ScienceDirect

Journal of Pure and Applied Algebra
Volume 219, Issue 9, September 2015, Pages 4061-4089

The stable category of a left hereditary ring ☆

Alex Martsinkovsky [a], Dali Zangurashvili [b]

Show more

Share Cite

https://doi.org/10.1016/j.jpaa.2015.02.007 Get rights and content

Abstract

The (co)completeness problem for the (projectively) stable module category of an associative ring is studied. (Normal) monomorphisms and (normal) epimorphisms in such a category are characterized. As an application, we give a criterion for the stable category of a left hereditary ring to be abelian. By a structure theorem of Colby–Rutter, this leads to an explicit description of all such rings.

Introduction

Throughout the paper, a ring will always be an associative ring with identity, and a module a left module. Given a ring, the corresponding stable module category has modules as objects, while its morphisms are equivalence classes of module homomorphisms modulo homomorphisms factoring through projectives. This category was introduced by Eckmann and Hilton in the 1950s, with a goal to build an algebraic “prototype” for duality in homotopy theory ([10, Ch. 13]). Soon it found uses in modular representation theory of finite groups and, eventually, in representation theory of artin algebras.

One may think that the stable category is not abelian (and this is sometimes claimed in the folklore), but if Λ is semisimple, then the corresponding stable category consists of zero objects only and is thus trivially abelian. There now arises a natural question of whether there is a nontrivial example of an abelian stable category. Notice that if we are allowed to replace the category of all modules by a subcategory, then it is not difficult to come up with such an example. It is easy to see that the trivial group is the only projective in the abelian category of finite abelian groups, and therefore the corresponding stable category is isomorphic to the original category. Thus, we keep the entire module category in the statement of the problem and assume that the ring Λ is not semisimple.

Our first nontrivial example of an abelian stable category comes from representations of quivers, more precisely, those of $A_2 : 1 \longrightarrow 2$ Let k be a field. The path algebra kA_2 is isomorphic to the algebra of triangular 2×2 matrices with entries in k. Let P_i and S_i denote, respectively, the projective and the simple module corresponding to the vertex $i = 1, 2$. It is well-known that kA_2 is of finite representation type, with only indecomposables being $P_2 = S_2$, P_1, and S_1. Passing to the quotient modulo projectives, we have that, up to isomorphism, S_1 is the only nonzero indecomposable module in the stable category of finitely generated kA_2-modules, making that category equivalent to the category of finite-dimensional k-vector spaces, and hence abelian. To describe the stable category of the entire category of kA_2-modules, we recall ([15, Corollary 4.4]) that, for algebras of finite representation type, each module, finitely generated or not, is a direct sum of indecomposables. It now follows that the stable category of all kA_2-modules is equivalent to the abelian category of all k-vector spaces.

The above example fits a more general pattern. Let A_n, $n \geq 2$ be the equioriented quiver $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n$ By analyzing the Auslander–Reiten quiver of kA_n, one can see that the stable category of finitely generated kA_n-modules is equivalent to the category of finitely generated modules over kA_{n-1}, and is thus abelian. Since kA_n is also of finite representation type, we can use [15, Corollary 4.4] again and deduce that the stable category of all kA_n-modules is equivalent to the abelian category of all kA_{n-1}-modules.

With the motivating examples above, our goal in this paper is to determine when the stable category is abelian. In the case of a left hereditary ring, we give a complete answer. Our main result, Theorem 9.5, says that the stable category of such a ring is abelian if and only if the injective envelope of the ring viewed as a left module over itself is projective. By a structure theorem of Colby–Rutter, these are precisely finite direct products of complete blocked triangular matrix algebras over division rings.

The paper is organized as follows. In Section 2, we set up notation and recall basic facts about stable categories.

In Section 3, we deal with the quotient functor from modules to modules modulo projectives. In particular, we characterize rings over which the quotient functor has left or right adjoints.

In Section 4, we characterize monomorphisms in stable categories. The case of a left hereditary ring is considered in more detail, which leads to seven new characterizations of such rings. As a consequence, we have that the stable category of a left hereditary ring is finitely complete.

Section 5 deals with epimorphisms in stable categories. Theorem 5.7 characterizes epimorphisms in stable categories in terms of null-homotopy of chain maps associated with certain pushouts. For left hereditary rings, we provide yet another criterion for a homomorphism to give rise to an epimorphism in the stable category. For left hereditary rings, Theorem 5.14 gives a new necessary and sufficient condition for a homomorphism to represent an isomorphism in the stable category. Unlike Heller's general criterion, our condition is formulated in terms of submodules, rather than overmodules.

In Section 6, we show that the study of epimorphisms in the stable category reduces, to a large extent, "modulo torsion". Also, at the end of the section, we show that the full subcategory determined by the torsionfree modules is reflective in the stable category of the ring.

In Section 7, we study normal monomorphisms in the stable category. In particular, we show (Theorem 7.4) that all monomorphisms in the stable category of a left hereditary ring are normal if and only if the injective envelope of the ring, viewed as a left module over itself, is projective. As a consequence, the stable category of such a ring is well-powered.

Section 8 deals with normal epimorphisms. Compared with all epimorphisms, they give rise to null-homotopies of chain maps associated with certain additional pushouts (Proposition 8.7). Over left hereditary rings, the existence of such null-homotopies implies the normality (Theorem 8.9). Lemma 8.10 shows that if the injective envelope of a left hereditary ring is projective, then the corresponding stable category is conormal. To determine when the converse is true, we show (Theorem 8.12) that when a left hereditary ring has the DCC on direct summands of itself and has a non-projective injective envelope, then there exists a nonzero projective module with a stable injective envelope; this construction gives rise to a bimorphism in the stable category which is not an isomorphism. At the end of the section, we briefly mention factorization systems, and give a necessary condition for the stable category of a left hereditary ring to admit epi-mono factorizations.

In the last Section 9, we show that if the injective envelope of a left hereditary ring is projective, then the corresponding stable category is cocomplete. This leads to Theorem 9.5, the main result of the paper.

Since this paper straddles the area between module theory and category theory, a substantial effort has been made to present the results in a way accessible to a wide audience. Yet, if the reader needs more background from category theory, we recommend [1], [4], and [12]. For a concise and focused treatment of ring- and module-theoretic concepts, the reader is referred to [2].

The authors thank Kiyoshi Igusa for a helpful comment on the above examples. Special thanks go to Oana Veliche, who carefully read the initial drafts of the paper and helped improve its readability. The second author expresses her sincere gratitude to Alex Martsinkovsky and Oana Veliche for their extraordinary kindness and hospitality during her visit to Northeastern University.

Section snippets

Notation and preliminaries

Blanket assumptions. As we mentioned in the introduction, throughout this paper, a ring means an associative ring with identity, and a module is a left module. The symbol $\ker f$ (respectively, $\mathbf{coker} f$) will denote the kernel (respectively, the cokernel) of the morphism f. The symbol $\mathrm{Ker} f$ (respectively, $\mathrm{Coker} f$) will denote the domain (respectively, the codomain) of the kernel (respectively, of the cokernel) of f. Also, a (co)limit in a category will always mean a small (co)limit.

Given a ring Λ, the …

The quotient functor $\mathcal{Q}$

In this section we give criteria for the existence of adjoints of the quotient functor $\mathcal{Q} : \Lambda\text{-}\mathrm{Mod} \to \Lambda\text{-}\underline{\mathrm{Mod}}$. To describe the class of rings for which the quotient functor has a left adjoint, we first need a preliminary result characterizing rings over which $\mathcal{Q}$ preserves products.

Proposition 3.1

Given a ring Λ, the following are equivalent:
(1) *the quotient functor $\mathcal{Q} : \Lambda\text{-}\mathrm{Mod} \to \Lambda\text{-}\underline{\mathrm{Mod}}$ preserves products;…*

(2) *the direct product of any family of projectives is projective;…*

(3) *Λ is left perfect and right coherent.…*

…

Proof

(1) ⇒ (2). Since $\mathcal{Q}$…

...

Monomorphisms in the stable category

We begin with a simple observation, which is valid over an arbitrary ring and will be used several times.

Lemma 4.1

Let $0 \to A \stackrel{f}{\to} B \stackrel{g}{\to} C \to 0$ *be a split exact sequence. Then:*

(a) $\underline{g}$ *is an isomorphism in the stable category if and only if A is projective.…*

(b) $\underline{f}$ *is an isomorphism in the stable category if and only if C is projective.…*

...

Proof

(a) The above split exact sequence gives rise to a c-exact sequence of functors $0 \to \underline{h}^C \stackrel{\underline{h}^g}{\to} \underline{h}^B \stackrel{\underline{h}^f}{\to} \underline{h}^A \to 0$. By relations (i) and (ii) on page 4064 and Yoneda's lemma, $\underline{g}$ is an isomorphism if and only…

...

Epimorphisms in the stable category

We begin by general remarks about epimorphisms in the stable categories. Their proofs are analogous to those of their counterparts in the previous section, and are therefore omitted.

Lemma 5.1

Let Λ *be a ring and* $f : A \to B$ *a homomorphism of* Λ*-modules. If* $\underline{f}$ *is an epimorphism in the stable category, then* $\mathbf{coker} f$ *factors through a projective. If, in addition, A is projective, then so is B.* □…

Lemma 5.2

Let Λ *be a ring and* $f : A \to B$ *a homomorphism of* Λ*-modules. If B is totally stable or if* Λ *is left hereditary and B is stable,…*

...

Epimorphisms in the stable category and torsion

We continue to assume that Λ is an arbitrary ring and A a Λ-module. Recall that the torsion submodule $t(A)$ is defined as the kernel of the canonical map $e_A : A \to A^{**}$. Equivalently, this is the intersection of the kernels of all linear forms on A.[4]...

Normal monomorphisms in the stable category

Recall that a monomorphism (resp., epimorphism) in a category is said to be normal if it is a kernel (resp., cokernel) of some morphism. For example, in any preadditive category, all split monomorphisms and split epimorphisms are normal.

Lemma 7.1

Let Λ be an arbitrary ring and A a Λ-module. Suppose $A \to E$ is an essential extension such that the corresponding morphism $\underline{p} : E \to E/A$ is a kernel, in the stable category, of $\underline{f} : E/A \to X$ for some $f : E/A \to X$. Then f is a monomorphism....

Proof

Under the assumptions, $\ker f$ lifts over p ...

...

Normal epimorphisms in the stable category

Recall that a weak kernel of a morphism in a pointed category (i.e., in a category with a zero object) is defined as a weak equalizer of that morphism and the zero map, which is in turn defined by removing the uniqueness requirement from the definition of equalizer.

Lemma 8.1

Let Λ be an arbitrary ring, and $f : A \twoheadrightarrow B$ an epimorphism of Λ-modules. Then $\underline{\ker f}$ is a weak kernel of $\underline{f}$ ....

Proof

This is just a reformulation of the fact that the functor $\underline{h}^X$ is half-exact. □...

It is well-known that if an epimorphism is normal...

Main theorem

Recall that a set of objects S of a category **C** is said to be cogenerating if, for any pair of distinct morphisms $h, h' : A \to B$, there is an object X from S and a morphism $s : B \to X$ such that $sh \neq sh'$. As is well-known, if **C** has products, then the equivalent condition is that for any object C there are a set I and a monomorphism $C \rightarrowtail \prod_{i \in I} X_i$, where the X_i are in S. The next result is common knowledge, but for the convenience of the reader, we provide a proof.

Lemma 9.1

Any complete well-powered category **C** *with a…*

…

References (16)

A. Martsinkovsky
1-torsion of finite modules over semiperfect rings
J. Algebra (2010)

P.J. Freyd *et al.*
Categories of continuous functors. I
J. Pure Appl. Algebra (1972)

M. Auslander
Comments on the functor Ext
Topology (1969)

J. Adámek *et al.*
Abstract and Concrete Categories. The Joy of Cats
(1990)

F.W. Anderson *et al.*
Rings and Categories of Modules
(1992)

F. Borceux
Handbook of Categorical Algebra, vol. 1: Basic Category Theory
(1994)

C. Cassidy *et al.*
Reflective subcategories, localizations and factorization systems
J. Aust. Math. Soc. A (1985)

S.U. Chase

Direct products of modules

Trans. Am. Math. Soc. (1960)

There are more references available in the full text version of this article.

Cited by (7)

Injective stabilization of additive functors. II. (Co)torsion and the Auslander-Gruson-Jensen functor

2020, Journal of Algebra

> *Citation Excerpt :*
>
> ...Another obvious choice is the class of all rings whose injective envelope is projective. The (left) hereditary rings with this property have been classified [6, Theorem 3.2]; based on that result, it was later shown that those are precisely the left hereditary rings whose category of left modules modulo projectives is abelian [24, Theorem 9.5]. Image 31, i.e., the torsion functor is isomorphic to the colimit extension of the reject of flats restricted to finitely presented modules....

Show abstract

Injective stabilization of additive functors, I. Preliminaries

2019, Journal of Algebra

> *Citation Excerpt :*
>
> ...But stable categories don't seem to have been studied for their own sake. An attempt at a phenomenological study of categories modulo projectives was recently undertaken in [14]. It then became clear that there were surprisingly tight and unexpected connections between the properties of the ring and the properties of its projectively stable category....

Show abstract

Cokernels in the stable category of a left hereditary ring

2021, arXiv

On the quotient category of the module category of a finite group and its equivalence

2019, Journal of Zhejiang University, Science Edition

Co-Gorenstein Algebras

2019, Applied Categorical Structures

Co-Gorenstein algebras

2018, arXiv

View all citing articles on Scopus

Recommended articles (6)

Research article

Vertices of Lie modules

Journal of Pure and Applied Algebra, Volume 219, Issue 11, 2015, pp. 4816-4839

Show abstract

Research article

A new Frobenius exact structure on the category of complexes

Journal of Pure and Applied Algebra, Volume 219, Issue 7, 2015, pp. 2756-2770

Show abstract

Research article

Conjugate complex homogeneous spaces with non-isomorphic fundamental groups

Comptes Rendus Mathematique, Volume 353, Issue 11, 2015, pp. 1001-1005

Show abstract

Research article

Computing Hasse–Schmidt derivations and Weil restrictions over jets

Journal of Algebra, Volume 411, 2014, pp. 114-128

Show abstract

Research article

Growth of multiplicities of graded families of ideals

Journal of Algebra, Volume 452, 2016, pp. 311-323

Show abstract

Research article

An algebraic approach to computations with progress

Journal of Logical and Algebraic Methods in Programming, Volume 85, Issue 4, 2016, pp. 520-539

Show abstract

☆ The first author was supported by the Collaborative Research Centre 701 “Spectral Structures and Topological Methods in Mathematics” at the University of Bielefeld during his visit in May–July of 2014. He thanks the University of Bielefeld for providing ideal conditions for work. A major part of this paper was prepared during the second author's visit to Northeastern University (USA) in October–November 2012 under the financial support from the Short-Term Individual Travel Grant from Shota Rustaveli National Science Foundation (Ref. 03/109), which she gratefully acknowledges. She also gratefully acknowledges the Research Grant DI/18/5-113/13 from the same foundation and the Volkswagen Foundation Research Grant (Ref. 85 989).

View full text