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ON THE INNOVATION OF CONTINUOUS MULTIDIMENSIONAL SEMIMARTINGALE. III. INFORMATION MODELLING IN FINANCE

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Abstract. The paper is devoted to study of mean-variance hedging problem under the partial information which in our context means the following: the main model of risky asset contains an unobservable random element. The agent remove this nuisance parameter by filtering and as a result he get the process of "new information" so called innovation process. The main goal of the paper consists in construction of optimal hedging strategy under the partial information generated by the innovation process.

რეზიუმე. ნაშრომი ეძღვნება საშუალო-კვადრატული აზრით ჰეჯირების პრობლემას ნაწილობრივ დაკვირვების შემთხვევაში, რაც მოცემულ კონტექსტში ნიშნავს შემდეგს: რისკიანი აქტივის მოდელი შეიცავს დაუკვირვებად შემთხვევით ელემენტს. აგენტი ფილტრაციის მეშვეობით აცილებს სქემას ამ ხელშემშლელ პარამეტრს, რის შედეგადაც იღებს "ახალი ინფორმაციის" პროცესს – განმაახლებელ პროცესს. ნაშრომის მთავარი ამოცანაა გადაიჭრას საშუალო-კვადრატული აზრით ჰეჯირების პრობლემა ახალი ინფორმაციის პირობებში, ე.ი. იმ ინფორმაციაზე დაყრდნობით, რომელსაც შეიცავს განმაახლებელი პროცესი.

1. Introduction

In the paper we continue to study the problem of construction of an innovation process for continuous multidimensional semimartingale with further application to the mean-variance hedging problem of mathematical finance, namely, we consider the latter problem under so called partial information. Many authors studied the mean-variance hedging problem in different schemes: e.g., in cases when the stock prices are observed at discrete time moments, [3], [4], when the restriction on information is more general, but the stock price process is a martingale under the objective probability [5] [6], when the drift coefficient is unobservable [8], etc.

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Our approach concerns slightly different situation. We consider the partially observable financial market when the prices of risky assets of main (observable) component influenced by the some random stochastic factor which is modelled by random element which takes values in some measurable space or by price process of risky assets which is traded on non-observable component of "big" financial market.

In this situation unobservable factors play the role of nuisance parameter and the first what the agent can do is to remove this parameter from the scheme by filtering. After this filtration agent faces the new information which is "free" from unobservable factors and hence more dependable for trading decisions. Mathematically this means that the investor construct the so-called innovation process and as a main information considered the flow of σ-algebras generated by the values of innovation process. But in such situation, generally, the following inclusions are true:

$$F^{\overline{M}} \subset F^S \subset F,$$

where F is initial wide flow of σ-algebras and initially all objects are F-adapted; F^S is a flow generated by the values of stock S itself and finally $F^{\overline{M}}$ is the flow of information generated by the innovation process $\overline{M}$.

The problem is to solve the mean-variance hedging problem under the restriction that the optimal hedging strategy must be $F^{\overline{M}}$-predictable.

Our approach consist in following: we construct the object somewhat less popular (and less investigated) so-called strong innovation process $\overline{M}$ for the process S. The main feature of such process is that "it contains the same information as S", i.e., $F^{\overline{M}} = F^S$. Based on last property we study three schemes of markets: first two leads to the complete market (after "strong filtering", Section 2) and the last is a popular model of stochastic volatility which gives an example of incomplete market (Section 3).

The paper is based on the results obtained in [1], [2].

2. General Preliminaries and Statement of Problem

Consider a financial market M that operates under uncertain conditions describes by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $F = (\mathcal{F}_t)_{0\le t\le T}$, $T > 0$, satisfying the usual conditions and representing the flow of total information on $[0, T]$. For simplicity we assume that $\mathcal{F}_T = \mathcal{F}$.

Market M consists of two interacting components M_1 and M_2.

There are $d_1 + 1$ assets ($d_1 \ge 1$) in the market M_1. The 0-th asset is riskless, equal to 1 at any time. The last d assets could be risky (stocks) and their price process is given by an R^{d_1}-valued F-adapted process $S = (S_1, \dots, S_{d_1})$.

Component M_2 consists of some stochastic factors which influences the stock price process S. The elements of market M_2 may be some random element Y which takes values in the measurable space $(\mathbb{B}, \mathcal{B})$, or d_2-dimensional $(d_2 \geq 1)$ price process $Y = (Y_1, \dots, Y_{d_2})$ of some risky assets given by an R^{d_2}-valued F-adapted process Y, etc.

An investor (agent) functions only in the market M_1: the stocks in the market M_1 only are available to agent for trading and price dynamics of assets traded in the market M_1 are observable. The stochastic factors of market M_2 are unobservable and will be considered as nuisance factors (parameters).

This model of financial market M we call partially observable with observable component M_1 and non-observable component M_2.

Such model seems reasonable, e.g., in following situations:

1) Let Y be the exogenous variable of economics, described by the market model M_1. This variable actions on the price dynamics of assets traded in the market M_1, but itself it is not observable;

2) The market M consists of the open market M_1 and the shadow market M_2. The prices of assets in the open and shadow markets are interacting. The agent functions in the open market M_1;

3) The information about unobservable factors not dependable and will be considered as nuisance parameter, and the investor, who invests in the risky assets S in the market M_1 does not wish to use this distort information to trading decisions. The investor's decisions are based on the dependable information available on the marked M_1.

To be more concrete introduce the models of markets M_1 and M_2.

a) Suppose the price dynamics of risky assets S are governed by the following system of stochastic differential equations (SDEs)

$$\begin{gathered} dS(t) = \operatorname{diag} S(t) \left(\mu(t, S, Y)\, dA_t + \overline{\sigma}(t, S)\, dM(t) \right), \\ S(0) = S^0 \in R_+^{d_1}, \end{gathered} \tag{2.1}$$

where $S = (S_1, \dots, S_{d_1})$, the coefficients $\mu(t, x, y) : [0, T] \times \mathcal{C}^{d_1}_{[0,T]} \times \mathbb{B} \to R^{d_1}$, $\mu = (\mu_1, \dots, \mu_{d_1})$, $\sigma(t, X) : [0, T] \times \mathcal{C}^{d_1}_{[0,T]} \to R^{d_1} \times R^d$, $\sigma = (\sigma_{ij})$, $1 \leq i \leq d_1$, $1 \leq j \leq d$, $d_1 \leq d$, are nonanticipative functionals and $\sigma = (\overline{\sigma}\, \overline{\sigma}')^{\frac{1}{2}}$ is nonsingular matrix, $M = (M_1, \dots, M_d)$ is a d-dimensional continuous martingale, with $\langle M \rangle = I^{d \times d} \cdot A$, where $A = (A_t)_{0 \leq t \leq T}$ is a continuous increasing deterministic function, $A_T < \infty$, $I^{d \times d}$ is unit $d \times d$-dimensional matrix, Y is a random element taking values in some measurable space $(\mathbb{B}, \mathcal{B})$ independent of M, $\operatorname{diag} S(t)$ is a matrix $\begin{pmatrix} S_1(t) & & 0 \\ & \ddots & \\ 0 & & S_{d_1}(t) \end{pmatrix}$.

b) Suppose the price dynamics of stocks S are the same as in a), i.e., governed by the SDEs (2.1) and d_2 stochastic factors $Y = (Y_1, \dots, Y_{d_2})$,

$d_1 + d_2 = d$, follows the SDE

$$dY_t = \widetilde{a}(t, S, Y)\, dA_t + \widetilde{b}(t, S, Y)\, dM(t), \quad Y(0) = Y^0 \in R^{d_2}, \tag{2.2}$$

where the coefficients $\widetilde{a}(t,x,y) : [0,T] \times \mathcal{C}^{d_1}_{[0,T]} \times \mathcal{C}^{d_2}_{[0,T]} \to R^{d_2}$, $\widetilde{b}(t,x,y) : [0,T] \times \mathcal{C}^{d_1}_{[0,T]} \times \mathcal{C}^{d_2}_{[0,T]} \to R^{d_2} \times R^d$, $\widetilde{a} = (\widetilde{a}_1, \dots, \widetilde{a}_{d_2})$, $\widetilde{b} = (\widetilde{b}_{ij})$, $1 \le i \le d_2$, $1 \le j \le d$, are nonanticipative functionals.

c) Suppose the stock price process S is given by the following stochastic volatility model:

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)\left(\mu(t,S,Z,Y)\, dA_t + \overline{\sigma}_1(t,S,Z)\right) dM(t), \\ &\qquad\qquad S(0) = S^0 \in R^n_+, \\ dZ(t) &= K(t,S,Z,Y)\, dA_t + \overline{\sigma}_2(t,S,Z)\, dM(t), \\ &\qquad\qquad Z(0) = Z^0 \in R^m, \end{aligned} \tag{2.3}$$

where the nuisance parameter Y are as in a) or b), i.e., Y is a random element with values in the space $(\mathbb{B}, \mathcal{B})$ independent of the process M, or $Y = (Y_1, \dots, Y_{d_2})$ is a stochastic process governed by the SDE (2.2). Here the coefficients $\mu = (\mu_1, \dots, \mu_n)$, $\overline{\sigma}_1 = (\overline{\sigma}_{1,ij})$, $1 \le i \le n$, $1 \le j \le d$, $K = (K_1, \dots, K_m)$, $\overline{\sigma_2} = (\overline{\sigma}_{2,ij})$, $1 \le i \le m$, $1 \le j \le d$, $m + n = d_1$, $d_1 + d_2 = d$, $\mu(t,x,z,y)$ and $K(t,x,z,y)$ are nonanticipative functionals defined on $[0,T] \times \mathcal{C}^n_{[0,T]} \times \mathcal{C}^m_{[0,T]} \times \mathbb{B}$ (or $\mathcal{C}^{d_2}_{[0,T]}$) and $\sigma_1(t,x,z)$, $\sigma_2(t,x,z)$ are nonanticipative functionals defined on $[0,T] \times \mathcal{C}^n_{[0,T]} \times \mathcal{C}^m_{[0,T]}$, the process $M = (M_1, \dots, M_d)$ and the function A are as in a) and b). Suppose also that $\left(\begin{smallmatrix}\sigma_1\\ \sigma_2\end{smallmatrix}\right) = \left(\left(\begin{smallmatrix}\overline{\sigma}_1\\ \overline{\sigma}_2\end{smallmatrix}\right)\left(\begin{smallmatrix}\overline{\sigma}_1\\ \overline{\sigma}_2\end{smallmatrix}\right)'\right)^{\frac{1}{2}}$ is a nondegenerated $d_1 \times d_1$ matrix.

In all schemes a), b) and c) suppose there exist unique strong solutions of corresponding SDEs (2.1), (2.1)–(2.2), (2.3)–(2.2).

Let us remove the unobservable nuisance parameter Y from the equations (2.2) and (2.3) by the filtration.

We get:

$$dS(t) = \operatorname{diag} S(t)\left(m(t,S)\, dA_t + \sigma(t,S)\, d\overline{M}(t)\right), \quad S(0) = S^0 \in R^{d_1}_+, \tag{2.4}$$

where the innovation process $\overline{M} = (\overline{M}_1, \dots, \overline{M}_{d_1})$ is given by the formula

$$\overline{M}(t) = \int_0^t \sigma^{-1}(u,S)\,)(\operatorname{diag} S(u))^{-1}\left(dS(u) - \operatorname{diag} S(u) m(u,S)\, dA_u\right), \tag{2.5}$$

with $m(t,S) = E\left(\mu(t,S,Y) \mid \mathcal{F}_t^S\right)$ and $\sigma(t,S)=(\overline{\sigma}\,\overline{\sigma}\,')^{\frac{1}{2}}(t,S)$ in the schemes (2.1) or (2.1)–(2.2) and

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)\left(m_1(t,S,Z)\,dA_t + \sigma_1(t,S,Z)\,d\overline{M}(t)\right), \\ &\qquad\qquad\qquad S(0) = S^0 \in R_+^n, \\ dZ(t) &= m_2(t,S,Z)\,dA_t + \sigma_2(t,S,Z)\,d\overline{M}(t), \\ &\qquad\qquad\qquad Z(0) = Z^0 \in R^n, \end{aligned} \tag{2.6}$$

where the innovation process $\overline{M} = (\overline{M}_1 \ldots, \overline{M}_{d_1})$, is given by the relation

$$\overline{M}(t) = \int_0^t \sigma^{-1}(u,S,Z)\left(d\begin{pmatrix} S(u) \\ Z(u) \end{pmatrix} - \right. \\ \left. - \operatorname{diag}\begin{pmatrix} S(u) \\ 1 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}(u,S,Z)\,dA_u\right) \tag{2.7}$$

with

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}(t,S,Z) = E\left(\begin{matrix}\mu \\ K\end{matrix}(t,S,Z,Y) \mid \mathcal{F}_t^{S,Z}\right)$$

and

$$\sigma(t,S,Z) = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}(t,S,Z) = \left(\begin{pmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \end{pmatrix}\begin{pmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \end{pmatrix}'\right)^{\frac{1}{2}}(t,S,Z)$$

in the scheme (2.3)–(2.2).

Note that in the schemes described by (2.4) and (2.5)

$$F^{\overline{M}} \subseteq F^S, \tag{2.8}$$

and in the scheme (2.6)–(2.7)

$$F^{\overline{M}} \subseteq F^{S,Z}. \tag{2.9}$$

Suppose now that in the scheme (2.6)–(2.7) $\sigma_1\sigma_1'$ in continuous in (t,s,z) and for all (t,s) the functional $\sigma_1\sigma_1'(t,s,\cdot)$ is one-to-one from $\mathcal{C}_{[0,T]}^m$ into a subset Σ of the set of $n \times n$-dimensional positive definite matrices, and its inverse functional denoted $\mathcal{L}(t,s,\cdot)$ is continuous in $(t,s,\sigma) \in [0,T]\times\mathcal{C}_{[0,T]}^d \times \Sigma$. See, also [10], [11].

Under these conditions

$$Z(t) = \mathcal{L}(t,S,\sigma_1\sigma_1'(t,S,Z))$$

and

$$\langle S\rangle_t = \int_0^t \sigma_1\sigma_1'(u,S,Z)\,dA_u.$$

From this easily follows that

$$F^Z \subseteq F^S,$$

and, consequently,

$$F^{S,Z} = F^S. \tag{2.10}$$

In both cases the investor removes the "junk" information and passes to the new "clean" information containing in the innovation process $\overline{M}$ which the agent wishes to use for investment decisions.

Hence we get the following information flows

$$F^{\overline{M}} \subseteq F^S \subseteq F, \tag{2.11}$$

where $F^{\overline{M}}$ and F^S are the P-augmented flows of σ-algebras generated by the values of process $\overline{M}$ and S, respectively.

We consider the following investment problem: construct the mean-variance optimal hedging strategies of general contingent claim, under the partial information, i.e., when the investor used an information containing only in $F^{\overline{M}}$.

3. The Mean-Variance Hedging Problem under Restricted Information

Consider the models a) or b) described in the previous Section 2. As it follows from Lemma 5.2 from [2], the model b) can be reduces to the model a).

But to illustrate how the conditions can be verified, we consider the model (2.1), (2.2)

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t)\left(\mu(t,S,Y)\,dA_t + \overline{\sigma}(t,S)\,dM(t)\right), \\ &\qquad\qquad\qquad\qquad S(0) = S^0 \in R_+^{d_1}, \\ dY(t) &= \widetilde{a}(t,S,Y)\,dA_t + \widetilde{b}(t,S,Y)\,dM(t), \quad \eta(0) = \eta^0 \in R^{d_2}, \end{aligned} \tag{3.1}$$

with innovation

$$\overline{M}(t) = \int_0^t \sigma^{-1}(u,S)\,(\operatorname{diag} S(u))^{-1}\left(dS(u) - \operatorname{diag} S(u) m(u,S)\,dA_u\right), \tag{3.2}$$

where the coefficients and the other object are the same as in subsections a) and b) of Section 2.

Let us formulate the mean-variance hedging problem under restricted information. The space of admissible trading strategies $\Theta(F)$ consists of all R^d-valued F-predictable process θ, which are S-integrable, such that $\int_0^T \theta(t)\,dS(t) \in L^2(P,\mathcal{F}_T)$ and the stochastic integral $\int \theta\,dS$ is a Q-martingale under any $Q \in \mathcal{P}(F)$. Here $\mathcal{P}(F) = \{Q \sim P$ on $(\Omega, \mathcal{F}_t)$: $\frac{dQ}{dP}\big|_{\mathcal{F}_T} \in L^2(P, \mathcal{F}_T)$, and S is a Q local martingale$\}$.

It is assumed that there is no arbitrage, i.e., $\mathcal{P}(F) \neq 0$. The process $\theta = (\theta(t))$ for each t represents the number of shares of stocks held at time t, based on information $\mathcal{F}_t$. For a given initial investment $x \in R^1_+$ and trading strategy $\theta \in \Theta(F)$, the self-financed wealth process is defined as $V_t^{x,\theta} = x + \int_0^t \theta_s dS(s)$, $0 \leq t \leq T$. The $\mathcal{F}_T$-measurable r.v. $H \in L^2(P, \mathcal{F}_T)$ models the payoff from financial product at maturity time T. If a hedger starts with the initial investment x and uses the trading strategy θ, the mean-variance hedging problem means to find a trading strategy $\theta^{*,F}(x)$ solution of

$$\mathcal{J}_F(x) = \min_{\theta \in \Theta(F)} E(H - V_T^{x,\theta})^2.$$

In our situation, i.e., under the innovation information restriction we have

$$F^{\overline{M}} \subset F^S \subset F$$

and our decisions (trading strategy or portfolio) must be based on information flow $F^{\overline{M}}$, i.e., θ must be $F^{\overline{M}}$-predictable process. Denoting by $g = (g(t, X, Y))$ any of coefficients of SDE (3.1), suppose that

1) $|g(t, X, Y)| \leq const$, $\forall (t, x, y) \in [0, T] \times \mathcal{C}^{d_2}_{[0,T]} \times \mathbb{B}$,

2) Introduce the following stopping time (for each $N = 1, 2, \dots$)

$$\tau_N(x^1, x^2) := \inf \left\{ t : t > 0, \sup_{0 \leq s \leq t} \max\left(|x^1(s)|^2, |x^2(s)|^2\right) > N \right\},$$

where $\inf\{\varnothing\} = +\infty$, $|\cdot|$ is a norm in R^{d_1}, and the set $\mathbb{D}_N$, where

$$\mathbb{D}_N = \left\{ (t, x^1, x^2) \in [0, T] \times \mathcal{C}^{d_1}_{[0,T]} \times \mathbb{B} : 0 < t < \tau_N(x^1, x^2) \right\},$$

and suppose that on the set $\mathbb{D}_N$

$$\left| g(t, x^2, y^2) - g(t, x^2, y^2) \right|^2 \leq \text{const}_N \left(\left(\|x^1 - x^2\|_t^{d_1} \right)^2 + \left(\|y^1 - y^2\|_t^{d_2} \right)^2 \right),$$

where $\|x\|_t^l = \sup_{0 \leq s \leq t} |x(s)|_l$, $|\cdot|_l$ is a norm in R^l, where $l = d_1$ or d_2.

3) The matrix $\sigma = (\overline{\sigma}\,\overline{\sigma}\,')$ is uniformly elliptic.

Under these conditions there exists unique strong solution (S, Y) of the system of SDEs (3.1). Indeed, introduce the process $X_t = \ln S(t)$ which is well defined since under conditions 1)–3) $\inf_{0 \leq t \leq T} S(t) > 0$, P-a.s. Then if we use the Itô formula we easily arrive at the scheme (5.1) of [2] with

$$\begin{aligned} \varphi(t, x, y) &= \mu(t, e^x, y) - \frac{1}{2}\, \vec{dg}(\overline{\sigma}\,\overline{\sigma}\,')(t, e^x), \\ \sigma_1(t, x) &= \overline{\sigma}(t, e^x), \\ \psi(t, x, y) &= \widetilde{a}(t, e^x, y), \end{aligned}$$

$$\sigma_2(t,x,y) = \widetilde{b}(t,e^x,y),$$

where $\vec{dg}\,r$ is the vector of diagonal elements $(\gamma_{11},\dots,\gamma_{d_1d_1})$ of the matrix $\Gamma = \|\gamma_{ij}\|$, $i,j=\overline{1,d_1}$.

Note now that using elementary inequality

$$|e^x - e^y| \le \frac{e^x+e^y}{2}\,|x-y|, \quad x,y\in R_1,$$

and the Lipschitz condition we easily get that the new coefficients φ, σ_1, ψ, σ_2 satisfy the local Lipschitz condition in variable x and the global one in variable y. All coefficients are bounded and $\sigma_1\sigma_1'$ is uniform elliptic. Thus using Theorem 5.2 of [2] we arrive at the following

Theorem 3.1. *Under conditions* 1), 2) *and* 3) *the SDE*

$$dS(t) = \operatorname{diag} S(t)\left(m(t,S)\,dA_t + \sigma(t,S)\,d\overline{M}(t)\right), \quad S(0)=S^0\in R_+^{d_1}, \tag{3.3}$$

has a unique strong solution.

Remark 3.1. We start by the system of the SDEs (3.1) with a given coefficients μ, $\overline{\sigma}$, $\widetilde{a}$ and $\widetilde{b}$ and given driving process $M=(M(t))$. Under the conditions mentioned above this system has strong solution (S,Y). By the formula (3.2), we then constructed the new process $\overline{M}=(\overline{M}(t))$. Hence the right hand side of (3.2) is nothing more than the definition of the left hand side process $\overline{M}$.

By Theorem 3.1 we initially fixed the coefficients $m(t,\cdot)$ and $\sigma(t,\cdot)$ and driving process $\overline{M}=\overline{M}(t)$ with independent increments and uncorrelated components and construct the process $S(t)=F(t,\overline{M})$ as a substitution into nonanticipative functional F of a given process $\overline{M}$. Hence by (3.2) $F^{\overline{M}}\subset F^S$, but by (3.3) $F^S\subset F^{\overline{M}}$. Thus $F^{\overline{M}}=F^S$.

Now return to our L^2-hedging problem.

In fact we are in the complete market framework.

Denote $(P,F^{\overline{M}})$-projection of contingent claim $H\in L^2(P,\mathcal{F}_T)$ by

$$H_{F^{\overline{M}}} := E\left(H \mid \mathcal{F}_T^{\overline{M}}\right).$$

The problem of mean-variance hedging under $F^{\overline{M}}$-information, given an initial wealth $x\in R^1$, consists in finding a trading strategy $\theta^{*,F^{\overline{M}}}(x)$ solution of

$$\mathcal{J}_{F^{\overline{M}}}(x) = \min_{\theta\in\Theta(F^{\overline{M}})} E\left[H - V_T^{x,\theta}\right]^2.$$

The solution $x_{F^M}(H)$ of

$$\mathcal{J}_{F^{\overline{M}}} = \min_{x\in R_+^1} \mathcal{J}_{F_{\overline{M}}}(x)$$

is called $F^{\overline{M}}$-approximating price of H. This problem for the case when full information F is available for the agent is solved in [7], [9]; if information F^S containing only in the stock S is available for investment decisions the solution of the problem is given in [8].

Note that

$$\begin{aligned}\mathcal{J}_{F^{\overline{M}}}(x) &= \min_{\theta\in\Theta(F^{\overline{M}})} E\left(H - H_{F^{\overline{M}}} + H_{F^{\overline{M}}} - V_T^{x,\theta}\right)^2 = \\ &= \min_{\theta\in\Theta(F^{\overline{M}})} \Big[E(H - H_{F^{\overline{M}}})^2 + E(H_{F^{\overline{M}}} - V_T^{x,\theta})^2 + \\ &\quad + E\left(E\left((H - H_{F^{\overline{M}}})(H_{F^{\overline{M}}} - V_T^{x,\theta}) \mid \mathcal{F}_T^{\overline{M}}\right)\right)\Big].\end{aligned}$$

But

$$\begin{aligned}E\left((H - H_{F^{\overline{M}}})(H_{F^{\overline{M}}} - V_T^{x,\theta}) \mid \mathcal{F}_T^{\overline{M}}\right) &= \\ = \left(H_{F^{\overline{M}}} - V_T^{x,\theta}\right) E\left(H - H_{F^{\overline{M}}} \mid \mathcal{F}_T^{\overline{M}}\right) &= 0,\end{aligned}$$

by the definition of $H_{F^{\overline{M}}}$ and $\mathcal{F}_T$-measurability of $V_T^{x,\theta}$ for all $\theta \in \Theta(F^{\overline{M}})$.

Hence

$$\mathcal{J}_{F^{\overline{M}}}(x) = E(H - H_{F^{\overline{M}}})^2 + \widehat{\mathcal{J}}_{F^{\overline{M}}}(x),$$

with

$$\widetilde{\mathcal{J}}_{F^{\overline{M}}}(x) = \min_{\theta\in\Theta(F^{\overline{M}})} E\left(H_{F^{\overline{M}}} - V_T^{x,\theta}\right)^2.$$

Denote $\widetilde{P}$ the variance-optimal equivalent local martingale measure (ELMM). It is unique element of $\mathcal{P}(F^{\overline{M}}) \neq \varnothing$ (the standing assumption) which minimizes $\left\|\frac{dQ}{dP}\right\|_{L^2(P)}$ over all $Q \in \mathcal{P}(F^{\overline{M}})$.

It is well-known that

$$\widetilde{Z}_t := \widetilde{E}\left[\frac{d\widetilde{P}}{dP}\Big|\mathcal{F}_t\right] = \widetilde{Z}_0 + \int_0^t \widetilde{\zeta}_u \, dS(u)$$

for some $\widetilde{\zeta} \in \Theta(F^{\overline{M}})$.

The following statements give us the solution of mean-variance problem.

Let $H \in L^2(P)$; calculate $H_{F^{\overline{M}}}$ and write the Galtchouk–Kivita–Watanabe decomposition of $H_{F^{\overline{M}}}$ under measure $\widetilde{P}$ with respect to S, i.e.

$$H_{F^{\overline{M}}} = \widetilde{E}[H_{F^{\overline{M}}}] + \int_0^T \xi_u^{H_{F^{\overline{M}}},\widetilde{P}} \, dS(u) + L_T^{H_{F^{\overline{M}}},\widetilde{P}} = V_T^{H_{F^{\overline{M}}},\widetilde{P}},$$

with

$$V_t^{H_{F^{\overline{M}}},\widetilde{P}} = \widetilde{E}\left(H_{F_{\overline{M}}} \mid \mathcal{F}_t^{\overline{M}}\right) =$$

$$= \widetilde{E}H_{F^{\overline{M}}} + \int_0^t \xi_u^{H_{F^{\overline{M}}},\widetilde{P}}\, dS(u) + L_t^{H_{F^{\overline{M}}},\widetilde{P}},$$

$$0 \le t \le T.$$

Then optimal initial investment (approximating price) is given by the formula

$$x_{F^{\overline{M}}}(H) = \widetilde{E}H_{F^{\overline{M}}} \tag{3.4}$$

and optimal trading strategy is given by the relation

$$\theta_t^{*,F^{\overline{M}}} = \xi_t^{H_{F^{\overline{M}}},\widetilde{P}} - \frac{\widetilde{\zeta}_t}{\widetilde{Z}_t}\left(V_t^{H_{F^{\overline{M}}},\widetilde{P}} - \widetilde{E}H_{F^{\overline{M}}} - \int_0^t \theta_u^{*,F^{\overline{M}}}\, dS(u)\right). \tag{3.5}$$

The minimal total risk of $H_{F^{\overline{M}}}$ is given by the formula

$$R^* = E\left(H_{F^{\overline{M}}} - V_T^{x_{F^{\overline{M}}}(H),\theta^*,F^{\overline{M}}}\right)^2 = E\left[\int_0^T \frac{Z_u^{\widetilde{P}}}{\widetilde{Z}_u}\,[L^{H_{F^{\overline{M}}},\widetilde{P}}]_u\right],$$

with $Z_t^{\widetilde{P}} = E\left(\frac{d\widetilde{P}}{dP}|\mathcal{F}_t^{\overline{M}}\right)$, $0 \le t \le T$.

Denote

$$\lambda(t,S) = \sigma(t,S)^{-1}m(t,S)$$

and introduce the measure P^* on the σ-algebra $\mathcal{F}_T^{\overline{M}}$ by the formula

$$dP^* = \mathcal{E}_T(-\lambda\cdot\overline{M})\,dP,$$

i.e.,

$$\frac{dP^* \mid \mathcal{F}_T^{\overline{M}}}{dP \mid \mathcal{F}_T^{\overline{M}}} = \exp\left(-\int_0^T \lambda(t,S)\,d\overline{M}(t) - \frac{1}{2}\int_0^T |\lambda(t,S)|^2\,dA_t\right).$$

From the boundedness of λ follows that $E_T\mathcal{E}(-\lambda\cdot\overline{M}) = 1$. Hence $P^* \sim P$ and the set

$$\mathcal{P}(F^{\overline{M}}) = \left\{Q \sim P \text{ on } \mathcal{F}_T^{\overline{M}},\ \frac{dQ}{dP}\Big|_{\mathcal{F}_T^{\overline{M}}} \in L^2(P,F^{\overline{M}}),\right.$$
$$\left.\text{and } S \text{ is a } Q\text{-local martingale}\right\} = \{P^*\},$$

since $\mathcal{P}(F^{\overline{M}})$ is nonempty if and only if $E(\mathcal{E}_T(-\lambda\cdot\overline{M}))^2 < \infty$ which is satisfies in considered case.

Since $H \in L^2(P,\mathcal{F}_T)$, $H_{F^{\overline{M}}} = E(H \mid \mathcal{F}_T^{\overline{M}}) \in L^2(P,\mathcal{F}_T^{\overline{M}})$.

Define $(P^*,F^{\overline{M}})$ martingale

$$V_t^{\overline{M}} = E^{P^*}(H_{F^{\overline{M}}} \mid \mathcal{F}_t^{\overline{M}}), \quad 0 \le t \le T.$$

By the martingale representation property we deduce

$$V_t^{\overline{M}} = E^{P^*}(H_{F^{\overline{M}}}) + \int_0^t \eta_t' \, d\overline{M}(t), \quad 0 \le t \le T, \tag{3.6}$$

with $F^{\overline{M}}$-adapted, R^{d_1}-valued process $\eta = (\eta_t)$, $\int_0^T |\eta_t|^2 dA_t < \infty$.

Introducing the process $\zeta^{\overline{M}} = (\sigma(t,S))^{-1}\eta_t$ we rewrite (3.6) in the form

$$V_t^{\overline{M}} = E^{P^*} H_{F^{\overline{M}}} + \int_0^t \zeta_u^{\overline{M}} \, dS(u), \quad 0 \le t \le T. \tag{3.7}$$

Note now that $\int \zeta^{\overline{M}} dS$ is a $(P^*, F^{\overline{M}})$ martingale and $\zeta^{\overline{M}} \in \Theta(F^{\overline{M}})$ since $\int_0^T \zeta_u^{\overline{M}} dS(u) = H_{F^{\overline{M}}} - E^{P^*}(H_{F^{\overline{M}}}) \in L^2(P, F^{\overline{M}})$. Considering (3.7) at the moment $t = T$ we get the Galtchouk–Kunita–Watenabe decomposition. This is the key point. From this decomposition by the usual way (using (3.4) and (3.5)) we get that

$$\mathcal{J}_{F^{\overline{M}}}(x) = \frac{(E^{P^*} H_{F^{\overline{M}}} - x)^2}{E^2 \mathcal{E}_T(-\lambda \cdot \overline{M})} + E(H - H_{F^{\overline{M}}})^2 \tag{3.8}$$

and

$$x_{F^{\overline{M}}}(H) = E^{P^*}[H_{F^{\overline{M}}}]. \tag{3.9}$$

Thus we prove the following

Theorem 3.2. *In the schemes* (2.2)–(2.3) *under conditions* 1), 2) *and* 3) *the minimal risk and approximative price of general contingent claim H under the restricted information, when the flow $F^{\overline{M}}$ is available to hedger for investment decisions is given by the formulas* (3.8) *and* (3.9). *The optimal hedging strategy can be find by the formula* (3.5).

Consider now the scheme described in subsection c) of Section 2.2. Denoting $g = (g(t,x,z,y))$ any of coefficients of scheme (2.3)–(2.2) suppose g satisfies the conditions 1), 2) and 3) (i.e., in 3) it is assumed that the matrix $\sigma^2 = \sigma_1\sigma_1'$ with $\sigma_1 = \binom{\overline{\sigma}_1}{\overline{\sigma}_2}$ is uniformly elliptic) of Section 3. Then there exists a unique strong solution (S, Z, Y) of system (2.2)–(2.3). Introduce the process $X = (X^1, X^2) = (\ln S, Z)$. Then using the Itô formula we easily see that the process (X, Y) satisfies system (5.1) from [2] with

$$\varphi(t,x,y) = \left(\mu(t, e^{x^1}, z, y) - \frac{1}{2}\,\vec{dg}(\overline{\sigma}_1\overline{\sigma}_1')(t, e^{x^1}, z), K(t, e^{x^1}, z, y)\right),$$

$$\sigma_1(t,x) = \left(\overline{\sigma}_1(t, e^{x^1}, z), \overline{\sigma}_2(t, e^{x^1}, z)\right),$$

$$\psi(t,x,y) = \widetilde{a}(t, e^{x^1}, y), \quad \sigma_2(t,x,y) = \widetilde{b}(t, e^{x^1}, y),$$

where $x = (x^1, z)$, x^1 is an n-dimensional and z is an m-dimensional vectors.

It is easy to verify that all coefficients are bounded, the Lipschitz continuous (locally in x and globally in y) and matrix $\sigma_1\sigma_1'$ is uniformly elliptic.

Hence using Theorem 5.2 we get the following

Theorem 3.3. *Under the above-mentioned conditions* 1)–3) *the system of SDE* (2.6) *has a unique strong solutions and hence there exists strong information process* (*given by* (2.7) *with*

$$F^{S,Z} = F^{\overline{M}}.$$

Now rewriting this system in triangle form we finally get

$$\begin{aligned} dS(t) &= \operatorname{diag} S(t) \left(m_1(t,S,Z)\, dA_t + \sigma(t,S,Z)\, d\overline{N}(t) \right), \\ &\qquad S(0) = S^0, \\ dZ(t) &= m_2(t,S,Z)\, dA_t + \rho(t,S,Z)\, d\overline{N}(t) + \gamma(t,S,Z)\, d\overline{M}(t), \\ &\qquad Z(0) = Z^0, \end{aligned} \tag{3.10}$$

where $\overline{N}$ and $\overline{M}$ are mutually independent n and m-dimensional processes with independent increments and independent components. Under the additional technical assumption (see Section 2, formula (2.10)) $F^{S,Z} = F^S$ and, finally,

$$F^{\overline{N},\overline{M}} = F^S.$$

We may now use the method described earlier in this section to get the full solution of mean-variance hedging problem in the case of partial information (i.e., when information $F^{\overline{M}}$ is only available for agent for trading decisions) for the scheme (3.10) as well.

4. The Markov Dilation of the Path Dependent Processes and the "Explicit" Solution of the Mean-Variance Hedging Problem under Partial Information

In this section we consider the scheme described by the systems of SDEs (2.3)–(2.2).

As it is well-known, when the processes describing market prices dynamics have the Markov property, then the "explicit" solution (describing in the terms of solutions of appropriate differential equations) of the mean-variance hedging problem may be constructed.

But the feature of our methods given in this paper is such that even if the initial scheme (2.3)–(2.2) has a Markov property, after the construction of the innovation process we necessarily arrived at the scheme (3.10) with path dependent coefficients of equations.

Hence to overcome this problem we used the so-called Markov dilation method, which is based on the following observation: if X is some continuous stochastic process with the sample paths from $\mathcal{C}^d_{[0,T]}$, $d \geq 1$, then the process X^t, $0 \leq t \leq T$, defined for each $t \in [0,T]$ by the relation

$$X^t = (X_{t\wedge s}, \ \ s \in [0,T])$$

has a Markov property, i.e., for any Borel set $B \in \mathcal{B}(\mathcal{C}^d_{[0,T]})$

$$\text{Prob}\left(X^t \in B \mid X^{t_1}, X^{t_1}, \dots, X^{t_n}\right) = \text{Prob}\left(X^t \in B \mid X^{t_n}\right), \quad 0 < t_1 < \dots < t_n < T,$$

since the σ-algebra $\sigma(X^t) = \sigma(X_s, \ s \leq t)$ increases as t increases.

Now consider the scheme (3.10) and suppose that $\rho(t,s,z) \equiv 0$, and the coefficients of system depend only on the second component Z of the process (S,Z). For simplicity consider the case when the both processes S and Z are one-dimensional, $\overline{N}$ and $\overline{M}$ are Wiener processes, hence $dA_t = dt$. We assume also that the coefficients of initial scheme (2.3)–(2.2) satisfy the conditions 1)–3) introduced in Section 3, hence $F^S = F^{S,Z} = F^{\overline{N},\overline{M}}$.

Thus we consider the following scheme: the process (S,Z) is a strong solution (thanks to Theorem 3.3) of the SDE

$$\begin{aligned} &(1)\ \ \frac{dS(t)}{S(t)} = m_1(t,Z)\,dt + \sigma(t,Z)\,d\overline{N}(t), \quad S(0) = S^0 \in R^1_+, \\ &(2)\ \ dZ(t) = m_2(t,Z)\,dt + \gamma(t,Z)\,d\overline{M}(t), \quad Z(0) = Z^0 \in R^1, \end{aligned} \tag{4.1}$$

where the coefficients m_1, σ, m_2, γ : $[0,T] \times \mathcal{C}_{[0,T]} \to R^1$ are bounded, continuous, nonanticipative functionals and, in addition, the SDE (4.1) (2) has a unique strong solution Z, and hence $F^Z = F^{\overline{M}}$.

Remember two main steps of solution of mean-variance hedging problem (in particular, under the restricted information):

1. determine the variance-optimal ELMM $\widetilde{P}$ and find the dynamics of (S,Z) under $\widetilde{P}$;

2. find the Galtchouk–Kivita–Watanabe decomposition of contingent claim $H_{F^{\overline{N},\overline{M}}} = E(H \mid \mathcal{F}^{\overline{N},\overline{M}}_T)$ w.r.t. S under $\widetilde{P}$ (see Section 3).

We need the following auxiliary results.

Lemma 4.1. *Let $\xi^{t,\varphi}$ be the solution of the SDE*

$$\begin{aligned} \xi^{t,\varphi}_s &= \varphi_s + \int_t^s m_2(u,\xi^{t,\varphi})\,du + \int_t^s \gamma(u,\xi^{t,\varphi})\,d\overline{M}(u), \quad s > t, \\ \xi^{t,\varphi}_s &= \varphi_s, \quad s < t. \end{aligned} \tag{4.2}$$

Denote

$$\xi = \xi^{0,c} \quad \text{and} \quad \xi^t = \xi(t \wedge \cdot),$$

where c is a constant.

Suppose m_2, $\gamma : [0,T]\times \mathcal{C}_{[0,T]} \to R^1$ are nonanticipative functionals such that (4.2) has a unique strong solution.

Then

$$\xi^{t,\xi^t} = \xi \quad P\text{-a.s. for all } t \in [0,T].$$

Proof. If $s < t$, then $\xi_s^{t,\xi^t} = \xi_s$ by definition. If $s > t$, then

$$\xi_s^{t,\xi^t} = \xi(t\wedge s) + \int_t^s m_2(u,\xi^{t,\xi^t})\,du + \int_t^s \gamma(u,\xi^{t,\xi^t})\,d\overline{M}(u) =$$
$$= c + \int_0^t m_2(u,\xi)\,du + \int_0^t \gamma(u,\xi)\,d\overline{M}(u) + \int_t^s m_2(u,\xi^{t,\xi^t})\,du +$$
$$+ \int_t^s \gamma(u,\xi^{t,\xi^t})\,d\overline{M}(u).$$

Since ξ substituted instead of ξ^{t,ξ^t} also satisfies the latter equation by unicity of solution, we obtain that $\xi_s^{t,\xi^t} = \xi_s$, $s > t$. Lemma is proved. □

Proposition 4.1. *Let $C : [0,T]\times\mathcal{C}_{[0,T]} \to R^1$ be nonanticipative bounded continuous functional and $g : \mathcal{C}_{[0,T]} \to R$ be founded continuous function.*

Then the martingale

$$M_t^\xi = E\Big[g(\xi)e^{\int_0^T C(s,\xi)\,ds} \mid \mathcal{F}_t^\xi\Big]$$

can be represented as

$$v(t,\xi^t)e^{\int_0^t C(s,\xi)\,ds},$$

where $v(t,\varphi) = Eg(\xi^{t,\varphi})e^{\int_t^T C(u,\xi^{t,\varphi})\,du}$.

Proof. It is easy to see that

$$M_t^\xi = e^{\int_0^t C(u,\xi)\,du} E\Big[g(\xi)e^{\int_t^T C(u,\xi)\,du} \mid \mathcal{F}_t^\xi\Big] \equiv e^{\int_0^t C(u,\xi)\,du} V_t.$$

Using the fact that $\mathcal{F}^\xi = \sigma(\xi^t)$ and Lemma 4.1 we get

$$V_t = E\Big[g(\xi^{t,\xi^t})e^{\int_t^T C(u,\xi^{t,\xi^t})\,du} \mid \sigma(\xi^t)\Big] =$$
$$= Eg(\xi^{t,u})e^{\int_t^T C(u,\xi^{t,\varphi})\,du}\Big|_{\varphi=\xi^t} = v(t,\xi^t).$$

Proposition is proved. □

It is well-known that the variance-optimal ELMM $\widetilde{P}$ for the system (4.1) has the density of the form

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}_T\Big(-\int_0^\cdot \theta(s,Z)\,d\overline{N}(s) - \int_0^\cdot v_s\,d\overline{M}(s)\Big),$$

where $\theta = \frac{m_1}{\sigma}$ and $V_t = \frac{g_t}{c+\int_0^t g_s\, d\overline{M}(s)}$, with

$$c + \int_0^t g_s\, d\overline{M}(s) = E\Big[e^{-\int_0^T \theta^2(s,Z)\, ds} \mid \mathcal{F}_t^Z\Big].$$

Prove now the following

Theorem 4.1. *Let $z^{t,\varphi}$ be the solution of the following SDE*

$$z_s^{t,\varphi} = \varphi_s + \int_t^s m_2(u, z^{t,\varphi})\, du + \int_t^s \gamma(u, z^{t,\varphi})\, d\overline{M}(u), \quad s > t,$$

$$z_s^{t,\varphi} = \varphi_s, \quad s < t.$$

Then

$$E\Big[e^{-\int_0^T \theta^2(s,Z)\, ds} \mid \mathcal{F}_t^Z\Big] = v(t, Z(t\wedge\cdot))e^{-\int_0^T \theta^2(s,Z)\, ds}, \tag{4.3}$$

where

$$v(t,\varphi) = Ee^{-\int_0^T \theta^2(s, z^{t,\varphi})\, ds}, \quad (t,\varphi) \in [0,T] \times \mathcal{C}_{[0,T]}.$$

Moreover, if $\theta(t,\varphi)$, $m_2(t,\varphi)$, $\gamma(t,\varphi)$ are twice Frechet differentiable, then $v(t, Z(t\wedge\cdot))$ admits the Itô decomposition with martingale part

$$\int_0^t \gamma(s, Z(s\wedge\cdot))\, \frac{\partial v(s, Z(s\wedge\cdot))}{\partial j_s}\, d\overline{M}(s),$$

where $j_s = 1_{[s,T]}$.

In this case v can be found by the relation

$$v_t = \gamma(t,Z)\, \frac{\partial v(t, Z(t\wedge\cdot))}{\partial j_t} \Big/ v(t, Z(t\wedge\cdot)),$$

or equivalently

$$v_t = \gamma(t,Z)\, \frac{\partial q}{\partial j_t}\, (t, Z(t\wedge\cdot)),$$

where

$$q(t,\varphi) = \ln v(t,\varphi).$$

Proof. The formula (4.3) is the simple consequence of Proposition 4.1.

Further, by the Itô formula we have

$$e^{-\int_0^t \theta^2(s,Z)\, ds} v(t, Z(t\wedge\cdot)) =$$

$$= c + \int_0^t e^{-\int_0^s \theta^2(u,Z)\, du} \gamma(s,Z)\, \frac{\partial v(s, Z(s\wedge\cdot))}{\partial j_s}\, d\overline{M}(s).$$

Since, see [12], [13],

$$Z(t\wedge\cdot)=Z(0)+\int_0^t m_2(s,Z)j_s ds+\int_0^t \gamma(s,Z)j_s d\overline{M}(s),$$

the Itô formula for $v(t,Z(t\wedge\cdot))$ is of the form

$$v(t,Z(t\wedge\cdot))=v(0,c)+\int_0^t \gamma(s,Z)\,\frac{\partial v}{\partial j_s}\,(s,Z(s\wedge\cdot))\,d\overline{M}(s)+$$

$$+\int_0^t \bigg[\frac{\partial v}{\partial s}\,(s,Z(s\wedge\cdot))+m_2(s,Z)\,\frac{\partial v}{\partial j_s}\,(s,Z(s\wedge\cdot))+$$

$$+\frac{1}{2}\,\gamma(s,Z(s\wedge\cdot))\,\frac{\partial^2 v}{\partial j_s^2}\,(s,Z(s\wedge\cdot))\bigg]\,ds.$$

Therefore

$$v_t=\frac{g_t}{c+\int_0^t g_s\,d\overline{M}(s)}=\frac{e^{-\int_0^t \theta^2(u,Z)\,du}\gamma(t,Z)\,\frac{\partial v}{\partial j_t}\,(t,Z(t\wedge\cdot))}{e^{-\int_0^t \theta^2(u,Z)\,du}v(t,Z(t\wedge\cdot))}=$$

$$=\gamma(t,Z)\,\frac{\partial q}{\partial j_t}\,(t,Z(t\wedge\cdot)).$$

Theorem is proved. □

Now for the variance-optimal ELMM $\widetilde{P}$ we can write

$$\frac{d\widetilde{P}}{dP}=\mathcal{E}_T\bigg(-\int_0^{\cdot} \theta(s,Z)\,d\overline{N}(s)-\int_0^{\cdot} v_s\,d\overline{M}(s)\bigg),$$

with

$$v_t=\gamma(t,Z)\,\frac{\partial q(t,Z^t)}{\partial j_t},$$

$$q(t,\varphi)=\ln Ee^{-\int_0^t \theta^2(s,z^{t,\varphi})ds},\quad Z^t=Z(t\wedge\cdot).$$

Using the Girsanov theorem we can define the Wiener processes w.r.t measure $\widetilde{P}$

$$\widetilde{N}_t=\overline{N}(t)+\int_0^t \theta(s,Z)\,ds \quad\text{and}\quad \widetilde{M}_t=\overline{M}(t)+\int_0^t v_s\,ds.$$

Hence w.r.t. measure $\widetilde{P}$ the price process dynamics of the process (S,Z) is given by the SDE

$$\frac{dS(t)}{S(t)}=\sigma(t,Z)\,d\widetilde{N}_t,$$

$$dZ_t = \widetilde{m}(t,Z)\,dt + \gamma(t,Z)\,d\widetilde{M}_t,$$

with

$$\widetilde{m}(t,Z) = m_2(t,Z) - \gamma^2(t,Z)\,\frac{\partial q}{\partial j_t}\,(t,Z^t),$$

and thus we pass the first step of our program.

Let the contingent claim $H_{F^{\overline{N},\overline{M}}}$ which we get after the projection of contingent claim $H \in L^2(P,\mathcal{F}_T)$ on the σ-algebra $\mathcal{F}_T^{\overline{N},\overline{M}} = \mathcal{F}_T^S = \mathcal{F}_T^{S,Z}$ has the following form

$$H_{F^{\overline{N},\overline{M}}} = h\big(S(T), Z_0^T\big),$$

where $h : (0,\infty) \times \mathcal{C}_{[0,T]} \to R$ be bounded continuous function. By results of [12] $(S(t), Z^t)$ is a Markov process which satisfies the following SDE w.r.t $\widetilde{P}$

$$\begin{aligned} dS(t) &= S(t)\,\sigma(t,Z^t)\,d\widetilde{N}_t,\\ dZ^t &= \widetilde{m}(t,Z^t) j_t\,dt + \gamma(t,Z^t) j_t\,d\widetilde{M}_t. \end{aligned}$$

Hence by the Markov property we get

$$\begin{aligned} V_t^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}} &= E^{\widetilde{P}}\big[H_{F^{\overline{N},\overline{M}}} \mid \mathcal{F}_t^{S,Z}\big] =\\ &= E^{\widetilde{P}}\big[h(S(T), Z_0^T) \mid (S_t, Z^t)\big] = v^h(t, S(t), Z^t), \end{aligned}$$

where $v^h(t,x,\varphi)$, $(t,x,\varphi) \in [0,T] \times (0,\infty) \times \mathcal{C}_{[0,T]}$ is some bounded continuous function.

Theorem 4.2. *Let the coefficients $\widetilde{m}$, σ, γ have bounded continuous Frechet differential up to the second order w.r.t. third variable $\varphi \in \mathcal{C}_{[0,T]}$. Then v^h satisfies the following "PDE", see* [12],

$$\begin{gathered} \frac{\partial v^h}{\partial t} + \widetilde{m}\,\frac{\partial v^h}{\partial x} + \frac{1}{2}\,x^2\sigma^2\,\frac{\partial^2 v^h}{\partial x^2} + \frac{1}{2}\,\gamma^2\,\frac{\partial^2 v^h}{\partial j_t^2} = 0,\\ v^h(T,x,\varphi) = h(x,\varphi). \end{gathered}$$

Now by the Itô formula

$$\begin{aligned} v^h(t,S_t,Z^t) = v^h(0,S_0,c) &+ \int_0^t \frac{\partial v^h}{\partial x}\,(s,S(s),Z^s)\,dS(s) +\\ &+ \int_0^t \frac{\partial v^h}{\partial j_s}\,(s,S(s),Z^s)\gamma(s,Z)\,d\widetilde{M}(s) +\\ &+ \int_0^t \Big[\frac{\partial v^h}{\partial s}\,(s,S(s),Z^s) + \frac{\partial v^h}{\partial j_s}\,(s,S(s),Z^s) m_2(s,Z) + \end{aligned}$$

$$+ \frac{1}{2} \frac{\partial^2 v^h}{\partial j_s^2} (s, S(s), Z^s) \gamma^2(s, Z) \Big] \, ds$$

one can easily obtain the Galtchouk–Kivita–Watanabe decomposition of the martingale $V^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}}$:

$$V_t^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}} = \widetilde{E} H_{F^{\overline{N},\overline{M}}} + \int_0^t \xi_s^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}} \, dS(s) + L_t^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}}, \quad 0 \le t \le T,$$

where

$$\xi_t^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}} = \frac{\partial v^h}{\partial x} (t, S(t), Z^t),$$

$$L_t^{H_{F^{\overline{N},\overline{M}}},\widetilde{P}} = \int_0^t \gamma(s, Z) \frac{\partial v^h}{\partial j_s} (s, S(s), Z^s) \, d\widetilde{M}(s).$$

Thus we finished the second step of our program. The mean-variance hedging problem is solved.

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