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Mathematics

The Weighted Reverse Poincare Inequality for the Difference of Two Weak Subsolutions

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ABSTRACT. We consider weak subsolutions of the linear second order uniformly elliptic partial differential equation of general type in a ball.

We establish a new type weighted reverse Poincare inequality for the difference of two continuous weak subsolutions.

The prototype of this inequality for univariate convex functions was proved by Shashiashvili (2005). *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *weak L-subsolutions, reverse Poincare inequality, gradient estimate.*

1. Introduction

Consider two arbitrary finite convex functions $f(x)$ and $\varphi(x)$ on a closed interval $[a,b]$.

The following energy inequality has been established by K. Shashiashvili and M. Shashiashvili in [1] (see Theorem 2.1)

$$\int_a^b (x-a)^2(b-x)^2\left(f'(x-)-\varphi'(x-)\right)^2 dx \le$$

$$\le \frac{8}{9}\sqrt{3}\sup_{x\in(a,b)}\left|f(x)-\varphi(x)\right|\sup_{x\in(a,b)}\left|f(x)+\varphi(x)\right|(b-a)^3+\frac{4}{3}\left(\sup_{x\in(a,b)}\left|f(x)-\varphi(x)\right|^2\right)(b-a)^3. \tag{1.1}$$

Later this kind of estimate with a family of weight functions and on an infinite interval $[0,\infty)$ was proved by Hussain, Peĉariè and Shashiashvili [2].

The natural generalization of univariate convex functions to several variables case are subharmonic functions that share many convenient attributes of the former ones. An extensive study of the properties of subharmonic functions is conducted in the manual [3] by Lars Hörmander (see Chapter 3 of it).

Consider a sequence of subharmonic functions $u_m(x)$, $m=1,2,\dots$, in a ball $B=B(x_0,R)$, which converges to subharmonic function $u(x)$ in $L^1_{loc}(B)$. Then the Theorem 3.2.13 in Hörmander [3] asserts that the weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$, $i=1,\ldots,n$, tend to $\frac{\partial u(x)}{\partial x_i}$, $i=1,\ldots,n$ in $L^p_{loc}(B)$ for the exponents p with $1\le p<\frac{n}{n-1}$.

The next Proposition 3.4.19 in the same manual concerns a sequence of bounded nonpositive subharmonic functions $u_m(x)$ in a ball *B*, such that $u_m(x)|_{\partial B}=0$ and $\operatorname{supp}\Delta u_m(x)$ is contained in a fixed compact set $K\subset B$ (here Δ denotes the famous Laplace operator).

It is proved that if $u_m(x)\downarrow u(x)$ when $m\to\infty$, then the weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$, $i=1,\ldots,n$ converge to $\frac{\partial u(x)}{\partial x_i}$, $i=1,\ldots,n$ in $L^2(B)$.

So it seems natural to ask whether the mapping $u(x)\to \operatorname{grad} u(x)$ possesses certain Hölder continuity property when restricted to the family of subharmonic functions on a ball *B*.

Throughout the paper $B=B(x_0,R)$ will denote the open ball in $\mathbb{R}^n$ with center x_0 and radius *R* and by $\overline{B}=\overline{B}(x_0,R)$ its closure.

Further *C*(*B*) will denote the space of continuous functions on *B* and $L^\infty(B)$ is the space of (a.e.) bounded functions on *B*.

$C_0^k(B)$ will mean the space of *k* times continuously differentiable functions with compact support in *B*, where $k=1,2,\ldots\infty$.

Littman [4] gave a very fruitful generalization of the notion of subharmonic function to the case of general type second order linear elliptic partial differential operators. According to Littman [4] the locally integrable function $u(x)$ defined in a ball *B* is called generalized subharmonic function if for all nonnegative functions $v(x)\in C_0^2(B)$ the following inequality does hold

$$\int_B u(x)L^*v(x)dx\ge 0 \tag{1.2}$$

(that is $Lu(x)\ge 0$ in the sense of the theory of distributions), where

$$Lu(x)=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i}+c(x)u(x) \tag{1.3}$$

and $L^*u(x)$ is the adjoint to the operator $Lu(x)$

$$L^*u(x)=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i^*(x)\frac{\partial u(x)}{\partial x_i}+c^*(x)u(x), \tag{1.4}$$

where

$$\begin{gathered} b_i^*(x)=-b_i(x)+\sum_{j=1}^n\frac{\partial a_{ij}(x)}{\partial x_j},\quad i=1,\ldots,n,\\ c^*(x)=c(x)-\sum_{i=1}^n\frac{\partial b_i(x)}{\partial x_i}+\sum_{i,j=1}^n\frac{\partial^2 a_{ij}(x)}{\partial x_i\partial x_j}. \end{gathered} \tag{1.5}$$

We'll assume that the operator *L* is uniformly elliptic

$$\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \geq \alpha |y|^2, \quad x \in B, \ y \in \mathbb{R}^n \tag{1.6}$$

and $\alpha > 0$ is the ellipticity constant, and the coefficients satisfy the following smoothness conditions

$$a_{ij}(x) \in C^{2+\beta}(\overline{B}), \quad b_i(x) \in C^{1+\beta}(\overline{B}), \quad c(x) \in C^{\beta}(\overline{B}), \ i, j = 1, \ldots, n \tag{1.7}$$

with some Hölder exponent β, $0 < \beta \leq 1$.

We shall use the name weak *L*-subsolution instead of Littman's generalized subharmonic function.

The objective of the present article is to establish an estimate analogous to one-dimensional inequality (1.1) – namely the reverse Poincare inequality for the difference of two continuous weak *L*-subsolutions.

2. Formulation of the Basic Results

Consider now the linear space *S* of locally integrable functions $u(x)$ in a ball *B* which have the weak (Sobolev) derivatives $\dfrac{\partial u(x)}{\partial x_i}$, $i = 1, \ldots, n$.

Define the weight functions

$$\hat{h}(x) = \overset{2}{\text{dist}}(x, \partial B), \quad \overline{h}(x) = R^2 - |x - x_0|^2 \tag{2.1}$$

in a ball $B = B(x_0, R)$, where $\text{dist}(x, \partial B)$ denotes the distance from the point $x \in \overline{B}$ to the boundary ∂B.

Let us introduce the subspace $H^1(B; \hat{h})$ of the space *S* consisting of those functions $u(x) \in S$ for which the following integral sum is finite

$$\|u\|^2_{H^1(B;\hat{h})} \equiv \int_B u^2(x)dx + \sum_{i=1}^{n} \int_B \left(\frac{\partial u(x)}{\partial x_i} \right)^2 \hat{h}(x)dx. \tag{2.2}$$

It is easy to check that $H^1(B; \hat{h})$ is a complete linear space which is called the weighted Sobolev space. The following inclusion is obvious

$$H^1(B) \subseteq H^1(B; \hat{h}),$$

where $H^1(B)$ is the usual first order Sobolev space.

Our first result is formulated in the following manner

Proposition 2.1. *Suppose that conditions* (1.6)-(1.7) *are satisfied. Then any continuous weak L-subsolution* $u(x)$ *possesses weak partial derivatives* $\dfrac{\partial u(x)}{\partial x_i}$, $i = 1, \ldots, n$ *in a ball* $B = B(x_0, R)$.

The next result concerns the continuous weak *L*-subsolutions bounded in a ball *B*.

Proposition 2.2. Assume *that conditions* (1.6)-(1.7) *are satisfied. Consider any weak L-subsolution* $u(x)$ *in a ball B, such that*

$$u(x) \in C(B) \cap L^{\infty}(B). \tag{2.3}$$

Then the function $u(x)$ belongs to the weighted Sobolev space $H^1(B; \hat{h})$.

Now we formulate the main result of this article.

Proposition 2.3 (The weighted reverse Poincare inequality). *Let the conditions* (1.6)-(1.7) *be satisfied. Consider* *two weak L-subsolutions* $u_i(x),\ i=1,2$ *in a ball B, such that*

$$u_i(x)\in C(B)\cap L^\infty(B),\ i=1,2\,.$$

Then the following reverse Poincare type inequality holds true for the difference $(u_2(x)-u_1(x))$ *of two weak L-subsolutions*

$$\|u_2-u_1\|^2_{H^1(B;\hat h)}\le\left(\frac{c}{\alpha}+\operatorname{meas}B\right)\left[2\|u_2-u_1\|_{L^\infty(B)}\left(\|u_1\|_{L^\infty(B)}+\|u_2\|_{L^\infty(B)}\right)+\|u_2-u_1\|^2_{L^\infty(B)}\right],\tag{2.4}$$

where

$$c=\int_B\left(\left|L^*\bar h(x)\right|+\left|c(x)\right|\bar h(x)\right)dx\tag{2.5}$$

and $\alpha>0$ *is the constant of the uniform ellipticity.*

We note that (2.5) is the Hölder type estimate which asserts that if two bounded continuous weak *L*-subsolutions in a ball B are close in the uniform norm, then they are close in the weighted Sobolev norm as well.

The proof of the Propositions 2.1-2.3 requires the approximation of arbitrary continuous weak *L*-subsolution by a sequence of smooth *L*-subsolutions. It turns out that this is a non-trivial task for elliptic differential operators with variable coefficients as standard mollification arguments work only for the case of constant coefficients.

Fortunately enough, this kind of approximation techniques was developed by Littman in [4] and we've been based on it essentially.

3. An application

The particular case of subharmonic functions is of special interest. Theorem 3.2.11 in Hörmander [3] states the equivalence between the notion of subharmonic function and the notion of the weak Δ-subsolution, where Δ is the famous Laplace operator. In this case it is easy to calculate constant *c* defined by the equality (2.5)

$$c=2n\ \operatorname{meas}B\,.\tag{3.1}$$

Wilson and Zwick [5] studied the problem of best approximation in the norm of $L^\infty(B)$ of a given function $f(x)$ by subharmonic functions. For continuous function in $\bar B$ they characterized best continuous subharmonic approximations. It turns out that the best subharmonic approximation of continuous function $f(x)$ is just the greatest subharmonic minorant of it adjusted by a constant.

In problems for which it is known a priori that the analytically unknown continuous exact solution $u(x)$ must be subharmonic in a ball *B* it makes sense to seek numerical approximations $v_h(x)$ (*h* is some small parameter) that are subharmonic themselves. One expects that they will better mimic an unknown solution $u(x)$ than somehow constructed continuous uniform approximation $u_h(x)$. The nice idea of Wilson and Zwick [5] consists in replacement of $u_h(x)$ by its greatest subharmonic minorant $v_h(x)$ defined by

$$v_h(x)=\sup\{g(x):\ g(x)\ \text{is subharmonic in } B \text{ and } g(x)\le u_h(x)\}\,.\tag{3.2}$$

We state the following important result at the end of this article as an application of the basic Proposition 2.3.

Proposition 3.1. *Consider analytically unknown subharmonic function* $u(x)$ *in a ball B and its known uniform approximation* $u_h(x)$. *We assume that they are continuous and bounded in a ball B. Then the following estimate of* $\operatorname{grad}u(x)$ *through* $\operatorname{grad}v_h(x)$ *is valid*

$$\|\operatorname{grad}v_h-\operatorname{grad}u\|^2_{L^2(B;\hat h)}\le 8n\ \operatorname{meas}B\left[\|u_h-u\|_{L^\infty(B)}\|u\|_{L^\infty(B)}+\|u_h-u\|^2_{L^\infty(B)}\right].\tag{3.3}$$