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ON A SIMPLE STOCHASTIC MODEL OF HAIL CLOUDS EMERGING OVER A CIRCULAR DOMAIN

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Abstract. Under a simple stochastic model of hail clouds emerging over a circular domain C centered at 0 which identifies a cloud with its plane projection rectangle Δ_ω of fixed sizes $2l \times 2h$, $h < l$, centered at ω and is based on the uniformity of the location of ω in the outer parallel set of the basic rectangle Δ_0, the Minkowski sum $\Delta_0 \oplus C$, and the isotropy of rectangle orientation on the one hand and identity of probabilities and stochastic independence of covering the center of C by random rectangles Δ_ω on the other hand, and using the normal approximation for the binomial probability distribution of the random number of such a covering the confidence interval is constructed which gives the bounds for the unknown number of hail clouds over the domain by the number of hail clouds observed over the center.

Encouraged by the advanced studies in stochastic modelling completed by R. Chitashvili and E. Khmaladze at I.Vekua Institute of Applied Mathematics, the author performed the present research at the same institute in early 1970ies. The real problem was posed by G. Sulakvelidze.

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A number of hail clouds is to be estimated over a circular domain C with the center O and radius R at the time interval $[0,T]$ by a number ξ of clouds registered over the point O at the same time interval.

Assume that there are n clouds observable from C each of which at random and independently from others covers the point O with the same probability p. This assumption leads to the following binomial distribution with the parameters n and p for the random variable ξ

$$P(\xi = m) = b(m; n, p) = C_n^m p^m (1-p)^{n-m}, \quad m = 0, 1, \ldots, n.$$

Below we will assign a value to p according to a simple model of hail clouds emerging over C.

It is easy to check that maximum likelihood estimator $\widehat{n}$ for n under the observed value ξ when p is known, i.e., $\arg\max_{n=1,2,\ldots} b(\xi; n, p)$ equals to

$$\widehat{n} = \left[\frac{\xi}{p}\right], \tag{1}$$

where $[x]$ is the integral part of a real number x.

Assume that an unknown n is large enough. The De Moivre–Laplace theorem enables us to write that

$$P\left\{\left|\frac{\xi - np}{\sqrt{npq}}\right| < t\right\} \approx 2\Phi(t)$$

with $q = 1 - p$ and $\Phi(t) = (2\pi)^{-1/2}\int\limits_0^t e^{-u^2/2}du$. Choosing t_α such that $\Phi(t_\alpha) = \alpha/2$, we obtain

$$|\xi - np| < \sqrt{npq}\, t_\alpha$$

with probability α. Solving this inequality with respect to n, we have the following asymptotic confidence interval for n

$$(\xi/p - a(\xi, p, \alpha), \xi/p + b(\xi, p, \alpha))$$

with the confidence probability α, i.e.,

$$P\left(\xi/p - a(\xi, p, \alpha) < n < \xi/p + b(\xi, p, \alpha)\right) \approx \alpha, \tag{2}$$

where

$$a(\xi, p, \alpha) = \frac{\sqrt{t_\alpha^2 q(t_\alpha^2 q + 4\xi)} - t_\alpha^2 q}{2p}\,,\; b(\xi, p, \alpha) = a(\xi, p, \alpha) + \frac{t_\alpha^2 q}{p}\,. \tag{3}$$

Now we assign a meaningful value to the probability p using the notion of geometric probability. Let us identify a cloud with its orthogonal projection onto the plane and assume that the latter is a rectangle having the length $2l$ and width $2h$.

Assume that the cloud is observable from the circle C if the above-mentioned rectangle intersects with the circle. Under registration of the cloud over the point O let us mean the hitting of the point O into the rectangle. Thus we have to calculate the probability that the rectangle $2l \times 2h$ $(l > h)$, randomly chosen from those rectangles which intersect with the circle C of radius R, will cover the center of the circle.

The position of the rectangle on the plane is characterized by that of its center and angle between the fixed line, passing through the point O, and the rectangle basis. By the symmetry, we can fix this angle (and as we will see below, the conditional probability given the angle does not depend on the angle, i.e., by the formula of total probability the unconditional probability is equal to the conditional one).

For any $u = (s, t) \in R^2$ denote $\Delta_u = [s - l, s + l] \times [t - h, t + h]$ the rectangle of fixed sizes with the center at u, Δ_0 being the basic rectangle $[-l, l] \times [-h, h]$. Evidently, $u + \Delta_0 = \Delta_u$ and the inclusions $u \in \Delta_v$ and $v \in \Delta_u$ are equivalent for any two $u, v \in R^2$.

Let us now construct the set Ω of positions of the rectangle center $\omega = (x, y)$ when the rectangle Δ_ω intersects with the circle C of radius R. Place the origin of the Cartesian coordinate system at O and assume that the Ox-axis is a straight line for the angle counting out. For the sake of simplicity, we assume that the angle between the rectangle basis and the Ox-axis is equal to zero.

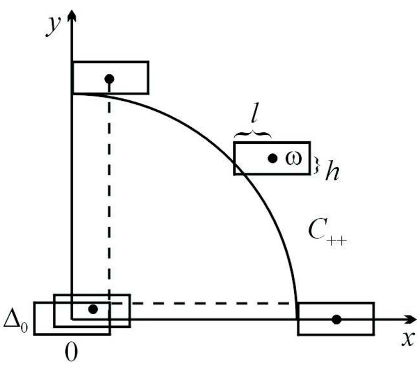

Fig. 1

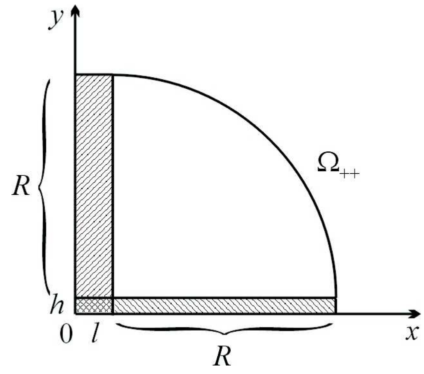

Fig. 2

From the definition of the Minkowski sum $A \oplus B = \{a+b | a \in A, b \in B\}$ of two sets A and B in Euclidean space (see, e.g., [1], [2], [3]) it is easy to derive the representation

$$\Omega = C \oplus \Delta_0$$

for the set

$$\Omega = \{\omega | \Delta_\omega \cap C \neq \varnothing\}$$

as Minkowski sum of the basic rectangle Δ_0 and the given circle C called an outer parallel set of Δ_0 [2].

Indeed, $C \oplus \Delta_0 = \cup_{c \in C}(\Delta_0 + c) = \cup_{c \in C} \Delta_c$ and we obtain the following sequence of equivalent assertions, which proves the desired representation.

$$\omega \in C \oplus \Delta_0 \Leftrightarrow [\exists\, c \in C : \ \omega \in \Delta_c] \ \Leftrightarrow [\exists\, c \in C : \ c \in \Delta_\omega] \ \Leftrightarrow C \cap \Delta_\omega \neq \varnothing \Leftrightarrow \omega \in \Omega.$$

If instead of Δ_0 a general convex set K is meant and K_R denotes its outer parallel set on the distance R, then according to [2, Ch. I, §2] we have the following formulas for the perimeter L_R and area (Lebesgue measure) of K_R:

$$L_R = L + 2\pi R, \quad F_R = F + LR + \pi R^2, \tag{4}$$

where L is the perimeter of K and F is its area.

Figures 1 and 2 visualize the definition and structure of Ω. It is done for the first quadrant and C_{++} and Ω_{++} stand, respectively, for the parts of C and Ω from this quadrant. When ω belongs to the basic rectangle Δ_0 shown on Fig. 1 the suitable part of which is twice shaded on Fig. 2, then the rectangle Δ_ω covers the point O.

Thus if we assume that all the positions of ω are uniformly distributed on Ω for the probability that a random rectangle Δ_ω covers the point O we obtain from (4)

$$p_{l,h} = \frac{F}{F_R} = \frac{4lh}{4lh + 4(l+h)R + \pi R^2}\,. \tag{5}$$

(The notation $p_{l,h}$ emphasizes that the probability is calculated for rectangles of fixed sizes.) If extra randomness is introduced assuming that l and h are random variables with a known joint distribution, then the unknown probability would be equal to mathematical expectation $p = E(p_{l,h})$.

Note that if we can indicate a priori the numbers $\boldsymbol{l}_0$, $\boldsymbol{l}$, $\boldsymbol{h}_0$, $\boldsymbol{h}$, such that

$$\boldsymbol{l}_0 < l < \boldsymbol{l}, \quad \boldsymbol{h}_0 < h < \boldsymbol{h}, \quad \boldsymbol{l} \ll R, \quad \boldsymbol{h} \ll R, \tag{6}$$

then

$$p \approx 4\,\frac{E(lh)}{\pi R^2}\,. \tag{7}$$

But if l and h are not correlated, then

$$p \approx 4\,\frac{E(l)E(h)}{\pi R^2}\,. \tag{8}$$

The expectations $E(lh)$, $E(l)$, $E(h)$ may be unknown but on the basis of suitable sampling data they can be approximated reliably by the empirical means $\overline{lh}$, $\bar{l}$ and $\bar{h}$. Thus with a high reliability

$$p \approx 4\,\frac{\overline{lh}}{\pi R^2}\,, \tag{7$'$}$$

and in the case of uncorrelated l and h, when

$$p \approx 4\,\frac{\bar{l}\,\bar{h}}{\pi R^2}\,, \tag{8$'$}$$

(8′) can be obtained by the choice from the very beginning of a rectangle of sizes $2\bar{l} \times 2\bar{h}$ by passing from

$$p \approx \frac{4\bar{l}\,\bar{h}}{4\bar{l}\,\bar{h} + 4(\bar{l} + \bar{h})R + \pi R^2} \tag{5$'$}$$

to (8′) under the condition (6).

The set of formulas (1)–(3), (5), (5′) and (8′) allow us to estimate the unknown number n.

Remark 1. According to my best knowledge no proper application of the proposed technique was done while there exists an example of misuse of some meteorological data. G. Sulakvelidze has had intentions to collect data to test model quality by comparison the values of areas damaged by hail and its model values. Many serious but unsuccessful efforts were undertaken by J. Mdinaradze to collect the official and research data to be treated by the presented technique.

Remark 2. Note that if in a role of basic set Δ_0 one takes the circle of radius l or ellipse with half-axes l and h ($h < l$) one obtains some meaningful extensions of our model which may have an interest for, say, biological, ecological and even meteorological modelling. For the case of circle our ratio equals to

$$p_l = \frac{F}{F_R} = \left(\frac{l}{R + l}\right)^2.$$

As for ellipse, we have

$$p_{l,h} = \frac{F}{F_R} = \frac{\pi lh}{\pi lh + 4lE(e)R + \pi R^2} ,$$

where $E(e)$ stands for the complete elliptic integral of the second kind and $e = \frac{\sqrt{l^2 - h^2}}{l}$ for the eccentricity of ellipse.

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