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"CORRECTION" OF THE MATRIX OF PARTIAL SUMS AND TRIGONOMETRIC FOURIER SERIES

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Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx+b_k\sin kx \tag{1}$$

be the Fourier series of a summable on $[0,2\pi]$ function $f(x)$.

By $S_n(x;f)$ we denote partial sums of the series (1), i.e.,

$$S_n(x;f)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx+b_k\sin kx.$$

Let the triangular matrix

$$\Lambda=\|\lambda_n(k)\|,$$

be such that $\lambda_n(0)=1$ and $\lambda_n(n+p)=0$ for $n\ge 0$ and $p\ge 1$.

Consider the means of the series (1),

$$t_n(x;f;\Lambda)=\frac{a_0}{2}+\sum_{k=1}^{n} \lambda_n(k)(a_k\cos kx+b_k\sin kx).$$

Following N. K. Bari (see [1] p.474), we say that Λ is a matrix of type F_c, if

$$\lim_{n\to\infty} t_n(x;f,\Lambda)=f(x) \tag{2}$$

for all continuous functions $f(x)$ at every point.

Analogously, we say that Λ is a matrix of type F, if (2) holds for all summable functions $f(x)$ at every Lebesgue point.

It is clear that if Λ is not a matrix of type F_c, then it is not likewise a matrix of type F.

Note (see [1], p.475) that for the matrices of type F_c the convergence in (2) is uniform on the segment $[0,2\pi]$.

Let the triangular matrix

$$\Gamma=\|\gamma_n(k)\|$$

be such that $\gamma_n(k)=1$ for $n\ge 0$ and $0\le k\le n$, and $\gamma_n(n+p)=0$ for $p\ge 1$.

Obviously,

$$t_n(x,f;\Gamma)=S_n(x;f).$$

We call Γ the matrix of partial sums. It is well-known that:

I) the matrix Γ is not of type F_c (Fejer's example; [1], p.132);

II) there exists $f\in L[0,2\pi]$, such that $\{t_n(x;f,\Gamma)\}_{n=0}^{\infty}$ diverges at every point (Kolmogorov's example; [1], p.412);

III) there exists $g\in L[0,2\pi]$, such that

$$\overline{\lim_{n\to\infty}}\|t_n(x;g,\Gamma)-g(x)\|_{L_1}>0;$$

(F. Riesz example; [1], p.599).

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N. N. Luzin and D. E. Men'shov established, respectively, the C-property and the strengthened C-property of "correction" of a function on a set of small measure.

Here we present D. E. Men'shov's theorem ([1], p. 448):

Let $f(x)$ be a measurable function, finite almost everywhere on $[0, 2\pi]$. For an arbitrary $\varepsilon > 0$ we can construct a function $g(x)$ coinciding with $f(x)$ on some set E of measure $mE > 2\pi - \varepsilon$, such that $\{S_n(x; g)\}_{n=0}^{\infty}$ converges uniformly on $[0, 2\pi]$.

Consequently, "a small correction" of a function guarantees uniform convergence of the Fourier series.

In the present work we consider "small correction" of the matrix of partial sums Γ.

The following proposition is valid:

Let $\varepsilon > 0$. In the n-th line $(n = 0, 1, 2, \dots)$ of the matrix Γ we can replace not more than $\varepsilon \cdot n$ units by some positive numbers, in such a way that the obtained matrix becomes of type F.

To formulate the obtained results exactly, we will need some definitions.

As we have mentioned above, in the matrix $\Lambda = \|\lambda_n(k)\|$

$$\lambda_n(0) = 1, \quad (n \geq 0).$$

For every $n \geq 0$, we denote

$$m(n) = \min_{0 \leq k \leq n} \{k : \lambda_n(k+1) \neq 1\}, \quad \text{i.e.}$$

$$\lambda_n(k) = 1 \quad \text{for} \quad 0 \leq k \leq m(n) \quad \text{and} \quad \lambda_n(m(n)+1) \neq 1.$$

We write the means $t_n(x; f; \Lambda)$ as follows:

$$t_n(x; f; \Lambda) = S_{m(n)}(x; f) + \sum_{k=m(n)+1}^{n} \lambda_n(k)(a_k \cos kx + b_k \sin kx).$$

The sequence of numbers $\{m(n)\}_{n=0}^{\infty}$ is called the sequence of indices of partial sums of the matrix Λ.

The number

$$\rho_n(\Lambda) = \frac{m(n)}{n}$$

will be called the density of the partial sum corresponding to $t_n(x; f, \Lambda)$.

Denote

$$\overline{\rho}(\Lambda) = \overline{\lim_{n\to\infty}} \frac{m(n)}{n}.$$

Clearly, $0 \leq \overline{\rho}(\Lambda) \leq 1$. We call the number $\overline{\rho}(\Lambda)$ the upper density of the sequence of partial sums of the matrix Λ.

If there exists the limit

$$\rho(\Lambda) = \lim_{n\to\infty} \rho_n(\Lambda),$$

then the number $\rho(\Lambda)$ will be called the density of the sequence of indices of partial sums of the matrix Λ.

If all elements of the matrix Λ are nonnegative, then we simply write $\Lambda \geq 0$.

The following theorem is valid.

Theorem 1. *For every number $\varepsilon > 0$, there exists a matrix $\Lambda \geq 0$, such that:*
a) *$\rho(\Lambda) > 1 - \varepsilon$;*
b) *Λ is a matrix of type F (and, respectively, of type F_c);*
c) *for every function $f \in L[0, 2\pi]$,*

$$\lim_{n\to\infty} \|t_n(x; f, \Lambda) - f(x)\|_{L_1} = 0.$$

On the other hand, the following theorem is true.

Theorem 2. *If a matrix $\Lambda \geq 0$ is such that $\overline{\rho}(\Lambda) = 1$, then:*
a) *the matrix Λ is not of type F_c;*

b) *there exists a function* $g \in L[0,2\pi]$, *such that*

$$\overline{\lim_{n\to\infty}} \|t_n(x;g;\Lambda) - g(x)\|_{L_1} > 0.$$

Theorem 2 leads to the following

Corollary. *Let* $\{n_k\}_{k=0}^{\infty}$ *be any increasing subsequence of natural numbers. Then there exists a continuous on* $[0,2\pi]$ *function* $f(x)$, *such that* $\{S_{n_k}(x;f)\}_{k=0}^{\infty}$ *diverges at some point* $x_0 \in [0,2\pi]$.

References

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