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Research Article

On Divergence of Fourier Series by Some Methods of Summability

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A new summability method of series is introduced and studied. The particular cases of this method are, for example, variable-order Cesaro and Riesz methods. Applications to divergence problem of Fourier series are given. An extension of Kolmogorov, Schipp, and Bočkarev's well-known theorems on divergence of Fourier trigonometric, Walsh, and orthonormal series is established.

1. A New Summability Method of Series

Let

$$\Lambda = \|\lambda_n(k)\|, \quad n = 0, 1, 2, \dots, \ k = 0, 1, 2, \dots, n, \tag{1.1}$$

be such triangular matrix which satisfies the following conditions:

$$\begin{aligned} &(1)\ 0 \le \lambda_n(k+1) \le \lambda_n(k) \le 1, \quad 0 \le k \le n; \\ &(2)\ \lambda_n(0) = 1, \qquad \lambda_n(k) = 0, \quad k \ge n+1. \end{aligned} \tag{1.2}$$

By s_n we denote a partial sum of a series

$$\sum_{k=0}^{\infty} u_k, \tag{1.3}$$

and by σ_n we denote a mean constructed by the Λ matrix, that is,

$$s_n = \sum_{k=0}^{n} u_k, \qquad \sigma_n = \sum_{k=0}^{n} \lambda_n(k) u_k. \tag{1.4}$$

Theorem 1.1. *Let matrix* (1.1) *satisfies an inequality*

$$\underline{\lim}_{n\to\infty} \lambda_n(n) > \frac{1}{2}. \tag{1.5}$$

Then for any series (1.3) *which satisfies the following condition:*

$$\overline{\lim}_{n\to\infty} |s_n| = +\infty, \tag{1.6}$$

an equality

$$\overline{\lim}_{n\to\infty} |\sigma_n| = +\infty \tag{1.7}$$

holds.

Below we prove a Lemma which is used to prove Theorem 1.1.

Lemma 1.2. *For every natural number n an inequality*

$$|s_n - \sigma_n| \leq 2(1 - \lambda_n(n)) \cdot \max_{1\leq k\leq n} |s_k| \tag{1.8}$$

holds.

Proof of the Lemma. Using Abel transformation and $\lambda_n(0) = 1$ we get

$$\begin{aligned} s_n - \sigma_n &= \sum_{k=0}^{n} u_k - \sum_{k=0}^{n} \lambda_n(k) u_k \\ &= \sum_{k=1}^{n} u_k - \sum_{k=1}^{n} \lambda_n(k) u_k \\ &= \sum_{k=1}^{n} (1 - \lambda_n(k)) u_k \\ &= \sum_{k=1}^{n-1} (\lambda_n(k+1) - \lambda_n(k)) s_k + (1 - \lambda_n(n)) s_n. \end{aligned} \tag{1.9}$$

Therefore,

$$
\begin{aligned}
|s_n - \sigma_n| &\le \sum_{k=1}^{n-1} |\lambda_n(k+1) - \lambda_n(k)| \cdot |s_k| + |1 - \lambda_n(n)| \cdot |s_n| \\
&\le \max_{1\le k\le n} |s_k| \cdot \left(\sum_{k=1}^{n-1} |\lambda_n(k+1) - \lambda_n(k)| + |1 - \lambda_n(n)| \right).
\end{aligned}
\tag{1.10}
$$

Thus, taking into account (1.1) we immediately get

$$
\begin{aligned}
|s_n - \sigma_n| &\le \max_{1\le k\le n} |s_k| \cdot \left(\sum_{k=1}^{n-1} (\lambda_n(k) - \lambda_n(k+1)) + 1 - \lambda_n(n) \right) \\
&= \max_{1\le k\le n} |s_k| \cdot (\lambda_n(1) - \lambda_n(n) + 1 - \lambda_n(n)) \\
&\le \max_{1\le k\le n} |s_k| \cdot (1 - \lambda_n(n) + 1 - \lambda_n(n)) \\
&= 2 \cdot (1 - \lambda_n(n)) \cdot \max_{1\le k\le n} |s_k|.
\end{aligned}
\tag{1.11}
$$

So the Lemma is proved. □

Proof of Theorem 1.1. According to the condition of Theorem 1.1 we have

$$
\varliminf_{n\to\infty} \lambda_n(n) = \frac{1}{2} + \delta \tag{1.12}
$$

for some $\delta > 0$. Note that inequalities $0 \le \lambda_n(n) \le 1$ which hold for every natural n imply $1/2 + \delta \le 1$, that is, $\delta \le 1/2$.

So, $0 < \delta \le 1/2$ holds.

According to (1.12) there exists a natural number n_0 such that for every natural number $n > n_0$ we have

$$
\lambda_n(n) > \frac{1}{2} + \frac{\delta}{2}. \tag{1.13}
$$

So according to the Lemma, for every $n > n_0$ an inequality

$$
|s_n - \sigma_n| < 2 \cdot \left(1 - \left(\frac{1}{2} + \frac{\delta}{2} \right) \right) \cdot \max_{1\le k\le n} |s_k| \tag{1.14}
$$

holds true; that is, if $n > n_0$, then

$$
|s_n - \sigma_n| < (1 - \delta) \cdot \max_{1\le k\le n} |s_k|. \tag{1.15}
$$

Thus for every $n > n_0$ an inequality

$$||s_n| - |\sigma_n|| < (1-\delta) \cdot \max_{1\le k\le n} |s_k| \tag{1.16}$$

holds.

So for every $n > n_0$ we have

$$|\sigma_n| > |s_n| - (1-\delta) \cdot \max_{1\le k\le n} |s_k|. \tag{1.17}$$

Note that for every natural n there exists at least one natural number $1 \le q \le n$, such that the partial sums of the series (1.3) satisfy the following condition:

$$\left|s_q\right| = \max_{1\le k\le n} |s_k|. \tag{1.18}$$

We define p_n by a formula:

$$p_n = \max\left\{ q : \ 1 \le q \le n \ \& \ \left|s_q\right| = \max_{1\le k\le n} |s_k| \right\}. \tag{1.19}$$

So p_n is maximal number among the above-mentioned natural q numbers. Consequently,

$$1 \le p_n \le n, \qquad \left|s_{p_n}\right| = \max_{1\le k\le n} |s_k|, \tag{1.20}$$

$$p_n \le p_{n+1}, \qquad \left|s_{p_n}\right| \le \left|s_{p_{n+1}}\right|. \tag{1.21}$$

According to the condition of Theorem 1.1,

$$\overline{\lim_{n\to\infty}} |s_n| = +\infty. \tag{1.22}$$

Therefore,

$$\lim_{n\to\infty} \left|s_{p_n}\right| = +\infty, \tag{1.23}$$

that is,

$$\lim_{n\to\infty} p_n = +\infty. \tag{1.24}$$

A consequence of (1.24) is that there exists such natural n_1 that if $n > n_1$ then $p_n > n_0$ and since (1.17) holds for every $n > n_0$, then (1.17) remains true for every p_n, where $n > n_1$.

So

$$\left|\sigma_{p_n}\right| > \left|s_{p_n}\right| - (1-\delta) \cdot \max_{1\le k\le p_n} |s_k|. \tag{1.25}$$

Since $1 \le p_n \le n$, therefore,

$$\max_{1\le k\le p_n} |s_k| \le \max_{1\le k\le n} |s_k|. \tag{1.26}$$

Note that the last one and (1.25) imply

$$\left|\sigma_{p_n}\right| > \left|s_{p_n}\right| - (1-\delta)\cdot \max_{1\le k\le n}|s_k|. \tag{1.27}$$

So according to (1.21) we have

$$\left|\sigma_{p_n}\right| > \left|s_{p_n}\right| - (1-\delta)\cdot \left|s_{p_n}\right|, \tag{1.28}$$

that is, for every $n > n_1$ an inequality

$$\left|\sigma_{p_n}\right| > \delta \cdot \left|s_{p_n}\right| \text{ holds,} \quad \text{where } 0 < \delta \le \frac{1}{2}. \tag{1.29}$$

Also, (1.23) and (1.29) imply

$$\lim_{n\to\infty} \left|\sigma_{p_n}\right| = +\infty. \tag{1.30}$$

So we have finished the proof of Theorem 1.1. □

Below we consider some consequences of Theorem 1.1.

Let $\Lambda = \|\lambda_n(k)\|$ be a triangular matrix, where the sequence $\{\alpha_n\}$ is from $[0,1]$ and for every $0 \le k \le n$ number $\lambda_n(k)$ is defined by the formula:

$$\lambda_n(k) = \frac{A_{n-k}^{\alpha_n}}{A_n^{\alpha_n}}, \quad \text{where } A_n^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)\cdots(\alpha_n+n)}{n!}. \tag{1.31}$$

If $\alpha_n = \alpha$, for every $n \ge 0$ and (1.31) holds true, then the Λ method is Cesaro (C,α) summability method, and if $\alpha_n \equiv 0$, then the Λ method coincides with convergence.

We introduce Cesaro summability method with variable orders, denoted by a symbol $(C,\{\alpha_n\})$, which coincides with Λ summability method defined by (1.31). Means of this method for series (1.3) we denoted by $\sigma_n^{\alpha_n}$.

For $(C,\{\alpha_n\})$ we have the following.

Theorem 1.3. *Let a sequences $\{\alpha_n\}$ be such that for some positive number m we have*

$$\alpha_n \le \frac{c}{\ln n}, \quad \textit{where } 0 \le c < \ln 2 \textit{ and } n > m. \tag{1.32}$$

Then for any series (1.3) *which satisfies the following condition:*

$$\overline{\lim_{n\to\infty}} |s_n| = +\infty, \tag{1.33}$$

an equality

$$\overline{\lim_{n\to\infty}} \left|\sigma_n^{\alpha_n}\right| = +\infty \tag{1.34}$$

holds.

Proof of Theorem 1.3. Note that every $\lambda_n(k)$ satisfies condition (1.1) and condition (1.3). Indeed,

$$\frac{\lambda_n(k+1)}{\lambda_n(k)} = \frac{A_{n-k-1}^{\alpha_n}}{A_{n-k}^{\alpha_n}} = \frac{n-k}{\alpha_n + n - k} \leq 1 \tag{1.35}$$

and $\lambda_n(0) = 1$, when $n \geq 0$.

For every $n \geq 1$ we have

$$\lambda_n(n) = \frac{1}{A_n^{\alpha_n}}, \qquad \text{where } A_n^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)\cdots(\alpha_n+n)}{n!}, \tag{1.36}$$

that is,

$$A_n^{\alpha_n} = \left(1+\frac{\alpha_n}{1}\right)\left(1+\frac{\alpha_n}{2}\right)\cdots\left(1+\frac{\alpha_n}{n}\right). \tag{1.37}$$

Therefore,

$$\ln A_n^{\alpha_n} = \sum_{k=1}^{n} \ln\left(1+\frac{\alpha_k}{n}\right) < \sum_{k=1}^{n} \frac{\alpha_n}{k} = \alpha_n \cdot \sum_{k=1} \frac{1}{k} < \alpha_n(1+\ln n). \tag{1.38}$$

Note that the last one and (1.32) imply that

$$c = \ln\frac{2}{1+\gamma}, \qquad \text{for some } 0 < \gamma \leq 1, \tag{1.39}$$

and if $n > m$, we have

$$\begin{aligned} A_n^{\alpha_n} &< e^{\alpha_n(1+\ln n)} = e^{\alpha_n} \cdot e^{\alpha_n \ln n} \\ &\leq e^{\alpha_n} \cdot e^{c} = e^{\alpha_n} \cdot e^{\ln(2/(1+\gamma))} = e^{\alpha_n} \cdot \frac{2}{1+\gamma}, \end{aligned} \tag{1.40}$$

that is,

$$\lambda_n(n) = \frac{1}{A_n^{\alpha_n}} > \frac{1}{e^{\alpha_n}} \cdot \left(\frac{1}{2}+\frac{\gamma}{2}\right), \qquad \text{where } \gamma > 0. \tag{1.41}$$

Note that $\alpha_n \to 0$ implies the existence of such $\gamma_1 > 0$ and natural n_2, that if $n > n_2$, then

$$\frac{1}{e^{\alpha_n}} \cdot \left(\frac{1}{2} + \frac{\gamma}{2}\right) > \frac{1}{2} + \gamma_1, \tag{1.42}$$

that is, if $n > n_2$, then

$$\lambda_n(n) > \frac{1}{2} + \gamma_1. \tag{1.43}$$

A consequence of (1.43) is that if (1.32) holds, then the Λ matrix satisfies conditions of Theorem 1.1. This completes the proof of Theorem 1.3. □

Theorem 1.3 directly implies the following.

Theorem 1.4. *Let $\{\alpha_n\}$ be such sequence that*

$$\{\alpha_n\} = o\left(\frac{1}{\ln n}\right). \tag{1.44}$$

Then for every series (1.3) *which satisfies*

$$\overline{\lim_{n\to\infty}} |s_n| = +\infty, \tag{1.45}$$

we have

$$\overline{\lim_{n\to\infty}} \left|\sigma_n^{\alpha_n}\right| = +\infty. \tag{1.46}$$

2. On Divergence of Fourier Series

It is well known the following.

Theorem A (Kolmogorov [1]). *There exists such summable function f that Fourier trigonometric series of f*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{2.1}$$

unboundedly diverges everywhere.

Let $W = \{w_n(t)\}_{n=1}^{\infty}$ be the Walsh system. Below we formulate Theorem B which is analogous of Theorem A and holds for Fourier-Walsh series.

Theorem B (Schipp [2, 3]). *There exists such summable function g that Fourier-Walsh series of g*

$$\sum_{n=1}^{\infty} a_n w_n(t) \tag{2.2}$$

unboundedly diverges everywhere.

Let $\Phi = \{\varphi_n(t)\}$ be orthonormal functions system defined on $[0,1]$, such that

$$\left|\varphi_n(t)\right| \leq M, \quad t \in [0,1], \; n = 1, 2, \ldots \tag{2.3}$$

Then below-mentioned theorem holds.

Theorem C (Bočkarev [4]). *For every orthonormal system Φ which satisfies* (2.3)*, there exists such summable function h defined on $[0,1]$ that its Fourier series constructed by Φ system*

$$\sum_{n=1}^{\infty} a_n \varphi_n(t) \tag{2.4}$$

unboundedly diverges in any point of some set $E \subset [0,1]$ with positive measure.

Denote by $\sigma_n^{\alpha_n}(x; f)$, $\sigma_n^{\alpha_n}(t, g, W)$, and $\sigma_n^{\alpha_n}(t, h, \Phi)$ means of series (2.1), (2.2), and (2.4), respectively.

Theorem 1.3 implies that if $\{\alpha_n\}$ satisfies (1.32), then Theorems A, B, and C hold for $(C, \{\alpha_n\})$ summability method.

Namely, the following Theorems hold true.

Theorem 2.1. *Let a sequence $\{\alpha_n\}$ satisfies* (1.32)*. Then there exists such summable function f, that sequence $\{\sigma_n^{\alpha_n}(x; f)\}$ unboundedly diverges everywhere.*

Theorem 2.2. *Let a sequence $\{\alpha_n\}$ satisfies* (1.32)*. Then there exists such summable function g that sequence $\{\sigma_n^{\alpha_n}(t, g, W)\}$ unboundedly diverges everywhere.*

Theorem 2.3. *If orthonormal system Φ satisfies* (2.3) *and a sequence $\{\alpha_n\}$ satisfies* (1.32)*, then there exists such summable function h, defined on $[0,1]$, that sequence $\{\sigma_n^{\alpha_n}(t; h; \Phi)\}$ unboundedly diverges at every point of some set $E \subset [0,1]$ with positive measure.*

It is obvious that a consequence of Theorem 1.4 is that Theorems 2.1, 2.2, and 2.3 hold true if

$$\alpha_n = o\left(\frac{1}{\ln n}\right). \tag{2.5}$$

Remark 2.4. *If every number $\lambda_n(k)$ will be replaced by $(1 - k/(n+1))^{\alpha_n}$ in* (1.31)*, then we get a summability method defined by $\Lambda = \|\lambda_n(k)\|$ matrix, which we call Riesz summability method with variable orders and denote it by symbol $(R, \{\alpha_n\})$.*

It can be proved analogously that Theorems 2.1, 2.2, and 2.3 remain true for Riesz summability method with variable orders, that is, for $(R, \{\alpha_n\})$ method, where $\{\alpha_n\}$ satisfies (1.32).

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