Proceedings of A. Razmadze
Mathematical Institute
Vol. **150** (2009), 99–104

ON SOME LIMIT THEOREMS FOR SUMS AND PRODUCTS OF RANDOM VARIABLES

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Abstract. Central limit theorem for discounted sums of i.i.d. random vectors with a discount matrix [9] is extended to the case of a finite number of periodically varying discount matrices. Limiting behaviour of products of positive random variables is discussed which are conditionally independent and controlled by a finite Markov chain.

რეზიუმე. ცენტრალური ზღვარითი თეორემა დამოუკიდებელი ერთნაირად განაწილებული შემთხვევითი ვექტორების დისკონტირებული ჯამებისათვის მადისკონტირებელი მატრიცით [9] გადატანილია პერიოდულად ცვალებადი მადისკონტირებელი მატრიცების სასრული რაოდენობისათვის. განიხილება მარკოვის სასრული ჯაჭვით მართვადი პირობით დამოუკიდებელი დადებითი შემთხვევითი სიდიდეების ნამრავლთა ზღვარითი ყოფაქცევა.

1. Consider a sequence $X_0, X_1, \dots$ of i.i.d. m-dimensional random vectors on a probability space (Ω, F, P) such that $EX_0 = 0$, $\operatorname{cov}(X_0) = R$, $\operatorname{sp} R = \sigma^2 < \infty$.

For $0 < A < 1$ and $m = 1$ the so called discounted sum

$$\eta_A = \sum_{j=0}^{\infty} A^j X_j$$

may be interpreted as the present value of the consecutive payments X_0, $X_1, \dots$ with the discount factor A. In [2] Gerber proved that the probability distribution of the normalized sum $(1-A^2)^{1/2}\eta_A$ tends weakly to the normal law $N(0, \sigma^2)$ when $A \to 1^-$.

In the case $m > 1$ and an $m \times m$-matrix valued A the random vector η_A, where $A^0 = I$, $A^{j+1} = AA^j$, $j \geq 1$, and I stands for the identity $m \times m$-matrix, may have a lot of similar or other interpretations. Recently in [9] the following result was proved.

2000 *Mathematics Subject Classification.* 60F05, 60G50, 60G10, 91B70.

Key words and phrases. Limit theorems for discounted sums, periodically varying discount matrices, conditionally independent factors, markovian switching of distributions, limit theorem for products.

Theorem A. *If for fixed R and c, $1 \le c < \infty$, A takes its values in the set of $m \times m$-matrices*

$$\mathbf{A}(R,c) := \{A : \|A\| < 1,\ A = A^\top,\ AR = RA,\ \|I - A\| \le c(1 - \|A\|)\} \quad (1)$$

and $A \to I$ in the sense that $\|I - A\| \to 0$,then the probability distribution of $\zeta_A := (1 - A^2)^{1/2}\eta_A$ tends weakly to $N(0,R)$.

The theorem covers the case of positive scalar matrices $A = aI$ (with $0 < a < 1$ and $c = 1$) and diagonal ones with at least two different diagonal elements $a_{\max}$ and $a_{\min}$, in which case $c \ge (1 - a_{\min})/(1 - a_{\max}) > 1$, both (the latter is not less than $1 - c(1 - a_{\max})$) tending to 1 from the left.

When several discount matrices are chosen periodically, we have the following assertion (emphasizing scalar normalization in most transparent case of the above-mentioned scalar matrices).

Theorem 1. *If $k \ge 1$ and $B_l \in \mathbf{A}(R,c)$, the set defined by* (1) *in Theorem A, $B_l \to I$, $l = 1, \dots, k$, then for the discounted sum*

$$\eta_B := \sum_{j=0}^{\infty} A_j^j X_j,\ \ A_j = B_l, j \equiv (l-1) \mod k,\ \ l = 1, \dots, k, \qquad (2)$$

the probability distribution of the normalized sum

$$\zeta_B := \Big[\sum_{l=1}^{k} B_l^{2(l-1)}(I - B_l^{2k})^{-1}\Big]^{-1/2}\eta_B \qquad (3)$$

converges weakly to $N(0,R)$.

In the special case of scalar matrices $B_l = b_l I$, $b_l \to 1^-$, $l = 1, \dots, k$, the assertion holds for

$$\zeta_B = \Big[\sum_{l=1}^{k} b_l^{2(l-1)}(1 - b_l^{2k})^{-1}\Big]^{-1/2}\eta_B. \qquad (4)$$

Proof. Let $\xi_u, u \in T$, and $\eta_v, v \in T$, be two independent families of R^m-valued random variables with zero means and nonsingular covariance matrices, with an index set T being a closed subset of a metric space and let the convergence of u and v to an element $\theta \in T$ be considered in the sense of the suitable metric. Let there exist $m \times m$ matrices K_u, L_v such that the probability distributions of $K_u\xi_u$ and $L_v\eta_v$ tend in distribution to $N(0,R)$ as u and v tend to θ. If we assume $cov(K_u\xi_u) = cov(L_v\eta_v) = R$ where R is symmetric and positive definite, then both normalizing matrices K_u, L_v could be assumed symmetric and positive definite and if both commute with the covariance matrix R we have the relations $cov(\xi_u) = (K_u)^{-2}R$, $cov(\eta_v)(L_v)^{-2}R$.

Let us now ask for a matrix $M_{u,v}$ which ensures weak convergence of $M_{u,v}[\xi_u + \eta_v]$ to the same $N(0,R)$ as u and v simultaneously tend to θ.

Evidently it can be symmetric and if so should be such that

$$M_{u,v}[(K_u)^{-1}R(K_u)^{-1} + (L_v)^{-1}R(L_v)^{-1}]^{1/2} = R^{1/2}.$$

In the case when both K_u, L_v commute with R this leads to

$$M_{u,v} = [(K_u)^{-2} + (L_v)^{-2}]^{-1/2}. \tag{5}$$

As for scalar matrices $K_u = k_u I$, $L_u = l_u I$ we obtain $M_{u,v} = m_{u,v} I$ with $m_{u,v} = k_u l_v (k_u^2 + l_v^2)^{-1/2}$. The latter case is easy to treat by using characteristic functions (cf. [6]).

Similar relations hold for the sum of $k > 2$ independent sequences of random vectors and their normalizing matrices providing limiting $(0, R)$-normality for each of them.

Due to well-known facts of convergence of series of independent random vectors the convergent series η_B can be represented as the sum

$$\eta_B = \eta_{B_1}^{(1)} + \dots + \eta_{B_k}^{(k)} \tag{6}$$

of the independent random vectors(convergent series)

$$\eta_{B_l}^{(l)} := B_l^{l-1} \sum_{j=0}^{\infty} B_l^{kj} X_{kj+l-1}, \quad l = 1, \dots, k, \tag{7}$$

with covariance matrices $B_l^{2(l-1)}(I - B_l^{2k})^{-1}R$, $l = 1, \dots, k$, this representation being valid since each B_l is symmetric and commutes with R.

Theorem A yields the weak convergence of distribution of $B_l^{-(l-1)}\eta_{B_l}^{(l)}$ to $N(0, R)$ when $B_l \to I$ through $\mathbf{A}(R, c)$ (see(1)) and, as follows from $B_l^{-(l-1)} \to I$, the same for $\eta_{B_l}^{(l)}$, $l = 1, \dots, k$, given by (7). What about the limiting $(0, R)$-normality of the normalized sum (3), it follows by means of the above-declared statement similar to (5) from the representation (6) of (2) as the sum of k independent random vectors (7). As for (4) it readily follows from (3). □

A finite time horizon of discounted sums could be treated via criteria of normal convergence given in [7] devoted to the study of matrix-weighted sums of i.i.d. random vectors. This problem as well as combining the approach of both [7] and [9] to cover the case of infinite arrays of weight matrices and corresponding sums will be the subject of a forthcoming study of the second author.

2. Second task we deal with concerns a product of random variables, which, e.g., is applicable when treating reinvestment problem [5].

Consider a stationary two-component sequence (ξ_j, X_j), $j = 1, 2, \dots$, where ξ_j takes its values in $\{1, \dots, s\}$ and X_j is a real random variable;

denote

$$\xi = (\xi_1, \xi_2, \dots), \ \ \xi_{1n} = (\xi_1, \dots, \xi_n),$$
$$X = (X_1, X_2, \dots), \ \ X_{1n} = (X_1, \dots, X_n).$$

One says that X is a sequence of conditionally independent random variables controlled by a sequence ξ if for any natural n the conditional distribution $\mathcal{P}_{X_{1n}|\xi_{1n}}$ of X_{1n} given ξ_{1n} is the direct product of conditional distributions of X_j given only the corresponding ξ_j, $j = 1, \dots, n$, i.e.,

$$\mathcal{P}_{X_{1n}|\xi_{1n}} = \mathcal{P}_{\xi_1} \times \cdots \times \mathcal{P}_{\xi_n},$$

where $\mathcal{P}_i$ is the conditional distribution of X_1 given $\{\xi_1 = i\}$, $i = 1, \dots, s$ (see, e.g., [1, 8]). For $s = 1$ X becomes a sequence of i.i.d. random variables with $\mathcal{P}_1$ as a common distribution.

When the random variables X_j are positive consider the product $T_n = X_1 \cdots X_n$ in the case when the controlling sequence ξ is a regular Markov chain for which $\{1, \dots, s\}$ is the only ergodic class. A limiting behavior of this product is easy to describe using limit theorems for sums of such summands, called usually random variables defined on the Markov chain (we mention here the works by Ibragimov and Linnik (1965), Aleshkevichus (1966), O'Brien (1974), Koroliuk and Turbin (1976), Grigorescu and Oprisan (1976), Sirazhdinov and Formanov (1978), Silvestrov (1982), Anisimov (1982), Bokuchava (1984) and others; for the exact references see, e.g., [1]).

Let $\pi_i = P\{\xi_1 = i\}$, $i = 1, \dots, s$, be a common distribution of ξ_js, $Z = (z_{il},\, i, l = 1, \dots, s)$ be the fundamental matrix of the Markov chain ξ. Denote

$$\mu_i = E(\ln X_1 \mid \xi_1 = i), \ \ \sigma_i^2 = E[(\ln X_1 - \mu_i)^2 \mid \xi_1 = i], \ \ i = 1, \dots, s,$$
$$\mu = E \ln X_1 = \sum_{i=1}^{s} \pi_i \mu_i, \ \ \sigma_0^2 = \sum_{i=1}^{s} \pi_i \sigma_i^2,$$
$$t = \sum_{i,l=1}^{s} (\pi_i z_{il} + \pi_l z_{li} - \pi_i \delta_{il} - \pi_i \pi_l) \mu_i \mu_l$$

and for a real x let $N(x \mid 0, b)$ be the $(0, b)$-normal distribution function.

Consider the sum $S_n = \frac{1}{\sqrt{n}} \sum\limits_{j=1}^{n} (\ln X_j - \mu)$ and split it into two uncorrelated sums

$$S_n = S_{n1} + S_{n2},$$

where

$$S_{n1} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\ln X_j - \mu_{\xi_j}], \quad S_{n2} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\mu_{\xi_j} - \mu].$$

The following theorem is valid, where $\xrightarrow{w}$ stands for weak convergence as $n \to \infty$(see [1]).

Theorem B. *If $\sigma_0^2 < \infty$, then we have:*
1) $\mathcal{P}_{S_{n1}|\xi_{1n}} \xrightarrow{w} N(0, \sigma_0^2)$ *P-a.s.;*
2) $\mathcal{P}_{S_{n1}} \xrightarrow{w} N(0, \sigma_0^2)$*;*
3) $\mathcal{P}_{S_n} \xrightarrow{w} N(0, \sigma_0^2 + t)$*.*

This theorem yields the desired asymptotic behavior of T_n which refines convergence in probability in an analogue of the law of large numbers for the geometric mean of our sequence of conditionally independent random variables $X_1, \dots, X_n$ controlled by the Markov chain in the sense of markovian switching of distributions of X_j (which is easy to derive from the Khintchine theorem) asserting that $T_n^{1/n} e^{-\mu}$ tends to 1 in probability. In financial modelling the geometric mean $G_n = T_n^{1/n}$ can be interpreted as the mean discount factor or mean interest rate in the time horizon $[1, n]$.

Theorem 2. *If $\sigma_0^2 < \infty$, then for $x > 0$ we have as $n \to \infty$:*
1) $P\Big\{\Big[e^{-\sum_{j=1}^{n} \mu_{\xi_j}} T_n\Big]^{1/\sqrt{n}} < x \mid \xi_{1n}\Big\} \to N(\ln x \mid 0, \sigma_0^2)$ *P-a.s.;*
2) $P\Big\{\Big[e^{-\sum_{j=1}^{n} \mu_{\xi_j}} T_n\Big]^{1/\sqrt{n}} < x\Big\} \to N(\ln x \mid 0, \sigma_0^2)$*;*
3) $P\left\{(e^{-n\mu} T_n)^{1/\sqrt{n}} < x\right\} = P\left\{(e^{-\mu} G_n)^{\sqrt{n}} < x\right\} \to N(\ln x \mid 0, \sigma_0^2 + t)$.

Example. Let X_j, $j = 1, 2, \dots$, be i.i.d. positive random variables and ν_p be an independent on this sequence geometric random variable with a parameter p. In [5] motivated by the interpretation of T_{ν_p} as the total return after continued reinvestment in the same type of business beginning with unit capital $X_0 = 1$ with equal break-off probability at each step, the distributions of this and related products are studied, particularly for $p \to 0$.

Instead of independent environment let us consider the environment described by the above mentioned stationary Markov chain with $s = 2$ states and the transition matrix
$$\begin{pmatrix} 1-c & c \\ d & 1-d \end{pmatrix},$$
where $0 < d \le 1$, $0 < c \le 1$, $c + d < 2$. For $c + d = 2$, the chain reduces to the alternating sequence and for $c + d = 1$ to the independent Bernoulli sequence. For this chain $\pi_1 = d/(c+d)$, $\pi_2 = c/(c+d)$.

Let the corresponding conditional distributions be the uniform ones in $[0, \alpha]$ and $[0, \beta]$, respectively, $0 < \alpha < \beta \le 1$.Thus $\mu_1 = \ln\alpha - 1$, $\mu_2 = \ln\beta - 1$, $\sigma_1^2 = \sigma_2^2 = 1 = \sigma_0^2$, $\mu = \pi_1 \ln\alpha + \pi_2 \ln\beta - 1$ and as it follows from [4, Ch. IV]
$$t = cd(2 - c - d)(c + d)^{-3} \ln^2(\alpha/\beta).$$

For T_n Theorem 2 holds with these concrete μ_1, μ_2, μ, $\sigma_0^2 = 1$, t. Note that for the alternating sequence $t = 0$ and $\mu = \ln\sqrt{\alpha\beta} - 1$.

When $s = 1$ and the common distribution of i.i.d. X_js is the uniform one in $[0, 1]$, the random variable $-\ln T_n$ has the Erlang distribution with parameters n and 1 [3], which is approximated by Theorem 2 ($t = 0$).

Acknowledgement

The second author was partially supported by the Georgian National Science Foundation (grant GNSF/ST07/3-172).

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(Received 05.01.2009)

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