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FUBINI'S TYPE PHENOMENON FOR CONVERGENT IN PRINGSHEIM SENSE MULTIPLE FUNCTION SERIES

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Abstract. In the present paper ε-uniqueness multiple function systems are considered. A theorem representing a possibility of calculation of the limit of a convergent in the Pringsheim sense multiple function series with respect to an ε-uniqueness multiple function system via application of iterated limits is formulated.

Let $d \geq 2$ be a natural number, R^d be the d-dimensional Euclidean space, Z_0^d be the set of all points in R^d with integer nonnegative coordinates. By $x = (x_1, \dots, x_d)$ we denote the points of the unit cube $[0,1]^d$ and by $m = (m_1, \dots, m_d)$ and $n = (n_1, \dots, n_d)$ those from the set Z_0^d. The symbol $m \to \infty$ means that $m_j \to \infty$ for every j, $1 \leq j \leq d$ independently of each other. μ is the linear Lebesgue measure. $E_1 \times E_2 \times \dots \times E_d$ is the Cartesian product of the sets E_j, where $j = 1, 2, \dots, d$ and $E_j \subset [0,1]$.

Let $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ be a system of measurable and finite functions defined on $[0,1]$. So,

$$|\varphi_i(t)| < \infty, \quad t \in [0,1], \quad i = 0, 1, 2, \dots.$$

Definition 1. A set $A \subset [0,1]$ is called an U set of the system $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ if the convergence of a series $\sum\limits_{i=0}^{\infty} a_i \varphi_i(t)$ to zero on the set $[0,1] \backslash A$ implies that $a_i = 0$ for every $i \geq 0$.

Definition 2. The system $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ is called an ε-uniqueness system if the number $\varepsilon \in (0,1]$ and any set $A \subset [0,1]$ with $\mu A < \varepsilon$ is an U set of $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$.

The expression $\Phi \in U(\varepsilon)$ means, that Φ is an ε-uniqueness system.

Note, that if $0 < \varepsilon < \varepsilon_1 \leq 1$ and $\Phi \in U(\varepsilon_1)$, then $\Phi \in U(\varepsilon)$.

Examples of an ε-uniqueness systems are a lacunary trigonometric system defined on $[0,1]$, with $\varepsilon = 1$ (see [3]) and Rademacher system, with $\varepsilon = \frac{1}{2}$ (see [1]).

Let $\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}$ be a system of measurable and finite on $[0,1]$ functions for every j, where $1 \leq j \leq d$.

Let

$$\phi_n(x) = \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j), \quad x = (x_1, \dots, x_d) \in [0,1]^d$$

for every $n \in Z_0^d$.

Consider the d-multiple series with respect to the system $\overline{\phi} = \{\phi_n(x)\}_{n \in Z_0^d}$,

$$\sum_{n=0}^{\infty} a_n \phi_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\dots,n_d} \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j). \tag{1}$$

By $S_m(x)$ we denote rectangular partial sums of the series (1), i. e.,

$$S_m(x) = \sum_{n_1=0}^{m_1} \cdots \sum_{n_d=0}^{m_d} a_{n_1,\dots,n_d} \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j).$$

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The convergence of the series (1) at the point x means that there exists a finite Pringsheim limit, i. e.,

$$-\infty < \lim_{m\to\infty} S_m(x) < \infty.$$

Let $\{j_1, j_2, \dots, j_d\}$ be a rearrangement of $\{1, 2, \dots, d\}$, then it holds the following Fubini-type

Theorem. *Let for any j, $1 \le j \le d$, the system $\Phi^{(j)}$ be an ε_j-uniqueness system and a set $E_j \subset [0, 1]$ be such that $\mu E_j > 1 - \varepsilon_j$. If there exists*

$$\lim_{m\to\infty} S_m(x), \quad \textit{when} \quad x \in E_1 \times E_2 \times \cdots E_d,$$

then for any $\{j_1, j_2, \dots, j_d\}$ there exists iterated limit

$$\lim_{m_{j_1}\to\infty} \left(\lim_{m_{j_2}\to\infty} \left(\cdots \left(\lim_{m_{j_d}\to\infty} S_m(x) \right) \cdots \right) \right) \quad \textit{when} \quad x \in E_1 \times E_2 \times \cdots E_d$$

and

$$\lim_{m\to\infty} S_m(x) = \lim_{m_{j_1}\to\infty} \left(\lim_{m_{j_2}\to\infty} \left(\cdots \left(\lim_{m_{j_d}\to\infty} S_m(x) \right) \cdots \right) \right)$$

for any $x \in E_1 \times E_2 \times \cdots E_d$.

Remark. Note, that the theorem presented in [2] is a direct consequence of the above formulated theorem when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_d = \varepsilon$.

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References

1. S. B. Stečkin, P. L. Ul'janov, On sets of uniqueness. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 211–222.
2. Sh. Tetunashvili, On the Fubini's sets of multiple function series. *Proc. A. Razmadze Math. Inst.* **143** (2007), 138–139.
3. A. Zygmund, On lacunary trigonometric series. *Trans. Amer. Math. Soc.* **34** (1932), no. 3, 435–446.

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