

Research Article

On Divergence of Fourier Series by Some Methods of Summability

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A new summability method of series is introduced and studied. The particular cases of this method are, for example, variable-order Cesaro and Riesz methods. Applications to divergence problem of Fourier series are given. An extension of Kolmogorov, Schipp, and Bočkarov's well-known theorems on divergence of Fourier trigonometric, Walsh, and orthonormal series is established.

1. A New Summability Method of Series

Let

$$\Lambda = \|\lambda_n(k)\|, \quad n = 0, 1, 2, \dots, \quad k = 0, 1, 2, \dots, n, \quad (1.1)$$

be such triangular matrix which satisfies the following conditions:

$$\begin{aligned} (1) \quad & 0 \leq \lambda_n(k+1) \leq \lambda_n(k) \leq 1, \quad 0 \leq k \leq n; \\ (2) \quad & \lambda_n(0) = 1, \quad \lambda_n(k) = 0, \quad k \geq n+1. \end{aligned} \quad (1.2)$$

By s_n we denote a partial sum of a series

$$\sum_{k=0}^{\infty} u_k, \quad (1.3)$$

and by σ_n we denote a mean constructed by the Λ matrix, that is,

$$s_n = \sum_{k=0}^n u_k, \quad \sigma_n = \sum_{k=0}^n \lambda_n(k) u_k. \quad (1.4)$$

Theorem 1.1. *Let matrix (1.1) satisfies an inequality*

$$\underline{\lim}_{n \rightarrow \infty} \lambda_n(n) > \frac{1}{2}. \quad (1.5)$$

Then for any series (1.3) which satisfies the following condition:

$$\overline{\lim}_{n \rightarrow \infty} |s_n| = +\infty, \quad (1.6)$$

an equality

$$\overline{\lim}_{n \rightarrow \infty} |\sigma_n| = +\infty \quad (1.7)$$

holds.

Below we prove a Lemma which is used to prove Theorem 1.1.

Lemma 1.2. *For every natural number n an inequality*

$$|s_n - \sigma_n| \leq 2(1 - \lambda_n(n)) \cdot \max_{1 \leq k \leq n} |s_k| \quad (1.8)$$

holds.

Proof of the Lemma. Using Abel transformation and $\lambda_n(0) = 1$ we get

$$\begin{aligned} s_n - \sigma_n &= \sum_{k=0}^n u_k - \sum_{k=0}^n \lambda_n(k) u_k \\ &= \sum_{k=1}^n u_k - \sum_{k=1}^n \lambda_n(k) u_k \\ &= \sum_{k=1}^n (1 - \lambda_n(k)) u_k \\ &= \sum_{k=1}^{n-1} (\lambda_n(k+1) - \lambda_n(k)) s_k + (1 - \lambda_n(n)) s_n. \end{aligned} \quad (1.9)$$

Therefore,

$$\begin{aligned} |s_n - \sigma_n| &\leq \sum_{k=1}^{n-1} |\lambda_n(k+1) - \lambda_n(k)| \cdot |s_k| + |1 - \lambda_n(n)| \cdot |s_n| \\ &\leq \max_{1 \leq k \leq n} |s_k| \cdot \left(\sum_{k=1}^{n-1} |\lambda_n(k+1) - \lambda_n(k)| + |1 - \lambda_n(n)| \right). \end{aligned} \quad (1.10)$$

Thus, taking into account (1.1) we immediately get

$$\begin{aligned} |s_n - \sigma_n| &\leq \max_{1 \leq k \leq n} |s_k| \cdot \left(\sum_{k=1}^{n-1} (\lambda_n(k) - \lambda_n(k+1)) + 1 - \lambda_n(n) \right) \\ &= \max_{1 \leq k \leq n} |s_k| \cdot (\lambda_n(1) - \lambda_n(n) + 1 - \lambda_n(n)) \\ &\leq \max_{1 \leq k \leq n} |s_k| \cdot (1 - \lambda_n(n) + 1 - \lambda_n(n)) \\ &= 2 \cdot (1 - \lambda_n(n)) \cdot \max_{1 \leq k \leq n} |s_k|. \end{aligned} \quad (1.11)$$

So the Lemma is proved. \square

Proof of Theorem 1.1. According to the condition of Theorem 1.1 we have

$$\lim_{n \rightarrow \infty} \lambda_n(n) = \frac{1}{2} + \delta \quad (1.12)$$

for some $\delta > 0$. Note that inequalities $0 \leq \lambda_n(n) \leq 1$ which hold for every natural n imply $1/2 + \delta \leq 1$, that is, $\delta \leq 1/2$.

So, $0 < \delta \leq 1/2$ holds.

According to (1.12) there exists a natural number n_0 such that for every natural number $n > n_0$ we have

$$\lambda_n(n) > \frac{1}{2} + \frac{\delta}{2}. \quad (1.13)$$

So according to the Lemma, for every $n > n_0$ an inequality

$$|s_n - \sigma_n| < 2 \cdot \left(1 - \left(\frac{1}{2} + \frac{\delta}{2} \right) \right) \cdot \max_{1 \leq k \leq n} |s_k| \quad (1.14)$$

holds true; that is, if $n > n_0$, then

$$|s_n - \sigma_n| < (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \quad (1.15)$$

Thus for every $n > n_0$ an inequality

$$\|s_n\| - |\sigma_n| < (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k| \quad (1.16)$$

holds.

So for every $n > n_0$ we have

$$|\sigma_n| > |s_n| - (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \quad (1.17)$$

Note that for every natural n there exists at least one natural number $1 \leq q \leq n$, such that the partial sums of the series (1.3) satisfy the following condition:

$$|s_q| = \max_{1 \leq k \leq n} |s_k|. \quad (1.18)$$

We define p_n by a formula:

$$p_n = \max \left\{ q : 1 \leq q \leq n \ \& \ |s_q| = \max_{1 \leq k \leq n} |s_k| \right\}. \quad (1.19)$$

So p_n is maximal number among the above-mentioned natural q numbers. Consequently,

$$1 \leq p_n \leq n, \quad |s_{p_n}| = \max_{1 \leq k \leq n} |s_k|, \quad (1.20)$$

$$p_n \leq p_{n+1}, \quad |s_{p_n}| \leq |s_{p_{n+1}}|. \quad (1.21)$$

According to the condition of Theorem 1.1,

$$\overline{\lim}_{n \rightarrow \infty} |s_n| = +\infty. \quad (1.22)$$

Therefore,

$$\lim_{n \rightarrow \infty} |s_{p_n}| = +\infty, \quad (1.23)$$

that is,

$$\lim_{n \rightarrow \infty} p_n = +\infty. \quad (1.24)$$

A consequence of (1.24) is that there exists such natural n_1 that if $n > n_1$ then $p_n > n_0$ and since (1.17) holds for every $n > n_0$, then (1.17) remains true for every p_n , where $n > n_1$.

So

$$|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot \max_{1 \leq k \leq p_n} |s_k|. \quad (1.25)$$

Since $1 \leq p_n \leq n$, therefore,

$$\max_{1 \leq k \leq p_n} |s_k| \leq \max_{1 \leq k \leq n} |s_k|. \quad (1.26)$$

Note that the last one and (1.25) imply

$$|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \quad (1.27)$$

So according to (1.21) we have

$$|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot |s_{p_n}|, \quad (1.28)$$

that is, for every $n > n_1$ an inequality

$$|\sigma_{p_n}| > \delta \cdot |s_{p_n}| \text{ holds, where } 0 < \delta \leq \frac{1}{2}. \quad (1.29)$$

Also, (1.23) and (1.29) imply

$$\lim_{n \rightarrow \infty} |\sigma_{p_n}| = +\infty. \quad (1.30)$$

So we have finished the proof of Theorem 1.1. \square

Below we consider some consequences of Theorem 1.1.

Let $\Lambda = \|\lambda_n(k)\|$ be a triangular matrix, where the sequence $\{\alpha_n\}$ is from $[0, 1]$ and for every $0 \leq k \leq n$ number $\lambda_n(k)$ is defined by the formula:

$$\lambda_n(k) = \frac{A_{n-k}^{\alpha_n}}{A_n^{\alpha_n}}, \quad \text{where } A_n^{\alpha_n} = \frac{(\alpha_n + 1)(\alpha_n + 2) \cdots (\alpha_n + n)}{n!}. \quad (1.31)$$

If $\alpha_n = \alpha$, for every $n \geq 0$ and (1.31) holds true, then the Λ method is Cesaro (C, α) summability method, and if $\alpha_n \equiv 0$, then the Λ method coincides with convergence.

We introduce Cesaro summability method with variable orders, denoted by a symbol $(C, \{\alpha_n\})$, which coincides with Λ summability method defined by (1.31). Means of this method for series (1.3) we denoted by $\sigma_n^{\alpha_n}$.

For $(C, \{\alpha_n\})$ we have the following.

Theorem 1.3. *Let a sequences $\{\alpha_n\}$ be such that for some positive number m we have*

$$\alpha_n \leq \frac{c}{\ln n}, \quad \text{where } 0 \leq c < \ln 2 \text{ and } n > m. \quad (1.32)$$

Then for any series (1.3) which satisfies the following condition:

$$\overline{\lim}_{n \rightarrow \infty} |s_n| = +\infty, \quad (1.33)$$

an equality

$$\overline{\lim}_{n \rightarrow \infty} |\sigma_n^{\alpha_n}| = +\infty \quad (1.34)$$

holds.

Proof of Theorem 1.3. Note that every $\lambda_n(k)$ satisfies condition (1.1) and condition (1.3). Indeed,

$$\frac{\lambda_n(k+1)}{\lambda_n(k)} = \frac{A_{n-k-1}^{\alpha_n}}{A_{n-k}^{\alpha_n}} = \frac{n-k}{\alpha_n + n - k} \leq 1 \quad (1.35)$$

and $\lambda_n(0) = 1$, when $n \geq 0$.

For every $n \geq 1$ we have

$$\lambda_n(n) = \frac{1}{A_n^{\alpha_n}}, \quad \text{where } A_n^{\alpha_n} = \frac{(\alpha_n + 1)(\alpha_n + 2) \cdots (\alpha_n + n)}{n!}, \quad (1.36)$$

that is,

$$A_n^{\alpha_n} = \left(1 + \frac{\alpha_n}{1}\right) \left(1 + \frac{\alpha_n}{2}\right) \cdots \left(1 + \frac{\alpha_n}{n}\right). \quad (1.37)$$

Therefore,

$$\ln A_n^{\alpha_n} = \sum_{k=1}^n \ln \left(1 + \frac{\alpha_k}{n}\right) < \sum_{k=1}^n \frac{\alpha_n}{k} = \alpha_n \cdot \sum_{k=1}^n \frac{1}{k} < \alpha_n(1 + \ln n). \quad (1.38)$$

Note that the last one and (1.32) imply that

$$c = \ln \frac{2}{1 + \gamma}, \quad \text{for some } 0 < \gamma \leq 1, \quad (1.39)$$

and if $n > m$, we have

$$\begin{aligned} A_n^{\alpha_n} &< e^{\alpha_n(1 + \ln n)} = e^{\alpha_n} \cdot e^{\alpha_n \ln n} \\ &\leq e^{\alpha_n} \cdot e^c = e^{\alpha_n} \cdot e^{\ln(2/(1+\gamma))} = e^{\alpha_n} \cdot \frac{2}{1 + \gamma}, \end{aligned} \quad (1.40)$$

that is,

$$\lambda_n(n) = \frac{1}{A_n^{\alpha_n}} > \frac{1}{e^{\alpha_n}} \cdot \left(\frac{1 + \gamma}{2}\right), \quad \text{where } \gamma > 0. \quad (1.41)$$

Note that $\alpha_n \rightarrow 0$ implies the existence of such $\gamma_1 > 0$ and natural n_2 , that if $n > n_2$, then

$$\frac{1}{e^{\alpha_n}} \cdot \left(\frac{1}{2} + \frac{\gamma}{2} \right) > \frac{1}{2} + \gamma_1, \quad (1.42)$$

that is, if $n > n_2$, then

$$\lambda_n(n) > \frac{1}{2} + \gamma_1. \quad (1.43)$$

A consequence of (1.43) is that if (1.32) holds, then the Λ matrix satisfies conditions of Theorem 1.1. This completes the proof of Theorem 1.3. \square

Theorem 1.3 directly implies the following.

Theorem 1.4. *Let $\{\alpha_n\}$ be such sequence that*

$$\{\alpha_n\} = o\left(\frac{1}{\ln n}\right). \quad (1.44)$$

Then for every series (1.3) which satisfies

$$\overline{\lim}_{n \rightarrow \infty} |s_n| = +\infty, \quad (1.45)$$

we have

$$\overline{\lim}_{n \rightarrow \infty} |\sigma_n^{\alpha_n}| = +\infty. \quad (1.46)$$

2. On Divergence of Fourier Series

It is well known the following.

Theorem A (Kolmogorov [1]). *There exists such summable function f that Fourier trigonometric series of f*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad (2.1)$$

unboundedly diverges everywhere.

Let $W = \{w_n(t)\}_{n=1}^{\infty}$ be the Walsh system. Below we formulate Theorem B which is analogous of Theorem A and holds for Fourier-Walsh series.

Theorem B (Schipp [2, 3]). *There exists such summable function g that Fourier-Walsh series of g*

$$\sum_{n=1}^{\infty} a_n \omega_n(t) \quad (2.2)$$

unboundedly diverges everywhere.

Let $\Phi = \{\varphi_n(t)\}$ be orthonormal functions system defined on $[0, 1]$, such that

$$|\varphi_n(t)| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \dots \quad (2.3)$$

Then below-mentioned theorem holds.

Theorem C (Bočkarev [4]). *For every orthonormal system Φ which satisfies (2.3), there exists such summable function h defined on $[0, 1]$ that its Fourier series constructed by Φ system*

$$\sum_{n=1}^{\infty} a_n \varphi_n(t) \quad (2.4)$$

unboundedly diverges in any point of some set $E \subset [0, 1]$ with positive measure.

Denote by $\sigma_n^{\alpha_n}(x; f)$, $\sigma_n^{\alpha_n}(t, g, W)$, and $\sigma_n^{\alpha_n}(t, h, \Phi)$ means of series (2.1), (2.2), and (2.4), respectively.

Theorem 1.3 implies that if $\{\alpha_n\}$ satisfies (1.32), then Theorems A, B, and C hold for $(C, \{\alpha_n\})$ summability method.

Namely, the following Theorems hold true.

Theorem 2.1. *Let a sequence $\{\alpha_n\}$ satisfies (1.32). Then there exists such summable function f , that sequence $\{\sigma_n^{\alpha_n}(x; f)\}$ unboundedly diverges everywhere.*

Theorem 2.2. *Let a sequence $\{\alpha_n\}$ satisfies (1.32). Then there exists such summable function g that sequence $\{\sigma_n^{\alpha_n}(t, g, W)\}$ unboundedly diverges everywhere.*

Theorem 2.3. *If orthonormal system Φ satisfies (2.3) and a sequence $\{\alpha_n\}$ satisfies (1.32), then there exists such summable function h , defined on $[0, 1]$, that sequence $\{\sigma_n^{\alpha_n}(t; h; \Phi)\}$ unboundedly diverges at every point of some set $E \subset [0, 1]$ with positive measure.*

It is obvious that a consequence of Theorem 1.4 is that Theorems 2.1, 2.2, and 2.3 hold true if

$$\alpha_n = o\left(\frac{1}{\ln n}\right). \quad (2.5)$$

Remark 2.4. *If every number $\lambda_n(k)$ will be replaced by $(1 - k/(n+1))^{\alpha_n}$ in (1.31), then we get a summability method defined by $\Lambda = \|\lambda_n(k)\|$ matrix, which we call Riesz summability method with variable orders and denote it by symbol $(R, \{\alpha_n\})$.*

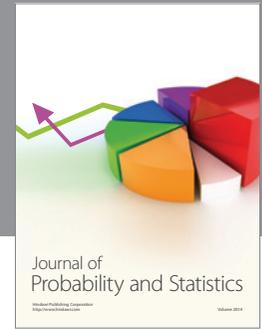
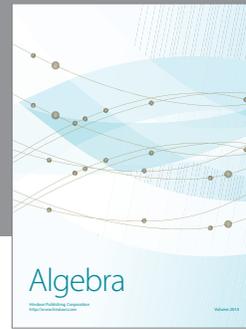
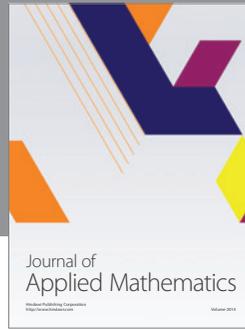
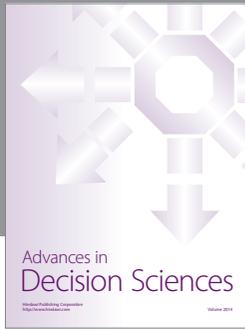
It can be proved analogously that Theorems 2.1, 2.2, and 2.3 remain true for Riesz summability method with variable orders, that is, for $(R, \{\alpha_n\})$ method, where $\{\alpha_n\}$ satisfies (1.32).

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