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ON THE ITERATED SUMMABILITY OF TRIGONOMETRIC FOURIER SERIES

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1}$$

be a Fourier series of the summable function $f(x)$.

By $S_n(x, f)$ we denote partial sums of the series (1), i.e.,

$$S_n(x; f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Let a triangular matrix

$$\Lambda = \|\lambda_n(k)\|$$

be such that $\lambda_n(0) = 1$ and $\lambda_n(n+p) = 0$ for $n \geq 0$ and $p \geq 1$.

Consider means of the series (1):

$$t_n^{(1)}(x; f) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_n(k)(a_k \cos kx + b_k \sin kx). \tag{2}$$

Following N. K. Bary ([1], p. 474), we say that Λ is a matrix of the type F_c if

$$\lim_{n\to\infty} t_n^{(1)}(x; f) = f(x) \tag{3}$$

for all continuous $f(x)$ at every point.

Analogously, we say that Λ is a matrix of the type F if (3) holds for all summable functions $f(x)$ at every Lebesgue point.

It is clear that if the matrix is not of the type F_c, then it is not of the type F. The means (2) can be written by a sequence of partial sums

$$\{S_n(x; f)\}_{n=0}^{\infty} \tag{4}$$

as follows:

$$t_n^{(1)}(x; f) = \sum_{k=0}^{n} (\lambda_n(k) - \lambda_n(k+1)) S_k(x; f). \tag{5}$$

$t_n^{(1)}(x; f)$ are called Λ-means (or $\Lambda^{(1)}$-means) of the series (1) (or of the sequence (4)).

Let $d \geq 2$ be any natural number, and assume that for every number j, where $1 \leq j \leq d-1$, the means $\{t_n^{(j)}(x; f)\}_{n=0}^{\infty}$ are already constructed.

By $t_n^{(d)}(x; f)$ we denote Λ-means for the sequence $\{t_n^{(d-1)}(x; f)\}_{n=0}^{\infty}$, i.e.,

$$t_n^{(d)}(x; f) = \sum_{k=0}^{\infty} (\lambda_n(k) - \lambda_n(k+1)) t_k^{(d-1)}(x; f). \tag{6}$$

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If $d \geq 2$, then $t_n^{(d)}(x; f)$ will be called the iterated $\Lambda^{(d)}$-means of the series (1), and if at some point x there exists the limit

$$\lim_{n\to\infty} t_n^{(d)}(x; f) = S, \tag{7}$$

then we will say that the series (1) is iterated $\Lambda^{(d)}$-summable at the point x to the number S.

Clearly, if Λ is a regular matrix, then from the relation

$$\lim_{n\to\infty} t_n^{(d-1)}(x; f) = S$$

it follows that the relation (7) is valid.

For every $d \geq 1$, using some triangular matrix

$$M^{(d)} = \|\mu_n^{(d)}(k)\|,$$

we write the iterated $\Lambda^{(d)}$-means of the series (1) as follows:

$$t_n^{(d)}(x; f) = \sum_{k=0}^{n} \mu_n^{(d)}(k)(a_k \cos kx + b_k \sin kx).$$

(Note that if $d = 1$, then $M^{(1)} = \Lambda$).

For the iterated $\Lambda^{(d)}$-summability the following theorem is valid.

Theorem 1. *Let $d \geq 1$ be any natural number and the matrix Λ be such that for every $n \geq 0$ and $0 \leq k \leq n$*

1) $0 \leq \lambda_n(k+1) \leq \lambda_n(k) \leq 1$,
2) $\lambda_n(k) \leq \lambda_{n+1}(k)$,
3) *for any natural number p*

$$\lim_{n\to\infty} \lambda_{pn}((p-1)n) = 1.$$

Then

a) *the matrix $M^{(d)}$ is not of the type F_c;*
b) *there exists the function $f \in L_1(0, 2\pi)$ such that*

$$\overline{\lim_{n\to\infty}} \|t_n^{(d)}(x; f) - f(x)\|_{L_1} > 0.$$

The above theorem can be generalized for $d = 1$. Thus the following theorem is valid.

Theorem 2. *Let the triangular matrices*

$$\Lambda = \|\lambda_n(k)\| \quad \textit{and} \quad M = \|\mu_n(k)\|$$

be such that for every $n \geq 0$ and $0 \leq k \leq n$

1) $\lambda_n(k \geq \mu_n(k))$, $\lambda_n(k) \geq \mu_n(k)$,
2) $0 \leq \mu_n(k+1) \leq \mu_n(k) \leq 1$,
3) *for any natural number p*

$$\lim_{n\to\infty} \mu_{pn}((p-1)n) = 1.$$

Then

a) *the matrix Λ is not of the type F_c;*
b) *there exists the function $f \in L_1(0, 2\pi)$ such that*

$$\lim_{n\to\infty} \|t_n^{(1)}(x; f) - f(x)\|_{L_1} > 0.$$

The above theorems result in different corollaries. Here we cite some of them.

Let $\{\alpha_n\}$ be a sequence of numbers from $[0, 1]$, $\alpha_{n+1} \leq \alpha_n$, and for $0 \leq k \leq n$

$$\lambda_n(k) = \frac{A_{n-k}^{\alpha_n}}{A_n^{\alpha_n}}, \quad \text{where} \quad A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \tag{8}$$

or

$$\lambda_n(k) = \left(1 - \frac{k}{n+1}\right)^{\alpha_n}. \tag{9}$$

Just as in case (8), the matrix $\Lambda = \|\lambda_n(k)\|$ in case (9) is completely regular.

If $\alpha_n = \alpha > 0$ for $n \geq 0$, then the matrix Λ in case (8) specifies the Cesaro method of summability (C, α) $\alpha > 0$, while the matrix Λ in case (9) spesifies the Riesz method of summability of order $\alpha > 0$.

It is known ([1], p. 482) that in these cases the matrix Λ is of the type F (and hence of the type F_c).

For the iterated $\Lambda^{(d)}$-summability, from Theorem 1 we have

Theorem 3. *Let $d \geq 1$ be any natural number, $\alpha_n \in [0, 1]$, $\alpha_n \downarrow 0$ as $n \to \infty$ and $\lambda_n(k)$, are defined by the relation* (8)*, or by the relation* (9)*.*

Then

a) *the matrix $M^{(d)}$ is not of the type F_c;*

b) *there exists the function $f \in L_1(0, 2\pi)$ such that*

$$\overline{\lim_{n\to\infty}} \|t_n^{(d)}(x; f) - f(x)\|_{L_1} > 0.$$

Consider the case $d = 1$.

Let the numbers $\alpha_{n,k} \in [0, 1]$, and for $0 \leq k \leq n$

$$\lambda_n(k) = \frac{A_{n-k}^{\alpha_{n,k}}}{A_n^{\alpha_{n,k}}} \tag{10}$$

or

$$\lambda_n(k) = \left(1 - \frac{k}{n+1}\right)^{\alpha_{n,k}}. \tag{11}$$

For the Λ-summability, from Theorem 2 we have

Theorem 4. *Let the numbers $\alpha_{n,k} \in [0, 1]$ be such that $\max_{0\leq k\leq n} \alpha_{n,k} \to 0$ as $n \to \infty$, and $\lambda_n(k)$ are defined by the relation* (10)*, or by the relation* (11)*.*

Then

a) *the matrix $\Lambda = \|\lambda_n(k)\|$ is not of the type F_c;*

b) *there exists the function $f \in L_1(0, 2\pi)$ such that*

$$\overline{\lim_{n\to\infty}} \|t_n^{(1)}(x; f) - f(x)\|_{L_1} > 0.$$

References

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