphi \subset \bar{M}\}.$$

From the positive definiteness of the energy form $E(\cdot, \cdot)$ with respect to the variables (2.3) it follows that

$$B(U, U) := \int_{\Omega} E(U, U) dx \geq 0. \quad (2.4)$$

Moreover, there exist positive constants C_1 and C_2 , depending only on the material parameters, such that the following Korn's type inequality (see (Part I, Section 12, [32]))

$$B(U, U) \geq C_1 \|U\|_{[H^1(\Omega)]^6}^2 - C_2 \|U\|_{[H^0(\Omega)]^6}^2 \quad (2.5)$$

holds for an arbitrary real-valued vector function $U \in [H^1(\Omega)]^6$.

Remark 2.2. If $U \in [H^1(\Omega)]^6$ and on some part $S^* \subset \partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{S^*}\{U\}^+ = 0$, we have the strict Korn's inequality

$$B(U, U) \geq C \|U\|_{[H^1(\Omega)]^6}^2 \quad (2.6)$$

with some positive constant $C > 0$ which does not depend on the vector U . This follows from (2.5) and the fact that in this case $B(U, U) > 0$ for $U \neq 0$ (see, e.g., [38], [32, Ch. 2, Exercise 2.17]).

Remark 2.3. By the standard limiting arguments, Green's formula (2.2) can be extended to the Lipschitz domains and to the vector function $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see [38,36]),

$$\int_{\Omega} [L(\partial)U \cdot U' + E(U, U')] dx = \langle \{T(\partial, n)U\}^+, \{U'\}^+ \rangle_{\partial\Omega}, \quad (2.7)$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$ which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By virtue of this relation, the generalized trace of the stress operator $\{T(\partial, n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is correctly determined.

3. Contact problems with a friction

3.1. Pointwise and variational formulation of the contact problem

Let the boundary S of the domain Ω be divided into two open, connected and nonoverlapping parts S_1 and S_2 of positive measure, $S = \bar{S}_1 \cup \bar{S}_2$, $S_1 \cap S_2 = \emptyset$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2 .

Definition 1. A vector function $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ is said to be a weak solution of the equation

$$L(\partial)U + \mathcal{G} = 0, \quad \mathcal{G} \in [L_2(\Omega)]^6 \tag{3.1}$$

in the domain Ω if

$$B(U, \Phi) = \int_{\Omega} \mathcal{G} \cdot \Phi \, dx \quad \forall \Phi \in [\mathcal{D}(\Omega)]^6,$$

where the bilinear form $B(\cdot, \cdot)$ is given by formula (2.4).

For the normal and tangential components of the force stress vector we will use, respectively, the following notation:

$$(\mathcal{T}U)_n := \mathcal{T}U \cdot n, \quad (\mathcal{T}U)_s := \mathcal{T}U - n(\mathcal{T}U)_n.$$

Further, let

$$\mathcal{G} = (\rho F, \rho \Psi)^\top \in [L_2(\Omega)]^6, \quad \varphi \in [H^{-1/2}(S_2)]^3 \quad \text{and} \quad g \in L_\infty(S_2), \quad g \geq 0.$$

Consider the following contact problem of statics with a friction.

Problem A. Find a vector function $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of Eq. (3.1) and satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \quad \text{on} \quad S_1, \tag{3.2}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \quad \text{on} \quad S_2, \tag{3.3}$$

$$r_{S_2}\{u_n\}^+ = 0 \quad \text{on} \quad S_2, \tag{3.4}$$

$$\text{if } |r_{S_2}\{(\mathcal{T}U)_s\}^+| < g, \quad \text{then } r_{S_2}\{u_s\}^+ = 0, \tag{3.5}$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1 r_{S_2}\{u_s\}^+ = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+. \tag{3.6}$$

The conditions (3.2) and (3.4) are understood in the usual trace sense, whereas (3.3) is understood in the generalized functional sense described in Remark 2.3.

This problem can be reformulated as a variational inequality. To this end, let us introduce on the space $[H^1(\Omega)]^3$ the following continuous convex functional

$$j(v) = \int_{S_2} g | \{v_s\}^+ | \, dS, \quad v \in [H^1(\Omega)]^3. \tag{3.7}$$

Next, we define the closed convex subset of $[H^1(\Omega)]^6$,

$$K(\Omega) := \{V = (v, \omega)^\top \in [H^1(\Omega)]^6 : r_{S_1}\{V\}^+ = 0, r_{S_2}\{v_n\}^+ = 0\}.$$

Consider the following variational inequality: Find $U = (u, \omega)^\top \in K(\Omega)$ such that the variational inequality

$$B(U, V - U) + j(v) - j(u) \geq \int_{\Omega} \mathcal{G} \cdot (V - U) \, dx + \langle \varphi, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \tag{3.8}$$

holds for all $V = (v, w)^\top \in K(\Omega)$.

Here and in what follows, the symbol $\langle \cdot, \cdot \rangle_M$ denotes the duality relation between the corresponding dual pairs $X(M)$ and $X^*(M)$. In particular, the brackets $\langle \cdot, \cdot \rangle_{S_2}$ in (3.8) denote the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\tilde{H}^{1/2}(S_2)]^3$.

3.2. Equivalence

Here we prove the following equivalence result.

Theorem 3.1. *If a vector function $U \in K(\Omega)$ solves the variational inequality (3.8), then it is a solution of Problem A, and vice versa.*

Proof. Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of Problem A. With the help of Green’s formula (2.7), we get

$$\langle \{T(\partial, n)U\}^+, \{V - U\}^+ \rangle_S - B(U, V - U) + \int_\Omega \mathcal{G} \cdot (V - U) dx = 0$$

for all $V = (v, w)^\top \in K(\Omega)$.

Since $r_{S_1}\{V - U\}^+ = 0$ and $r_{S_2}\{v_n - u_n\}^+ = 0$, these equations can be rewritten as

$$\begin{aligned} B(U, V - U) + j(v) - j(u) &= \int_\Omega \mathcal{G} \cdot (V - U) dx + \langle \varphi, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \\ &\quad + \int_{S_2} \left[\{(\mathcal{T}U)_s\}^+ \cdot \{v_s - u_s\}^+ + g(|\{v_s\}^+| - |\{u_s\}^+|) \right] dS. \end{aligned} \tag{3.9}$$

It is easy to see that if the conditions (3.5) and (3.6) hold, then

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ \cdot r_{S_2}\{v_s - u_s\}^+ + g(|r_{S_2}\{v_s\}^+| - |r_{S_2}\{u_s\}^+|) \geq 0 \quad \text{on } S_2.$$

Using this inequality, from (3.9) we obtain

$$B(U, V - U) + j(v) - j(u) \geq \int_\Omega \mathcal{G} \cdot (V - U) dx + \langle \varphi, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \quad \forall V = (v, w)^\top \in K(\Omega).$$

Thus, $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ is a solution of the variational inequality (3.8).

Let now $U \in K(\Omega)$ be a solution of the variational inequality (3.8). Substituting $U \pm \Phi$ for V in (3.8) with an arbitrary $\Phi \in [\mathcal{D}(\Omega)]^6$, we obtain

$$B(U, \Phi) = \int_\Omega \mathcal{G} \cdot \Phi dx \quad \forall \Phi \in [\mathcal{D}(\Omega)]^6$$

which implies that U is a weak solution of Eq. (3.1).

By virtue of the interior regularity theorems (see [32]), we have $U \in [H^2(\Omega')]^6$ for every $\overline{\Omega'} \subset \Omega$. Hence, the following equation holds in the domain Ω

$$L(\partial)U + \mathcal{G} = 0.$$

Using again Green’s formula, we have

$$\begin{aligned} B(U, V - U) - \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s\}^+ \rangle_{S_2} - \langle r_{S_2}\{\mathcal{M}U\}^+, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \\ = \int_\Omega \mathcal{G} \cdot (V - U) dx \quad \forall V = (v, w)^\top \in K(\Omega). \end{aligned} \tag{3.10}$$

We subtract (3.10) from inequality (3.8) and get

$$\begin{aligned} \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s\}^+ \rangle_{S_2} + \int_{S_2} g(|\{v_s\}^+| - |\{u_s\}^+|) dS \\ + \langle r_{S_2}\{\mathcal{M}U\}^+ - \varphi, r_{S_2}\{w - \omega\}^+ \rangle_{S_2} \geq 0 \end{aligned} \tag{3.11}$$

for all $V = (v, w)^\top \in K(\Omega)$.

Choose $V = (v, w)^\top \in K(\Omega)$ such that

$$r_{S_2}\{v\}^+ = r_{S_2}\{u\}^+ \quad \text{and} \quad r_{S_2}\{w\}^+ = r_{S_2}\{\omega\}^+ \pm r_{S_2}\psi,$$

where $\psi \in [\tilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Then (3.11) yields

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \quad \text{on } S_2,$$

i.e., (3.3) holds. The conditions (3.2) and (3.4) are satisfied automatically, since $U \in K(\Omega)$. Therefore the relation (3.11) can be rewritten as

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s\}^+ \rangle_{S_2} + \int_{S_2} g(|\{v_s\}^+| - |\{u_s\}^+|) dS \geq 0$$

for all $V = (v, w)^\top \in K(\Omega)$.

Let $\psi \in [\tilde{H}^{1/2}(S_2)]^3$. Since $\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s \rangle_{S_2} = \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi \rangle_{S_2}$ and $|r_{S_2}\psi_s| \leq |r_{S_2}\psi|$, therefore taking $r_{S_2}\psi_s$ in the place of $r_{S_2}\{v_s\}^+$, we obtain

$$\begin{aligned} \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s \rangle_{S_2} + \int_{S_2} g|\psi| dS - \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s \rangle_{S_2} \\ + \int_{S_2} g|\{u_s\}^+| dS \geq 0 \quad \forall \psi \in [\tilde{H}^{1/2}(S_2)]^3. \end{aligned} \tag{3.12}$$

Further, let $t \geq 0$ be an arbitrary number and take $\pm t\psi$ for ψ in (3.12),

$$\begin{aligned} t \left\{ \pm \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s \rangle_{S_2} + \int_{S_2} g|\psi| dS \right\} - \left\{ \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s \rangle_{S_2} + \int_{S_2} g|\{u_s\}^+| dS \right\} \geq 0 \\ \forall \psi \in [\tilde{H}^{1/2}(S_2)]^3 \end{aligned}$$

whence by sending t first to $+\infty$ and then to 0, we easily derive

$$\left| \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\varphi \rangle_{S_2} \right| \leq \int_{S_2} g|\psi| dS \quad \forall \psi \in [\tilde{H}^{1/2}(S_2)]^3, \tag{3.13}$$

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{u_s\}^+ \rangle_{S_2} + \int_{S_2} g|\{u_s\}^+| dS \leq 0. \tag{3.14}$$

Now we prove that $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$. Towards this end, we consider on the space $[\tilde{H}^{1/2}(S_2)]^3$ the linear functional

$$\Phi(\psi) = \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi \rangle_{S_2} \quad \forall \psi \in [\tilde{H}^{1/2}(S_2)]^3.$$

Inequality (3.13) shows that the functional Φ is continuous on the space $r_{S_2}[\tilde{H}^{1/2}(S_2)]^3$ with respect to the topology, induced by the space $[L_1(S_2)]^3$. Since the space $r_{S_2}[\tilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ can be continuously extended to the space $[L_1(S_2)]^3$ preserving the norm. Therefore, by the Riesz theorem, there is the function $\Phi^* \in [L_\infty(S_2)]^3$ such that

$$\Phi(\psi) = \int_{S_2} \Phi^* \cdot \psi dS \quad \forall \psi \in [L_1(S_2)]^3.$$

Thus,

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi \rangle_{S_2} = \int_{S_2} \Phi^* \cdot \psi dS \quad \forall \psi \in [\tilde{H}^{1/2}(S_2)]^3,$$

that is,

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+ - \Phi^*, r_{S_2}\psi \rangle_{S_2} = 0 \quad \text{for all } \psi \in [\tilde{H}^{1/2}(S_2)]^3,$$

which implies that

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ = \Phi^* \in [L_\infty(S_2)]^3.$$

It is well known that for an arbitrary function $\tilde{\psi} \in L_\infty(S_2)$ there is a sequence $\tilde{\psi}_\ell \in C^\infty(S_2)$ with $\text{supp } \tilde{\psi}_\ell \subset S_2$ such that (see, e.g., [39, Lemma 1.4.2])

$$\lim_{\ell \rightarrow \infty} \tilde{\varphi}_\ell(x) = \tilde{\psi}(x) \quad \text{for almost all } x \in S_2$$

and

$$|\tilde{\varphi}_\ell(x)| \leq \text{ess sup}_{y \in S_2} |\tilde{\psi}(y)| \quad \text{for almost all } x \in S_2.$$

Therefore, from inequality (3.13), by the Lebesgue dominated convergence theorem, it follows that

$$\int_{S_2} [\pm \{(\mathcal{T}U)_s\}^+ \cdot \psi - g|\psi|] dS \leq 0 \quad \forall \psi \in [L_\infty(S_2)]^3.$$

In the place of ψ we can put $\chi \psi$, where $\psi \in [L_\infty(S_2)]^3$ and χ is the characteristic function of an arbitrary measurable subset $\Gamma \subset S_2$. As a result, we arrive at the inequality $\pm \{(\mathcal{T}U)_s\}^+ \cdot \psi - g|\psi| \leq 0$ on S_2 for all $\psi \in [L_\infty(S_2)]^3$ and, consequently, by choosing $\psi = \{(\mathcal{T}U)_s\}^+$, we finally get

$$|r_{S_2}\{(\mathcal{T}U)_s\}^+| \leq g \quad \text{on } S_2. \tag{3.15}$$

In view of (3.14) and (3.15), we obtain

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ \cdot r_{S_2}\{u_s\}^+ + g|r_{S_2}\{u_s\}^+| = 0 \quad \text{on } S_2. \tag{3.16}$$

Now, it is evident that if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| < g$, then (3.16) implies $r_{S_2}\{u_s\}^+ = 0$. Also, if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then (3.16) can be rewritten as follows:

$$g|r_{S_2}\{u_s\}^+|(\cos \alpha + 1) = 0 \quad \text{on } S_2,$$

where α is the angle lying between the vectors $r_{S_2}\{(\mathcal{T}U)_s\}^+$ and $r_{S_2}\{u_s\}^+$ at the point $x \in S_2$. Therefore there exist the functions $\lambda_1(x) \geq 0$ and $\lambda_2(x) \geq 0$ such that $\lambda_1(x) + \lambda_2(x) > 0$ and

$$\lambda_1(x)r_{S_2}\{u_s(x)\}^+ = -\lambda_2(x)r_{S_2}\{(\mathcal{T}U)_s\}^+ \quad \text{on } S_2.$$

Moreover, we may assume that λ_1 belongs to the same class as $\{(\mathcal{T}U)_s\}^+$, and λ_2 belongs to the same class as $\{u_s\}^+$.

Thus, the conditions (3.5) and (3.6) of Problem A hold as well, and the proof is complete. \square

4. The existence and uniqueness theorems

Here we investigate the so-called coercive case, where the measure of the Dirichlet part of the boundary is positive, i.e., $\text{meas } S_1 > 0$.

Theorem 4.1. *The variational inequality (3.8) has at most one solution.*

Proof. Let $U = (u, \omega)^\top \in K(\Omega)$ and $U' = (u', \omega')^\top \in K(\Omega)$ be two solutions of the variational inequality (3.8). Then

$$B(U, U' - U) + j(u') - j(u) \geq (\mathcal{G}, U' - U) + \langle \varphi, r_{S_2}\{\omega' - \omega\}^+ \rangle_{S_2}$$

and

$$B(U', U - U') + j(u) - j(u') \geq (\mathcal{G}, U - U') + \langle \varphi, r_{S_2}\{\omega - \omega'\}^+ \rangle_{S_2}.$$

Summing these inequalities and applying the property (2.4), we easily derive that $B(U - U', U - U') = 0$. Therefore $U - U' = ([a \times x] + b, a)$ in Ω , where $a, b \in \mathbb{R}^3$ are arbitrary constant vectors (see Lemma 2.1). Since $r_{S_1}\{U - U'\}^+ = 0$, we conclude that $a = b = 0$, i.e., $U = U'$ in Ω . \square

To prove the existence result, we introduce on the set $K(\Omega)$ the following functional:

$$J(V) = \frac{1}{2} B(V, V) + j(v) - \int_{\Omega} \mathcal{G} \cdot V \, dx - \langle \varphi, r_{S_2} \{w\}^+ \rangle_{S_2} \quad \forall V = (v, w)^{\top} \in K(\Omega). \tag{4.1}$$

Due to the symmetry property of the form $B(U, V)$, it is not difficult to show that the variational inequality (3.8) is equivalent to the minimization problem for the functional (4.1) on the closed convex set $K(\Omega)$, i.e., the variational inequality (3.8) is equivalent to the following minimizing problem:

Find $U_0 \in K(\Omega)$ such that

$$J(U_0) = \inf_{V \in K(\Omega)} J(V).$$

In turn, in accordance with the general theory of variational inequalities (see, e.g., [30,40]), the solvability of the minimization problem immediately follows from the coercivity of the functional J , i.e., from the property

$$J(V) \rightarrow +\infty, \quad \text{when} \quad \|V\|_{[H^1(\Omega)]^6} \rightarrow \infty, \quad V \in K(\Omega). \tag{4.2}$$

Since B is positive and bounded below on $K(\Omega)$, due to (2.6) and the inequality $j(v) \geq 0$, it is easy to see by the trace theorem that

$$J(V) \geq C_1 \|V\|_{[H^1(\Omega)]^6}^2 - C_2 \|V\|_{[H^1(\Omega)]^6},$$

where C_1 and C_2 are some positive constants, independent of V . This inequality shows that the functional (4.1) is coercive on the set $K(\Omega)$. Therefore we have the following existence result for Problem A.

Theorem 4.2. *Let $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $g \in L_{\infty}(S_2)$ and $g \geq 0$. Then Problem A has a unique solution in $[H^1(\Omega)]^6$, depending continuously on the data \mathcal{G} , φ and g of the problem.*

Proof. The unique solvability follows from the equivalence Theorem 3.1, uniqueness Theorem 4.1 and the coercivity property (4.2) (see Theorem 2.1 in [40]).

Further, we establish the continuous dependence of solutions on the data of Problem A.

Let $U = (u, \omega)^{\top} \in [H^1(\Omega)]^6$ and $\tilde{U} = (\tilde{u}, \tilde{\omega})^{\top} \in [H^1(\Omega)]^6$ be two solutions of Problem A, corresponding to the data \mathcal{G} , φ , g and $\tilde{\mathcal{G}}$, $\tilde{\varphi}$, \tilde{g} , respectively. Thus we have two variational inequalities of type (3.8): the first inequality for U and the second one for \tilde{U} . Substituting $V = \tilde{U}$ into the first inequality and $V = U$ into the second one and taking their sum, we obtain

$$\begin{aligned} & -B(U - \tilde{U}, U - \tilde{U}) - \int_{S_2} (g - \tilde{g}) \left(|\{u_s\}^+| - |\{\tilde{u}_s\}^+| \right) dS \\ & - \int_{\Omega} (\mathcal{G} - \tilde{\mathcal{G}}) \cdot (U - \tilde{U}) \, dx - \langle \varphi - \tilde{\varphi}, r_{S_2} \{\omega - \tilde{\omega}\}^+ \rangle_{S_2}. \end{aligned}$$

Taking into account the last inequality, the inclusion $U, \tilde{U} \in K(\Omega)$ and the strong Korn’s inequality (2.6) (see Remark 2.2), we obtain

$$\|U - \tilde{U}\|_{[H^1(\Omega)]^6} \leq C \left(\|g - \tilde{g}\|_{L_2(S_2)} + \|\mathcal{G} - \tilde{\mathcal{G}}\|_{[L_2(\Omega)]^6} + \|\varphi - \tilde{\varphi}\|_{[H^{-1/2}(S_2)]^3} \right)$$

with some positive constant C , not depending on U and \tilde{U} and on the data of the problem under consideration. This estimate shows the desired Lipschitz dependence of the solution on the data of the problem. \square

5. The semicoercive case

Let $S_1 = \emptyset$, $S_2 = S$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S)]^3$, $g \in L_{\infty}(S)$ and $g \geq 0$. Consider the boundary contact problem.

Problem B. Find a vector function $U = (u, \omega)^{\top} \in [H^1(\Omega)]^6$ which is a weak solution of Eq. (3.1), satisfying the inclusion $\{(\mathcal{T}U)_s\}^+ \in [L_{\infty}(S)]^3$ and on the surface S the following boundary conditions:

$$\begin{aligned} & \{\mathcal{M}U\}^+ = \varphi, \quad \{u_n\}^+ = 0, \\ & \text{if } | \{(\mathcal{T}U)_s\}^+ | < g, \quad \text{then } \{u_s\}^+ = 0, \end{aligned}$$

if $|\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1 \{u_s\}^+ = -\lambda_2 \{(\mathcal{T}U)_s\}^+.$$

Analogously to the previous coercive case (see [Theorem 3.1](#)), we can show that [Problem B](#) is equivalent to the following variational inequality:

$$B(U, V - U) + j(v) - j(u) \geq \int_{\Omega} \mathcal{G} \cdot (V - U) dx + \langle \varphi, \{w - \omega\}^+ \rangle_S \tag{5.1}$$

which holds for all $V = (v, w)^\top \in [H^1(\Omega)]^6$. Here

$$j(v) = \int_S g |\{v_s\}^+| dS.$$

Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of the variational inequality (5.1). Substituting first $V = 0$ and then $V = 2U$ into inequality (5.1), we obtain

$$B(U, U) + j(u) = \int_{\Omega} \mathcal{G} \cdot U dx + \langle \varphi, \{\omega\}^+ \rangle_S. \tag{5.2}$$

By virtue of (5.2), from (5.1) we derive

$$B(U, V) + j(v) \geq \int_{\Omega} \mathcal{G} \cdot V dx + \langle \varphi, \{w\}^+ \rangle_S. \tag{5.3}$$

Thus inequality (5.1) is equivalent to the simultaneous relations (5.2) and (5.3). Substituting $-V$ in the place of V in (5.3), we get

$$\left| \int_{\Omega} \mathcal{G} \cdot V dx + \langle \varphi, \{w\}^+ \rangle_S - B(U, V) \right| \leq j(v) \tag{5.4}$$

for all $V = (v, \omega)^\top \in [H^1(\Omega)]^6$.

By \mathcal{R} we denote the set of solutions of the equation $B(U, U) = 0$ in the space $[H^1(\Omega)]^6$ (see [Lemma 2.1](#)),

$$\mathcal{R} := \left\{ \xi = (\rho, a)^\top \in [H^1(\Omega)]^6; \rho = [a \times x] + b, a, b \in \mathbb{R}^3 \right\}.$$

By the substitution of an arbitrary $\xi = (\rho, a)^\top \in \mathcal{R}$ in the place of V in (5.4), we derive the necessary condition of solvability of the variational inequality (5.1),

$$\left| \int_{\Omega} \mathcal{G} \cdot \xi dx + \langle \varphi, a \rangle_S \right| \leq \int_S g |\{\rho_s\}^+| dS \tag{5.5}$$

for all $\xi = (\rho, a)^\top \in \mathcal{R}$.

Let in (5.5) we have the strict inequality. Then taking into account the fact that the space \mathcal{R} has finite dimension, $\dim \mathcal{R} = 6$, it is easy to see that (5.5) is equivalent to the relation

$$\int_S g |\{\rho_s\}^+| dS - \left| \int_{\Omega} \mathcal{G} \cdot \xi dx + \langle \varphi, a \rangle_S \right| \geq C \|\xi\|_{[L_2(\Omega)]^6} \tag{5.6}$$

with some positive constant C , and for all $\xi \in \mathcal{R} \setminus \{0\}$.

Let $\mathcal{P}_{\mathcal{R}}$ be an orthogonal projection operator of the space $[H^1(\Omega)]^6$ on \mathcal{R} , in the sense of the space $[L_2(\Omega)]^6$, i.e., $\forall V \in [H^1(\Omega)]^6 : V = W + \xi$, where $\xi = (\rho, a)^\top = \mathcal{P}_{\mathcal{R}} V \in \mathcal{R}$, and

$$W = (\eta, \zeta)^\top \in \mathcal{R}^\perp := \left\{ U \in [H^1(\Omega)]^6 : \int_{\Omega} U \cdot \xi dx = 0, \forall \xi \in \mathcal{R} \right\}.$$

Due to inequality (2.5) and Lemma 5.1 in [19], the bilinear form B is semicoercive, i.e., there is a positive constant C_0 such that

$$B(V, V) \geq C_0 \|V - \mathcal{P}_{\mathcal{R}} V\|_{[H^1(\Omega)]^6}^2 = C_0 \|W\|_{[H^1(\Omega)]^6}^2 \quad \forall V \in [H^1(\Omega)]^6. \tag{5.7}$$

Therefore, for all $V \in [H^1(\Omega)]^6$, due to (5.6) and (5.7), we have

$$\begin{aligned} J(V) &= J(W + \xi) = \frac{1}{2} B(W, W) + j(\eta + \rho) - j(\rho) \\ &\quad - \int_{\Omega} \mathcal{G} \cdot W \, dx - \langle \varphi, \{\xi\}^+ \rangle_S - \int_{\Omega} \mathcal{G} \cdot \xi \, dx - \langle \varphi, a \rangle_S + j(\rho) \\ &\geq C_0 \|W\|_{[H^1(\Omega)]^6}^2 + C \|\xi\|_{[L_2(\Omega)]^6} - C_1 \|W\|_{[H^1(\Omega)]^6} + j(\eta + \rho) - j(\rho), \end{aligned}$$

for some positive constants C, C_0, C_1 .

Let us now estimate $j(\eta + \rho) - j(\rho)$. We have

$$j(\eta + \rho) - j(\rho) = \int_S g(|\{(\eta + \rho)_s\}^+| - |\{\rho_s\}^+|) dS \geq - \int_S g|\{\eta_s\}^+| dS \geq -C_2 \|W\|_{[H^1(\Omega)]^6},$$

where the positive constant C_2 is independent of η and ρ .

Taking into account this inequality, we finally have

$$J(V) \geq C_0 \|W\|_{[H^1(\Omega)]^6}^2 + C \|\xi\|_{[L_2(\Omega)]^6} - C_3 \|W\|_{[H^1(\Omega)]^6}$$

with some positive constants, whence it follows that

$$J(V) \rightarrow \infty, \quad \text{when } \|V\|_{[H^1(\Omega)]^6} \rightarrow \infty, \quad V \in [H^1(\Omega)]^6$$

i.e., the functional is coercive and the minimization problem for this functional is solvable. Consequently, the corresponding variational inequality (5.1) is solvable, as well (see [30,33]). Further, just as in Theorem 4.1, for the two possible solutions U and U^* to the variational inequality (5.1) of the class $[H^1(\Omega)]^6$, we easily derive $B(U - U^*, U - U^*) = 0$, which implies

$$U - U^* = ([a \times x] + b, a), \quad a, b \in \mathbb{R}^3.$$

Thus we have the following existence and uniqueness

Theorem 5.1. *Let $S_1 = \emptyset, \mathcal{G} \in [L_2(\Omega)]^6, \varphi \in [H^{-1/2}(S)]^3, g \in L_{\infty}(S), g \geq 0$ and the condition (5.6) be fulfilled. Then the variational inequality (5.1) is solvable in the space $[H^1(\Omega)]^6$. Moreover, the solutions are defined modulo generalized rigid displacement vectors.*

Remark 5.2. Analogously to the noncoercive case, we can study the problem, when on a part of the boundary S_1 instead of the Dirichlet condition (3.2) there is assigned the following fractional boundary condition

$$r_{S_1} \{T(\partial, n)U\}^+ = Q,$$

where $Q \in [\tilde{H}^{-1/2}(S_1)]^6$. Moreover, we assume that the vector φ appearing in the condition (3.3) belongs to the space $[\tilde{H}^{-1/2}(S_2)]^3$.

In this case, instead of (3.8) we have the following variational inequality: Find $U = (u, \omega)^{\top} \in [H^1(\Omega)]^6$ such that $\forall V = (v, w)^{\top} \in [H^1(\Omega)]^6$

$$B(U, V - U) + j(v) - j(u) \geq \int_{\Omega} \mathcal{G} \cdot (V - U) dx + \langle r_{S_1} Q, r_{S_1} \{V - U\}^+ \rangle_{S_1} + \langle r_{S_2} \varphi, r_{S_2} \{w - \omega\}^+ \rangle_{S_2}. \tag{5.8}$$

The necessary condition for the solvability of the variational inequality (5.8) reads now as

$$\left| \int_{\Omega} \mathcal{G} \cdot \xi \, dx + \langle r_{S_1} Q, r_{S_1} \{\xi\}^+ \rangle_{S_1} + \langle r_{S_2} \varphi, a \rangle_{S_2} \right| \leq \int_{S_2} g|\{\rho_s\}^+| dS,$$

where $\xi = (\rho, a)^{\top} \in \mathcal{R}$ is an arbitrary generalized rigid displacement vector.

Let us assume that in the necessary condition we have the strict inequality. Since \mathcal{R} is finite-dimensional, we can show that the strict inequality is equivalent to the condition: there is a positive constant C such that the inequality

$$\int_{S_2} g |\{\rho_s\}^+| dS - \left| \int_{\Omega} \mathcal{G} \cdot \xi dx + \langle r_{S_1} Q, r_{S_1} \{\xi\}^+ \rangle_{S_1} + \langle r_{S_2} \varphi, a \rangle_{S_2} \right| \geq C \|\xi\|_{[L_2(\Omega)]^6} \quad (5.9)$$

holds for all $\xi \in \mathcal{R} \setminus \{0\}$. This condition is sufficient for the variational inequality (5.8) to be solvable.

Thus, we have the following existence result.

Theorem 5.3. *Let mes $S_1 > 0$, $\mathcal{G} \in [L_2(\Omega)]^6$, $Q \in [\tilde{H}^{-1/2}(S_1)]^6$, $\varphi \in [\tilde{H}^{-1/2}(S_2)]^3$, $g \in L_\infty(S_2)$, $g \geq 0$ and the condition (5.9) be fulfilled. Then the variational inequality (5.8) is solvable and the solution minimizes the functional*

$$J(V) = \frac{1}{2} B(V, V) + j(V) - \int_{\Omega} \mathcal{G} \cdot V dx - \langle r_{S_1} Q, r_{S_1} \{V\}^+ \rangle_{S_1} - \langle r_{S_2} \varphi, r_{S_2} \{w\}^+ \rangle_{S_2},$$

$$V = (v, w)^\top \in [H^1(\Omega)]^6$$

on the space $[H^1(\Omega)]^6$. Solutions of the variational inequality (5.8) are defined modulo generalized rigid displacement vector.

Remark 5.4. Let the boundary $S = \partial\Omega$ fall into three mutually disjoint portions S_1 , S_T and S_2 such that $\bar{S}_1 \cup \bar{S}_T \cup \bar{S}_2 = S$, $\bar{S}_1 \cap \bar{S}_2 = \emptyset$. Analogously to the coercive case, we can study the problem, when on a part of the boundary S_T there is assigned the traction boundary condition

$$r_{S_T} \{T(\partial, n)U\}^+ = Q,$$

where $Q \in [H^{-1/2}(S_T)]^6$. The conditions on the boundaries S_1 and S_2 in this case remain the same as in Problem A.

In this case we have the following variational inequality:

Find $U = (u, \omega)^\top \in K(\Omega)$ such that $\forall V = (v, w)^\top \in K(\Omega)$,

$$B(U, V - U) + j(v) - j(u) \geq (\mathcal{G}, V - U) + \langle Q, r_{S_T} \{V - U\}^+ \rangle_{S_T} + \langle \varphi, r_{S_2} \{w - \omega\}^+ \rangle_{S_2},$$

where the functional j is defined by formula (3.7).

The proof of the existence and uniqueness theorems for this case can be carried out by repeating word for word the above arguments.

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