Proc. A. Razmadze Math. Inst. 150(2009), 150–152

SH. TETUNASHVILI

ON THE SUMMABILITY METHODS DEPENDING ON A PARAMETER

(Reported on 12.11.2008)

Let $\mathbb{R}^N$ be the Euclidean space of dimension N, $(N = 1, 2, 3, \dots)$. $x = (x_1, \dots, x_N)$ is a point of the space $\mathbb{R}^N$; n and k are natural numbers, and the number $R \in (0, +\infty)$. Z^N is the set of points in $\mathbb{R}^N$ with integer coordinates; $m = (m_1, \dots, m_N)$ is the vector from the set Z^N; $|m| = \sqrt{\sum_{k=1}^{N} m_k^2}$ is length of the vector m; $\mu_N E$ is the Lebesgue measure of the set $E \subset \mathbb{R}^N$; $\phi = \{f(x), x \in E\}$ is a normed functional space, and $\|\cdot\|_\phi$ is the norm of that space. $M(\alpha)$ is the summability method depending on the parameter α. $\tau_R^\alpha(x; f)$ are the means corresponding to the method $M(\alpha)$ for the function f at the point x.

$$Q_N = \{x : x \in \mathbb{R}^N, \quad -\pi < x_k \le \pi, \quad k = 1, \dots, N\}$$

is a cube in the space $\mathbb{R}^N$. $L_p(Q_N)$ $(1 \le p < \infty)$ is the space of functions $f(x)$ such that $|f(x)|^p$ are Lebesgue integrable on Q_N. $\|\cdot\|_p$ is the norm in the space $L_p(Q_N)$. $C(Q_N)$ is the space of continuous on Q_N functions; $\|\cdot\|_C$ is the norm in the space $C(Q_N)$. $\widehat{f}(m)$ are the Fourier coefficients of the function $f \in L(Q_N)$, i.e.,

$$\widehat{f}(m) = (2\pi)^{-N} \cdot \int\limits_{Q_N} f(x) \cdot e^{imx} dx \quad (m \in Z^N).$$

$\sum_{m \in Z^N} \widehat{f}(m) e^{imx}$ is the trigonometric Fourier series of the function f.

$$S_R^\alpha(x; f) = \sum_{|m| \le R} \Big(1 - \frac{|m|^2}{R^2}\Big)^\alpha \cdot \widehat{f}(m) e^{imx} dx$$

are the spherical Bochner-Riesz means of order $\alpha \ge 0$.

2000 *Mathematics Subject Classification:* 42B05, 42B08.
Key words and phrases. Fourier series, Bochner-Riesz summability method.

Definition. We say that a number α_0 is an indicator of the method $M(\alpha)$ for the space ϕ if for every function $f \in \Phi$

$$\lim_{R\to\infty} \|f(x) - \tau_R^\alpha(x;f)\|_\phi = 0 \quad \text{for} \quad \alpha > \alpha_0$$

and a number α_0 is an exact indicator of the method $M(\alpha)$ for the space ϕ if α_0 is an indicator of the method $M(\alpha)$ for the space ϕ and there exists the function $f_0 \in \phi$ such that

$$\overline{\lim}_{R\to\infty} \|f_0(x) - \tau_R^{\alpha_0}(x;f_0)\|_\phi > 0.$$

The following theorems are valid.

Theorem 1. *Let α_0 be the indicator of the method $M(\alpha)$ for the space ϕ. Then for every function $f \in \phi$ there exists $\{\alpha_{_R}\}_{R>0}$ with $\lim\limits_{R\to\infty} \alpha_{_R} = \alpha_0$ such that*

$$\lim_{R\to\infty} \|f(x) - \tau_R^{\alpha_R}(x;f)\|_\phi = 0.$$

Theorem 2. *Let α_0 be the indicator of the method $M(\alpha)$ for the space ϕ. Then for every function $f \in \phi$, any sequence $\{\alpha_k\}$ with $\alpha_k > \alpha_0$ and any set $M \subset (0, +\infty)$ with $\sup M = +\infty$ there exists $R_k \uparrow \infty$, $R_k \in M$ such that*

$$\sum_{k=1}^{\infty} \|f(x) - \tau_{R_k}^{\alpha_k}(x;f)\|_\phi \le 1.$$

The results obtained by Bochner ([1]) and Stein ([2], [3]) show that for the Bochner-Riesz method in the spaces $C(-\pi,\pi]^N$ and $L(-\pi,\pi]^N$ the number $\frac{N-1}{2}$ is the exact indicator, and for the spaces $L_p(-\pi,\pi]^N$ and $(1 < p < \infty)$ the number

$$\frac{N-1}{2} \cdot \left|\frac{2}{p} - 1\right|$$

is the indicator. Thus from Theorems 1 and 2 for every $N = 1, 2, 3, \dots$ it follows that the statements below are valid.

Theorem 3. *For every function $f \in C(-\pi,\pi]^N$ there exists $\{\alpha_{_R}\}_{R>0}$ such that $\lim\limits_{R\to\infty} \alpha_{_R} = \frac{N-1}{2}$ and*

$$\lim_{R\to\infty} \|f(x) - S_R^{\alpha_R}(x;f)\|_C = 0.$$

Theorem 4. *For every function $f \in L_p(-\pi,\pi]^N$ $(1 \le p < \infty)$ there exists $\{\alpha_R\}_{R>0}$ such that $\lim\limits_{R\to\infty} \alpha_R = \frac{N-1}{2}|\frac{2}{p} - 1|$ and*

$$\lim_{R\to\infty} \|f(x) - S_R^{\alpha_R}(x;f)\|_p = 0.$$

For the pointwise convergence following theorems are valid

Theorem 5. *For every function* $f \in L(-\pi,\pi]^N$ *there exists* $\{\alpha_{_R}(t)\}_{R>0}$ *such that* $\lim\limits_{R\to\infty} \alpha_{_R}(t) = \frac{N-1}{2}$, $t \in (-\pi,\pi]^N$ *and at every Lebesgue point of the function* f *(i.e., almost everywhere on* $(-\pi,\pi]^N$*) we have*

$$\lim_{R\to\infty} S_R^{\alpha_{_R}(x)}(x)(x;f) = f(x).$$

Theorem 6. *For every function* $f \in L(-\pi,\pi]^N$ *and any number* $\varepsilon > 0$ *there exist* $\{\alpha_{_R}\}_{R>0}$ *with* $\lim\limits_{R\to\infty} \alpha_{_R} = \frac{N-1}{2}$ *and the set* $F \subset (-\pi,\pi]^N$ *with* $\mu_N F > (2\pi)^N - \varepsilon$ *such that*

$$\lim_{R\to\infty} S_R^{\alpha_{_R}}(x;f) = f(x)$$

uniformly on the set F.

Theorem 7. *For every function* $f \in L(-\pi,\pi]^N$ *any sequence* $\{\alpha_k\}$ *with* $\alpha_k > \frac{N-1}{2}$ *and any set* $M \subset (0,+\infty)$ *with* $\sup M = +\infty$ *there exists* $R_k \uparrow \infty$, $R_k \in M$ *such that*

$$\lim_{K\to\infty} S_{R_K}^{\alpha_K}(x;f) = f(x)$$

almost everywhere on $(-\pi,\pi]^N$.

The above theorems are valid for every $N = 1,2,3,\dots$. In the case when the space dimension $N = 1$ and the method $M(\alpha)$ is the Cesaro (C,α) one, some of the above theorems have been mentioned in [4] and [5].

Acknowledgement

This work was supported by the Grant GNSF/STO7/3-169.

References

1. S. Bochner, Summation of multiple Fourier series by spherical means. *Trans. Amer. Math. Soc.* **40** (1936), No. 2, 175–207.
2. E. M. Stein, Localization and summability of multiple Fourier series. *Acta Math.* **100** (1958), 93–147.
3. E. M. Stein, On certain exponential sums arising in multiple Fourier series. *Ann. of Math. (2)* **73** (1961), 87–109.
4. Sh. Tetunashvili, On the summability of Fourier trigonometric series of variable order. *Proc. A. Razmadze Math. Inst.* **145** (2007), 130–131.
5. Sh. Tetunashvili, On the summability methods defined by matrix of functions. *Proc. A. Razmadze Math. Inst.* **148** (2008), 141–145.

Author's addresses:

A. Razmazde Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia

Department of Mathematics
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia