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ON CANTOR'S FUNCTIONALS

Let $A \subset [0,1]$ and $B \subset [0,1]$. The set $A \times B$ is Cartesian product of the sets A and B. Let μA be Lebesgue measure of A, i, j and k are non-negative integer numbers. The symbol $(i,j) \geq (k,k)$ means that $i \geq k$ and $j \geq k$.

Suppose that systems $\Phi = \{\varphi_i(\tau)\}_{i=0}^{\infty}$ and $\Psi = \{\psi_i(\tau)\}_{i=0}^{\infty}$ are systems of measurable and finite functions defined on $[0,1]$.

Consider a series in the system Φ:

$$\sum_{i=0}^{\infty} a_i \varphi_i(\tau) \tag{1}$$

Definition 1. We say that the system Φ has Cantor's property if there exists a set $A \subset [0,1]$, such that the only series (1) that converges to zero on A is the series all of whose coefficients are zero.

Definition 2. A measurable set A belongs to the class $U(\Phi)$ if the only series (1) that converges to zero on A is the series all of whose coefficients are zero.

Definition 3. We say that a finite function $f(\tau)$ belongs to the class $J(A;\Phi)$ if $A \in U(\Phi)$ and there exists a series (1) such that equality

$$\sum_{i=0}^{\infty} a_i \varphi_i(\tau) = f(\tau) \tag{2}$$

holds true for any $\tau \in A$.

Definition 4. We say that a sequence of functionals $\{G_i^{A,\Phi}(\cdot)\}_{i=0}^{\infty}$, defined on $J(A,\Phi)$, is Cantor's functionals sequence if for any function $f \in J(A,\Phi)$ and any coefficient a_i of the series (2) an equality

$$a_i = G_i^{A,\Phi}(f(\tau))$$

holds.

Definition 5. Let $\delta \in (0,1]$ be a number. We say that the system Φ is a system of δ linearly independent system, if every finite part of the system Φ is linearly independent on each set A, with $\mu A > 1 - \delta$.

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The following propositions hold.

Proposition 1. *If $A \in U(\Phi)$, then $J(A, \Phi)$ is linear space.*

Proposition 2. *If $\{G_i^{A,\Phi}(\cdot)\}_{i=0}^{\infty}$ is a sequence of Cantor's functionals, defined on linear space $J(A, \Phi)$, then $G_i^{A,\Phi}(\cdot)$ is linear functional, for any $i = 0, 1, 2, \dots$.*

Proposition 3. *If $A \in U(\Phi)$, $\varphi_0(\tau) \equiv 1$ and $f(\tau) \equiv c$, $\tau \in A$, then $G_0^{A,\Phi}(c) = 1$ and $G_i^{A,\Phi}(c) = 0$ for any $i = 1, 2, 3, \dots$.*

We consider a double series in the system $\Phi \times \Psi = \{\varphi_i(x) \cdot \Psi_j(y)\}_{(i,j) \geq (0,0)}$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_i(x) \psi_j(y). \tag{3}$$

The convergence of the series (3) will be understood as Pringsheim convergence.

Let as consider a finite function $F(x, y)$, where $(x, y) \in A \times B$. The symbol $G_i^{B,\Psi}(F(x, y))$ means that $G_i^{B,\Psi}$ acts on a function $F(x, y)$, where only $y \in B$ is an independent variable and x is a fixed point of the set A. Also, $F(x, y) \in J(B, \Psi)$ for any $x \in A$.

In the future we shall assume that Φ is δ linearly independence system, the set $A \in U(\Phi)$, $\mu A > 1 - \delta$ and the set $B \in U(\Psi)$.

We established that it is possible to calculate coefficients of convergent series (3) by iterated using of Cantor's functionals. Namely, the following holds true

Theorem. *Let*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_i(x) \cdot \psi_j(y) = F(x, y), \quad \text{when } (x, y) \in A \times B.$$

Then for every $(i, j) \geq (0, 0)$ an equality

$$a_{ij} = G_i^{A,\Phi}\left(G_j^{B,\Psi}(F(x, y))\right)$$

holds.

It follows from this theorem and propositions 1–3, the following corollaries:

Corollary 1. *Let $f(x)$ and $g(y)$ are finite functions and*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_i(x) \psi_j(y) = f(x) \cdot g(y), \quad \text{when } (x, y) \in A \times B.$$

Then for every $(i, j) \geq (0, 0)$ an equality

$$a_{ij} = G_i^{A,\Phi}(f(x)) \cdot G_j^{B,\Psi}(g(y))$$

holds.

Corollary 2. *Let* $\varphi_0(x) \equiv \psi_0(y) \equiv 1$*,* $f(x)$ *and* $g(y)$ *are finite functions and*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_i(x)\psi_j(y) = f(x) + g(y), \quad \textit{when} \ \ (x,y) \in A \times B.$$

Then for every $(i,j) \geq (1,1)$ *an equality*

$$a_{ij} = 0$$

holds.

Remark. As it is well known, the trigonometric system T=$\{1, \sqrt{2}\cos 2\pi i\tau, \sqrt{2}\sin 2\pi i\tau\}_{i=1}^{\infty}$ has Cantor's property (see [1]). At the same time the trigonometric system T is $\delta = 1$ linearly independence system. So the system T satisfies all conditions of above presented theorem and this theorem and corollaries are valid for double trigonometric series. Analogous propositions are valid also for d-multiple ($d \geq 3$) functional series.

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References

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