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Subsolutions That Are Close in the Uniform Norm Are Close in the Sobolev Norm as Well

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Abstract A new type weighted reverse Poincaré inequality is established for a difference of two continuous weak subsolutions of a linear second order uniformly elliptic partial differential equation in the ball.

This result is the key to deriving the error estimate for the gradient of the analytically unknown value function of the optimal stochastic control problem from the uniform error of the value function itself in the related numerical approximation problems.

1 Introduction

Consider two arbitrary finite convex functions $f(x)$ and $\varphi(x)$ on a closed interval $[a,b]$. The following energy inequality was established by K. Shashiashvili and M. Shashiashvili in [21, Theorem 2.1]

$$\begin{aligned}\int_a^b (x-a)^2(b-x)^2\big(f'(x-)-\varphi'(x-)\big)^2\,dx \\ \leq \frac{8}{9}\sqrt{3}\sup_{x\in(a,b)}|f(x)-\varphi(x)|\sup_{x\in(a,b)}|f(x)+\varphi(x)|(b-a)^3\end{aligned}$$

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$$+\frac{4}{3}\Big(\sup_{x\in(a,b)}|f(x)-\varphi(x)|\Big)^2(b-a)^3. \tag{1.1}$$

This kind of estimate with weight functions on an infinite interval $[0,\infty)$ was subsequently applied to hedging problems of mathematical finance in S. Hussain and M. Shashiashvili [12] (see also S. Hussain, J. Peĉariè and M. Shashiashvili [11]).

The natural generalization of univariate convex functions to the case of several variables are subharmonic functions that share many convenient attributes of the former functions. An extensive study of the properties of subharmonic functions was carried out by L. Hörmander in his manual [10, Chapter 3].

In reviewing some well-known results, we need the following notation.

Throughout the paper, we denote by $B=B(x_0,R)$ the open ball in $\mathbb{R}^n$ with center x_0 and radius R, and by $\overline{B}=\overline{B}(x_0,R)$ its closure.

$u(x)$, $v(x)$ are the real-valued functions defined in the ball B. We also use the following standard notation:

- $C(B)$ stands the space of continuous functions on B;
- $L^\infty(B)$ is the space of bounded (a.e.) functions on B;
- $C_0^k(B)$ is for the space of k-times continuously differentiable functions with compact support in B, where $k=1,2,\dots,\infty$;
- $\Delta=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

A locally integrable function $u(x)$ in the ball B is said to be a weak Δ-subsolution of the Laplace equation

$$\Delta u(x)=0 \quad \text{in the ball } B$$

if

$$\int_B u(x)\Delta v(x)\,dx\ge 0 \tag{1.2}$$

for all nonnegative $v(x)$, such that $v(x)\in C_0^2(B)$ (i.e. $\Delta u(x)\ge 0$ in the sense of the distribution theory).

Theorem 3.2.11 in [10] states the equivalence between the notion of a subharmonic function and the notion of a weak Δ-subsolution.

Consider a sequence of subharmonic functions $u_m(x)$, $m=1,2,\dots$, in the ball B, which converges to a subharmonic function $u(x)$ in $L^1_{loc}(B)$. Theorem 3.2.13 in L. Hörmander [10] asserts that weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$, $i=1,\dots,n$, tend to $\frac{\partial u(x)}{\partial x_i}$, $i=1,\dots,n$, in $L^p_{loc}(B)$ for an exponent p with $1\le p<\frac{n}{n-1}$.

Proposition 3.4.19 in [10] considers a sequence of bounded nonpositive subharmonic functions $u_m(x)$ in the ball B, such that $u_m(x)|_{\partial B}=0$ and $\operatorname{supp}\Delta u_m(x)$ is contained in a fixed compact set $K\subset B$. It is proved there that if

$$u_m(x)\downarrow u(x) \quad \text{when } m\to\infty,$$

then weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$, $i=1,\dots,n$, converge to $\frac{\partial u(x)}{\partial x_i}$, $i=1,\dots,n$, in $L^2(B)$.

So it seems reasonable to ask whether the mapping $u(x) \to \operatorname{grad} u(x)$ possesses some Hölder continuity property when restricted to the class of subharmonic functions defined in the ball B.

W. Littman [20] gave a very fruitful generalization of the notion of a subharmonic function to the case of general type (with variable coefficients) second order linear elliptic partial differential operators.

According to Littman [20], the locally integrable function $u(x)$ defined in the ball B is called a generalized subharmonic function if for all nonnegative functions $v(x) \in C_0^2(B)$ the following inequality holds

$$\int_B u(x) L^* v(x)\, dx \geq 0 \tag{1.3}$$

(i.e. $Lu(x) \geq 0$ in the sense of the distribution theory), where $L^* v(x)$ is the adjoint operator to $Lv(x)$

$$\begin{aligned} Lu(x) &= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x) u(x), \\ L^* u(x) &= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^*(x) \frac{\partial u(x)}{\partial x_i} + c^*(x) u(x), \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} b_i^*(x) &= -b_i(x) + 2 \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}, \\ c^*(x) &= c(x) - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \end{aligned} \tag{1.5}$$

with $a_{ij}(x) = a_{ji}(x)$, $i, j = 1, \ldots, n$. It is assumed that the operator L is uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \geq \alpha |y|^2, \quad x \in B,\ y \in \mathbb{R}^n, \tag{1.6}$$

where $\alpha > 0$ is the ellipticity constant and the coefficients satisfy the smoothness conditions

$$\begin{aligned} &a_{ij}(x) \in C^{2+\gamma}(\overline{B}), \qquad b_i(x) \in C^{1+\gamma}(\overline{B}), \\ &c(x) \in C^{\gamma}(\overline{B}), \quad i, j = 1, \ldots, n, \end{aligned} \tag{1.7}$$

with a Hölder exponent γ, $0 < \gamma \leq 1$.

Note that for the sake of simplicity we use the term a weak L-subsolution instead of the term Littman's generalized subharmonic function.

The Sobolev regularity of the weak subsolution in case of the Laplace operator is well-known classical result. Indeed for a bounded subharmonic function $u(x)$ the almost everywhere existence and the local L^2-integrability of the partial derivatives

$\frac{\partial u(x)}{\partial x_i}$, $i = 1, \dots, n$ dates back to Evans's paper ([5], 1935) for the particular case $n = 3$, while for arbitrary n the existence of the Sobolev gradient $\operatorname{grad} u(x)$ and its local L^2-integrability has been shown in Lindqvist [18, Theorem 3.4].

The essential difficulty arises in proving the Sobolev regularity of the weak (distributional) L-subsolution for general linear second order elliptic operator $Lu(x)$ with variable coefficients. Exactly at this point we need the techniques developed by Littman [20] to approximate a continuous weak L-subsolution by smooth ones as it gives the existence and local L^2-integrability of the Sobolev gradient of this L-subsolution.

The present paper pursues the double aim: firstly, to establish an estimate for a difference of two continuous weak L-subsolutions in an n-dimensional ball B, which is analogous to the one-dimensional estimate (1.1) and, secondly, to apply it to the approximation problem for the gradient of a solution of the Hamilton–Jacobi–Bellman equation.

The paper is organized as follows. In Sect. 2, we give the preliminary material on Green's identity and Green's formulas and state our basic result.

In Sect. 3, we prove several auxiliary propositions and the basic result, namely the weighted reverse Poincaré inequality.

In Sect. 4, our basic result is applied to the approximation problem of the gradient of the analytically unknown value function of the optimal stochastic control problem.

2 Preliminary Material and the Formulation of the Basic Result

Consider the twice continuously differentiable functions $u(x)$ and $h(x)$ in the ball $B = B(x_0, R)$. We start with the well-known Green's identity (see e.g. A. Friedman [7, Chapter 6, Section 4])

$$\begin{aligned} &h(x)Lu(x) - u(x)L^*h(x) \\ &\quad = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\left(h(x)a_{ij}(x)\frac{\partial u(x)}{\partial x_j} - u(x)a_{ij}(x)\frac{\partial h(x)}{\partial x_j}\right.\right. \\ &\qquad \left.\left. - u(x)h(x)\frac{\partial a_{ij}(x)}{\partial x_j}\right) + b_i(x)u(x)h(x)\right]. \end{aligned} \tag{2.1}$$

Suppose now that $u(x) \in C^2(\overline{B})$, $h(x) \in C^2(\overline{B})$ and integrate the identity (2.1) using the Gauss–Ostrogradski divergence theorem. We get

$$\begin{aligned} &\int_B Lu(x)h(x)\,dx \\ &\quad = \int_B u(x)L^*h(x)\,dx + \int_{\partial B}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(h(x)a_{ij}(x)\frac{\partial u(x)}{\partial x_j} - u(x)a_{ij}(x)\frac{\partial h(x)}{\partial x_j}\right.\right. \\ &\qquad \left.\left. - u(x)h(x)\frac{\partial a_{ij}(x)}{\partial x_j}\right)n_i(x) + b_i(x)u(x)h(x)n_i(x)\right]d\sigma, \end{aligned} \tag{2.2}$$

where $n(x) = (n_i(x))_{i=1,\dots,n}$ is the outward pointing unit normal vector at $x \in \partial B$, $d\sigma$ is an $(n-1)$-dimensional surface measure of the ball B.

We say that $h(x)$, $h(x) \in C(\overline{B})$, is a weight function if

$$h(x) > 0 \quad \text{in a ball } B \text{ and } h(x)|_{\partial B} = 0. \tag{2.3}$$

Let us consider a weight function $h(x) \in C^2(\overline{B})$. Then from the equality (2.2) we get the Green's second formula

$$\begin{aligned} &\int_B Lu(x)h(x)\,dx \\ &\quad = \int_B u(x)L^*h(x)\,dx - \int_{\partial B} u(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma, \end{aligned} \tag{2.4}$$

where

$$\operatorname{grad} h(x) = \left(\frac{\partial h(x)}{\partial x_i}\right)_{i=1,\dots,n}$$

and

$$\gamma_a(x) = (\gamma_{ai}(x))_{i=1,\dots,n},$$

where

$$\gamma_{ai}(x) = \sum_{j=1}^{n} a_{ji}(x)n_j(x), \quad i = 1, \dots, n. \tag{2.5}$$

We have

$$\big(\gamma_a(x), n(x)\big) = \sum_{i,j=1}^{n} a_{ij}(x)n_i(x)n_j(x) \geq \alpha|n(x)|^2 = \alpha > 0 \tag{2.6}$$

by the uniform ellipticity condition (1.6).

Hence for $x \in \partial B$

$$\big(\operatorname{grad} h(x), \gamma_a(x)\big) = \lim_{t\downarrow 0} \frac{h(x) - h(x - t\gamma_a(x))}{t} \leq 0. \tag{2.7}$$

Let us write the operator $Lu(x)$ in the variational form

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) - \sum_{i=1}^{n} b_i^*(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \tag{2.8}$$

and introduce the bilinear form $a(u, v)$ on the product space $C^1(\overline{B}) \times C^1(\overline{B})$

$$\begin{aligned} a(u, v) = \int_B \Bigg[&\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i} \\ &+ \sum_{i=1}^{n} b_i^*(x)\frac{\partial u(x)}{\partial x_i}v(x) - c(x)u(x)v(x)\Bigg]\,dx. \end{aligned} \tag{2.9}$$

In the sequel we will need the Green's first formula (see e.g. C. Baiocchi and A. Capelo [1, Chapter 18])

$$a(u,v) = -\int_B Lu(x)v(x)\,dx + \int_{\partial B} v(x)\big(\operatorname{grad} u(x), \gamma_a(x)\big)\,d\sigma \tag{2.10}$$

for $u(x) \in C^2(\overline{B})$ and $v(x) \in C^1(\overline{B})$.

Consider now the linear space S of locally integrable functions $u(x)$ in the ball B, which have weak (Sobolev) derivatives $\frac{\partial u(x)}{\partial x_i}$, $i = 1, \ldots, n$.

Define the weight functions

$$\begin{gathered}\widehat{h}(\beta) \equiv \widehat{h}(\beta; x) = R^{2-\beta}\ \operatorname{dist}^{\beta}(x, \partial B), \quad \beta \geq 1, \\ \overline{h}(x) = R^2 - |x - x_0|^2,\end{gathered} \tag{2.11}$$

where $\operatorname{dist}(x, \partial B)$ denotes the distance from a point $x \in \overline{B}$ to the boundary ∂B.

Introduce a subspace $H^1(B; \widehat{h}(\beta))$ of the space S consisting of functions $u(x) \in S$ for which the following integral is finite

$$\int_B u^2(x)\,dx + \sum_{i=1}^{n} \int_B \left(\frac{\partial u(x)}{\partial x_i}\right)^2 \widehat{h}(\beta; x)\,dx \equiv \|u\|^2_{H^1(B;\widehat{h}(\beta))}. \tag{2.12}$$

One can easily check that $H^1(B; \widehat{h}(\beta))$ is a complete linear space. We call it the weighted Sobolev space. The following inclusion is obvious

$$H^1(B) \subseteq H^1\big(B; \widehat{h}(\beta)\big) \subseteq H^1_{loc}(B), \tag{2.13}$$

where $H^1(B)$ and $H^1_{loc}(B)$ are respectively the first order Sobolev and the corresponding local Sobolev space.

Now we are ready to formulate the basic result of this paper.

Theorem 2.1 (the weighted reverse Poincaré inequality) *Assume that the conditions (1.6)–(1.7) are satisfied. Consider two weak L-subsolutions $u_i(x)$, $i = 1, 2$ in the ball B, such that*

$$u_i(x) \in C(B) \cap L^{\infty}(B), \quad i = 1, 2. \tag{2.14}$$

Then the functions $u_i(x)$ belong to the weighted Sobolev space $H^1(B; \widehat{h}(\beta))$, $\beta \geq 1$ and the following reverse Poincaré type inequality holds for the difference $(u_2(x) - u_1(x))$ of two weak L-subsolutions

$$\begin{aligned}&\|u_2 - u_1\|^2_{H^1(B;\widehat{h}(\beta))} \\ &\quad \leq \left(\frac{c}{\alpha} + \operatorname{meas} B\right) \\ &\qquad \times \big[2\|u_2 - u_1\|_{L^{\infty}(B)}(\|u_1\|_{L^{\infty}(B)} + \|u_2\|_{L^{\infty}(B)}) + \|u_2 - u_1\|^2_{L^{\infty}(B)}\big],\end{aligned} \tag{2.15}$$

where

$$\boldsymbol{c} = \int_B \big(|L^*\overline{h}(x)| + |c(x)|\overline{h}(x)\big)\,dx \tag{2.16}$$

and $\alpha > 0$ *is the constant of the uniform ellipticity.*

Note that (2.15) asserts that if two bounded continuous weak L-subsolutions in a ball B are close in the uniform norm, then they remain close in the weighted Sobolev norm as well.

3 Auxiliary Propositions and the Proof of the Basic Result

Consider a weight function $h(x) \in C^2(\overline{B})$ and two arbitrary smooth L-subsolutions $u_i(x) \in C^2(\overline{B})$ in the ball $B = B(x_0, r)$, $r > 0$, i.e.

$$Lu_i(x) \geq 0 \quad \text{for all } x \in B,\ i = 1, 2. \tag{3.1}$$

Proposition 3.1 *Suppose that the uniform ellipticity condition* (1.6) *is satisfied and the coefficients of the differential operator* $Lu(x)$ *are smooth*, *i.e.*

$$a_{ij}(x) \in C^2(\overline{B}), \quad b_i(x) \in C^1(\overline{B}), \quad c(x) \in C(\overline{B}), \quad i, j = 1, \ldots, n. \tag{3.2}$$

Then the following energy inequality is valid

$$\begin{aligned} &\int_B |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h(x)\,dx \\ &\quad \leq \frac{1}{\alpha}\int_B \big(|L^*h(x)| + |c(x)|h(x)\big)\,dx \\ &\qquad \times \big[2\|u_2 - u_1\|_{L^\infty(B)}(\|u_1\|_{L^\infty(B)} + \|u_2\|_{L^\infty(B)}) + \|u_2 - u_1\|^2_{L^\infty(B)}\big] \end{aligned} \tag{3.3}$$

for the difference $u_2(x) - u_1(x)$ *of smooth* L*-subsolutions* $u_i(x)$, $i = 1, 2$, *satisfying the inequality* (3.1).

Proof Define

$$u(x) = u_2(x) - u_1(x), \quad x \in \overline{B}. \tag{3.4}$$

Taking $u^2(x)$ instead of $u(x)$ in the Green's second formula (2.4), we have

$$\begin{aligned} &\int_B Lu^2(x)h(x)\,dx \\ &\quad = \int_B u^2(x)L^*h(x)\,dx - \int_{\partial B} u^2(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma. \end{aligned} \tag{3.5}$$

It is not difficult to calculate that

$$Lu^2(x) = 2\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j} + 2u(x)Lu(x) - c(x)u^2(x). \tag{3.6}$$

Thus we obtain

$$
\begin{aligned}
&2\int_B \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}h(x)\,dx + 2\int_B u(x)Lu(x)h(x)\,dx \\
&\quad = \int_B \big(L^*h(x)+c(x)h(x)\big)u^2(x)\,dx - \int_{\partial B} u^2(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma. \qquad (3.7)
\end{aligned}
$$

From the latter equality we can write

$$
\begin{aligned}
2\alpha &\int_B |\operatorname{grad} u(x)|^2 h(x)\,dx \\
&\le 2\sup_B |u(x)| \int_B |Lu(x)|h(x)\,dx \\
&\quad + \sup_B u^2(x)\int_B \big(|L^*h(x)|+|c(x)|h(x)\big)\,dx \\
&\quad + \sup_{\partial B} u^2(x)\int_{\partial B}\big|\big(\operatorname{grad} h(x),\gamma_a(x)\big)\big|\,d\sigma. \qquad (3.8)
\end{aligned}
$$

Taking $u(x)=1$ in the equality (3.7), we get

$$
\int_{\partial B}\big(\operatorname{grad} h(x),\gamma_a(x)\big)\,d\sigma = \int_B \big(L^*h(x)-c(x)h(x)\big)\,dx, \qquad (3.9)
$$

while we know from (2.7) that for any $x\in\partial B$

$$
\big(\operatorname{grad} h(x),\gamma_a(x)\big)\le 0. \qquad (3.10)
$$

Hence from the relation (3.8) we derive the estimate

$$
\begin{aligned}
\alpha &\int_B |\operatorname{grad} u(x)|^2 h(x)\,dx \\
&\le \sup_B |u(x)| \int_B |Lu(x)|h(x)\,dx \\
&\quad + \sup_B u^2(x)\int_B \big(|L^*h(x)|+|c(x)|h(x)\big)\,dx. \qquad (3.11)
\end{aligned}
$$

So far we have not used the particular structure (3.4) of the function $u(x)$, but it is needed from now on to bound the integral $\int_B |Lu(x)|h(x)\,dx$.

Indeed, we have

$$
|Lu(x)| = |Lu_2(x)-Lu_1(x)| \le L\big(u_1(x)+u_2(x)\big), \qquad (3.12)
$$

hence

$$
\int_B |Lu(x)|h(x)\,dx \le \int_B L\big(u_1(x)+u_2(x)\big)h(x)\,dx. \qquad (3.13)
$$

From the Green's second formula (2.4) we can write

$$
\begin{aligned}
&\int_B L\big(u_1(x)+u_2(x)\big)h(x)\,dx\\
&\quad=\int_B \big(u_1(x)+u_2(x)\big)L^*h(x)\,dx\\
&\qquad+\int_{\partial B}\big(u_1(x)+u_2(x)\big)\big(\operatorname{grad} h(x),-\gamma_a(x)\big)\,d\sigma.
\end{aligned}
\tag{3.14}
$$

But by (3.9)–(3.10) we know that

$$
\begin{gathered}
\big(\operatorname{grad} h(x),-\gamma_a(x)\big)\ge 0,\\
\int_{\partial B}\big(\operatorname{grad} h(x),-\gamma_a(x)\big)\,d\sigma=\int_B\big(-L^*h(x)+c(x)h(x)\big)\,dx,
\end{gathered}
\tag{3.15}
$$

therefore

$$
\begin{aligned}
&\int_B |Lu(x)|h(x)\,dx\\
&\quad\le 2\sup_B|u_1(x)+u_2(x)|\int_B\big(|L^*h(x)|+|c(x)|h(x)\big)\,dx.
\end{aligned}
\tag{3.16}
$$

From the estimates (3.11) and (3.16) we eventually obtain the desired inequality (3.3). □

Further, in order to extend the inequality (3.3) to the general case of weak L-subsolutions we need to approximate an arbitrary continuous weak L-subsolution by a sequence of smooth L-subsolutions. It turns out that in case of the variable coefficients of the differential operator $Lu(x)$ this is not a trivial task (since the standard mollification arguments work only for the case with constant coefficients). The technique of approximation of this kind was developed by W. Littman in [20] and we make essential use of it.

For an arbitrary continuous weak L-subsolution $u(x)$ (see the definition (1.3)) W. Littman constructed a monotonic nonincreasing sequence $u_m(x)$, $m=1,2,\dots$ of functions in the ball B, such that on each compact subset $K\subset B$

$$
\begin{gathered}
u_m(x)\in C^{2+\beta}(K),\qquad Lu_m(x)\ge 0,\quad x\in K,\\
\lim_{m\to\infty}\downarrow u_m(x)=u(x),\quad x\in K
\end{gathered}
\tag{3.17}
$$

for m sufficiently large (it depends on K).

Here we consider only the continuous weak L-subsolutions $u(x)$ in the ball B. By Dini's classical theorem the latter convergence is uniform

$$
\sup_K|u_m(x)-u(x)|\underset{m\to\infty}{\longrightarrow}0.
\tag{3.18}
$$

Let us introduce the balls $B_k = B(x_0, r_k)$,

$$r_k = R\frac{k}{k+1}, \quad k = 1, 2, \dots, \tag{3.19}$$

which are compactly imbedded in the original ball $B = B(x_0, R)$. We also introduce the smooth weight functions

$$\begin{aligned} &h_k(x) = r_k^2 - |x - x_0|^2, \quad x \in \overline{B}_k,\ k = 1, 2, \dots, \\ &h_\infty(x) = R^2 - |x - x_0|^2, \quad x \in \overline{B}, \\ &\quad \text{one for each ball } B_k,\ k = 1, 2, \dots. \end{aligned} \tag{3.20}$$

Now we will show that any continuous weak L-subsolution $u(x)$ in the ball B has all first order weak (Sobolev) derivatives

$$\frac{\partial u(x)}{\partial x_i}, \quad i = 1, \dots, n.$$

Proposition 3.2 *Suppose that the conditions (1.6)–(1.7) are satisfied. Then any continuous weak L-subsolution $u(x)$ possesses weak partial derivatives $\frac{\partial u(x)}{\partial x_i}$, $i = 1, \dots, n$, in the ball $B = B(x_0, R)$.*

Proof Let us consider the sequence $u_m(x)$ approximating the function $u(x)$. If we write the inequality (3.3) for

$$u_1(x) = u_m(x), \qquad u_2(x) = u_l(x)$$

and for the ball B_{k+1}, then we get

$$\begin{aligned} &\int_{B_{k+1}} |\operatorname{grad} u_m(x) - \operatorname{grad} u_l(x)|^2 h_{k+1}(x)\, dx \\ &\quad \le \frac{c_{k+1}}{\alpha}\big[2\|u_m - u_l\|_{L^\infty(B_{k+1})}\big(\|u_m\|_{L^\infty(B_{k+1})} + \|u_l\|_{L^\infty(B_{k+1})}\big) \\ &\qquad + \|u_m - u_l\|^2_{L^\infty(B_{k+1})}\big], \end{aligned} \tag{3.21}$$

where

$$c_k = \int_{B_k} \big(|L^* h_k(x)| + |c(x)| h_k(x)\big)\, dx. \tag{3.22}$$

Note that for $x \in B_k$ the following estimate is valid:

$$h_{k+1}(x) \ge \frac{R^2}{(k+1)(k+2)}. \tag{3.23}$$

Therefore if we restrict the integral on the left-hand side of (3.21) over the ball B_k, then we have

$$\frac{R^2}{(k+1)(k+2)} \int_{B_k} |\operatorname{grad} u_m(x) - \operatorname{grad} u_l(x)|^2\, dx$$

$$\leq \frac{c_{k+1}}{\alpha}\big[2\|u_m-u_l\|_{L^\infty(B_{k+1})}(\|u_m\|_{L^\infty(B_{k+1})}+\|u_l\|_{L^\infty(B_{k+1})})$$
$$+\|u_m-u_l\|^2_{L^\infty(B_{k+1})}\big]. \tag{3.24}$$

Since the sequence $u_m(x)$ converges to $u(x)$ in the norm $L^\infty(B_{k+1})$, we can write

$$\|u_m-u_l\|_{L^\infty(B_{k+1})} \longrightarrow 0 \quad \text{if } m,l\to\infty.$$

Passing to the limit in the inequality (3.24) as $m,l\to\infty$, we obtain

$$\lim_{m,l\to\infty}\sum_{i=1}^n\int_{B_k}\left(\frac{\partial u_m(x)}{\partial x_i}-\frac{\partial u_l(x)}{\partial x_i}\right)^2 dx=0. \tag{3.25}$$

By the completeness of the space $L^2(B_k)$, there exists a family of measurable functions $g_{k,i}(x)$, $i=1,\dots,n$, $k=1,2,\dots$, such that $g_{k,i}(x)\in L^2(B_k)$, $i=1,\dots,n$, and

$$\lim_{m\to\infty}\sum_{i=1}^n\int_{B_k}\left(\frac{\partial u_m(x)}{\partial x_i}-g_{k,i}(x)\right)^2 dx=0,\quad k=1,2,\dots. \tag{3.26}$$

Let us extend the functions $g_{k,i}(x)$ trivially outside B_k as follows

$$g_{k,i}(x)=\begin{cases} g_{k,i}(x) & \text{for } x\in B_k,\\ 0 & \text{for } x\in B\setminus B_k\end{cases}$$

and define the functions $g_i(x)$, $i=1,\dots,n$, on the ball B by

$$g_i(x)=\limsup_{k\to\infty} g_{k,i}(x),\quad i=1,\dots,n. \tag{3.27}$$

It is obvious that the functions $g_{k+l,i}(x)$, $l=0,1,2,\dots$, agree on the ball B_k and therefore

$$g_i(x)=g_{k,i}(x)\quad \text{(a.e.) on a ball } B_k. \tag{3.28}$$

Thus the functions $g_i(x)$, $i=1,\dots,n$, are locally square integrable on the ball B.

Let us check that $g_i(x)$, $i=1,\dots,n$, represent the weak partial derivatives of the function $u(x)$. Take any continuously differentiable function $\varphi(x)$ with compact support in B (i.e. $\varphi(x)\in C_0^1(B)$). Then $\operatorname{supp}\varphi(x)\subset B_k$ for some k. We have

$$\int_{B_k}\frac{\partial u_m(x)}{\partial x_i}\varphi(x)\,dx=-\int_{B_k}u_m(x)\frac{\partial\varphi(x)}{\partial x_i}\,dx.$$

But $u_m(x)$ converges uniformly to $u(x)$ on B_k, and $\frac{\partial u_m(x)}{\partial x_i}$ converges to $g_i(x)$ in $L^2(B_k)$. Hence, passing to the limit as $m\to\infty$ we obtain the equality

$$\int_{B_k}g_i(x)\varphi(x)\,dx=-\int_{B_k}u(x)\frac{\partial\varphi(x)}{\partial x_i}\,dx, \tag{3.29}$$

which means that $g_i(x)$, $i=1,\dots,n$, are indeed the weak partial derivatives of the function $u(x)$. □

Proposition 3.3 *Assume the conditions* (1.6)–(1.7) *to be satisfied. Then any continuous bounded weak L-subsolution $u(x)$ in the ball B belongs to the weighted Sobolev space $H^1(B;\widehat{h}(\beta))$, $\beta \geq 1$.*

Proof We write the inequality (3.3) for the functions $u_1(x) = 0$ and $u_2(x) = u_m(x)$ and the ball B_{k+l}, where the sequence $u_m(x)$ converges to $u(x)$. We obtain

$$\int_{B_{k+l}} |\operatorname{grad} u_m(x)|^2 h_{k+l}(x)\, dx \leq \frac{c_{k+l}}{\alpha} 3\|u_m\|^2_{L^\infty(B_{k+l})}. \tag{3.30}$$

Next, passing to the limit as $m \to \infty$, we get

$$\int_{B_{k+l}} |\operatorname{grad} u(x)|^2 h_{k+l}(x)\, dx \leq \frac{c_{k+l}}{\alpha} 3\|u\|^2_{L^\infty(B_{k+l})}.$$

Restricting the integral on the left-hand side of this inequality over the ball B_k and making the integer l tend to infinity, we obtain

$$\int_{B_k} |\operatorname{grad} u(x)|^2 h_\infty(x)\, dx \leq \frac{c_\infty}{\alpha} 3\|u\|^2_{L^\infty(B)} < \infty. \tag{3.31}$$

Since the left-hand side of (3.31) is increasing with respect to k and bounded, it has the finite limit so that

$$\int_{B} |\operatorname{grad} u(x)|^2 h_\infty(x)\, dx \leq \frac{3c_\infty}{\alpha} \|u\|^2_{L^\infty(B)}. \tag{3.32}$$

But

$$\begin{aligned} h_\infty(x) &= R^2 - |x - x_0|^2 \\ &\geq R^2 \left(\frac{\operatorname{dist}(x, \partial B)}{R} \right) \geq R^{2-\beta} \operatorname{dist}^\beta(x, \partial B), \quad \beta \geq 1, \end{aligned} \tag{3.33}$$

hence we get the energy estimate

$$\int_{B} |\operatorname{grad} u(x)|^2 \widehat{h}(\beta; x)\, dx \leq \frac{3c_\infty}{\alpha} \|u\|^2_{L^\infty(B)} < \infty, \tag{3.34}$$

where

$$c_\infty = \int_B \big(|L^* h_\infty(x)| + |c(x)| h_\infty(x) \big)\, dx. \tag{3.35}$$

□

Proof of Theorem 2.1 We consider the sequences of smooth L-subsolutions $u_{m,i}(x)$, $i = 1, 2$, $m = 1, 2, \ldots$, converging on the balls B_{k+l} uniformly to weak L-subsolutions $u_i(x)$, $i = 1, 2$. By the assumption of the theorem the functions $u_i(x)$, $i = 1, 2$, are continuous and bounded on the ball B, i.e.

$$u_i(x) \in C(B) \cap L^\infty(B), \quad i = 1, 2.$$

Let us apply the inequality (3.3) to the functions $u_{m,1}(x)$ and $u_{m,2}(x)$ and the balls B_{k+l}, $k, l = 1, 2, \ldots$. We have

$$\begin{aligned}
&\int_{B_{k+l}} |\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)|^2 h_{k+l}(x)\,dx \\
&\quad \le \frac{c_{k+l}}{\alpha}\big[2\|u_{m,2}(x) - u_{m,1}(x)\|_{L^\infty(B_{k+l})}(\|u_{m,2}\|_{L^\infty(B_{k+l})} + \|u_{m,1}\|_{L^\infty(B_{k+l})}) \\
&\qquad + \|u_{m,2} - u_{m,1}\|^2_{L^\infty(B_{k+l})}\big].
\end{aligned} \tag{3.36}$$

Passing to the limit as $m \to \infty$ in this inequality, by Proposition 3.2 we get

$$\begin{aligned}
&\int_{B_{k+l}} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h_{k+l}(x)\,dx \\
&\quad \le \frac{c_{k+l}}{\alpha}\big[2\|u_2 - u_1\|_{L^\infty(B_{k+l})}(\|u_2\|_{L^\infty(B_{k+l})} + \|u_1\|_{L^\infty(B_{k+l})}) \\
&\qquad + \|u_2 - u_1\|^2_{L^\infty(B_{k+l})}\big].
\end{aligned} \tag{3.37}$$

Now, restricting the integral on the left-hand side of (3.37) over the ball B_k and then passing to the limit as $l \to \infty$, we obtain

$$\begin{aligned}
&\int_{B_k} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h_\infty(x)\,dx \\
&\quad \le \frac{c_\infty}{\alpha}\big[2\|u_2 - u_1\|_{L^\infty(B)}(\|u_2\|_{L^\infty(B)} + \|u_1\|_{L^\infty(B)}) \\
&\qquad + \|u_2 - u_1\|^2_{L^\infty(B)}\big],
\end{aligned} \tag{3.38}$$

where we have used the assumption on the boundedness of $u_i(x)$, $i = 1, 2$, on the ball B.

By the energy estimates (3.32) and (3.34) we get that $u_i(x)$, $i = 1, 2$, belong to the weighted Sobolev spaces $H^1(B; h_\infty)$ and $H^1(B; \widehat{h}(\beta))$, $\beta \ge 1$.

Passing to the limit in the inequality (3.38) as $k \to \infty$, we obtain

$$\begin{aligned}
&\int_{B} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h_\infty(x)\,dx \\
&\quad \le \frac{c_\infty}{\alpha}\big[2\|u_2 - u_1\|_{L^\infty(B)}(\|u_2\|_{L^\infty(B)} + \|u_1\|_{L^\infty(B)}) \\
&\qquad + \|u_2 - u_1\|^2_{L^\infty(B)}\big],
\end{aligned} \tag{3.39}$$

from which taking into account the inequality (3.33) the desired estimate (2.15) follows. □

Remark 3.4 The particular case of bounded continuous subharmonic functions (i.e. of weak Δ-subsolutions) is of special interest.

It can be easily calculated that $\Delta\overline{h}(x) = -2n$ and therefore the constant $\boldsymbol{c}$ in (2.15) is equal to

$$\boldsymbol{c} = 2n\,\text{meas}(B). \tag{3.40}$$

Wilson and Zwick [22] studied the problem of best approximation in the norm of $L^\infty(B)$ of a given function $f(x)$ by subharmonic functions. For a continuous function in $\overline{B}$ they characterized best continuous subharmonic approximations. It turned out that the best subharmonic approximation of a continuous function $f(x)$ is just the greatest subharmonic minorant of $f(x)$ adjusted by a constant.

In problems for which it is known a priori that the analytically unknown continuous exact solution $u(x)$ must be subharmonic in the ball B it makes sense to seek for numerical approximations $v_h(x)$ (h is some small parameter) that are subharmonic themselves. One expects that they will better imitate the unknown solution $u(x)$ than the somehow constructed continuous uniform approximation $u_h(x)$.

Suppose we are given some continuous uniform approximation $u_h(x)$ to the unknown subharmonic function $u(x)$ in the ball $\overline{B}$. The nice idea of Wilson and Zwick [22] consists in replacing $u_h(x)$ by its greatest subharmonic minorant $v_h(x)$ defined by

$$v_h(x) = \sup\big\{g(x) : g(x) \text{ is subharmonic in } B \text{ and } g(x) \le u_h(x)\big\}. \tag{3.41}$$

Denote

$$\delta = \|u_h - u\|_{L^\infty(B)},$$

then we obtain

$$u_h(x) - \delta \le u(x), \qquad u(x) - \delta \le u_h(x).$$

Thus

$$v_h(x) - \delta \le u_h(x) - \delta \le u(x)$$

and as the subharmonic function $u(x) - \delta$ is the minorant of $u_h(x)$, we have

$$u(x) - \delta \le v_h(x).$$

Hence we get

$$\|v_h - u\|_{L^\infty(B)} \le \|u_h - u\|_{L^\infty(B)}. \tag{3.42}$$

So, both functions $v_h(x)$ and $u(x)$ are subharmonic in B (and we assume they are bounded and continuous), so that we can apply the energy inequality (3.39) and obtain the following important estimate

$$\begin{aligned} &\|\text{grad}\, v_h - \text{grad}\, u\|^2_{L^2(B;\widehat{h}(\beta))} \\ &\quad \le 2n\,\text{meas}\, B\big[4\|u_h - u\|_{L^\infty(B)}\|u\|_{L^\infty(B)} + 3\|u_h - u\|^2_{L^\infty(B)}\big]. \end{aligned} \tag{3.43}$$

Thus, the subharmonic approximation $v_h(x)$ indeed better imitates the unknown exact solution $u(x)$ than the initial uniform approximation $u_h(x)$.

Our results established for a ball B can be generalized to the case of bounded smooth domains. We shall formulate here only the extension of Theorem 2.1.

Theorem 3.5 (The weighted reverse Poincare inequality for smooth domains) *Let the conditions* (1.6)–(1.7) *be satisfied in a bounded domain* D, *such that* $D \in C^{2+\gamma}$, $0 < \gamma \leq 1$. *Assume that*

$$L^*1 = c^*(x) \leq 0 \quad \text{in } D. \tag{3.44}$$

Let the weight function $h(x)$ *be the unique smooth solution of the Dirichlet problem*

$$\begin{cases} L^*h(x) = -1 & \text{in } D, \\ h(x) = 0 & \text{on } \partial D. \end{cases} \tag{3.45}$$

Consider two weak L-*subsolutions* $u_i(x)$, $i = 1, 2$ *in the domain* D, *such that*

$$u_i(x) \in C(D) \cap L^\infty(D), \quad i = 1, 2. \tag{3.46}$$

Then the functions $u_i(x)$ *belong to the weighted Sobolev space* $H^1(D; h)$ *and the following reverse Poincare inequality holds for the difference* $(u_2(x) - u_1(x))$ *of two weak* L-*subsolutions*

$$\begin{aligned} &\|u_2 - u_1\|_{H^1(D;h)} \\ &\quad \leq \left(\frac{c}{\alpha} + \text{meas}\, D\right) \\ &\qquad \times \left[2\|u_2 - u_1\|_{L^\infty(D)}(\|u_1\|_{L^\infty(D)} + \|u_2\|_{L^\infty(D)}) + \|u_2 - u_1\|^2_{L^\infty(D)}\right], \end{aligned} \tag{3.47}$$

where

$$c = \int_D \big(1 + |c(x)|h(x)\big)\, dx \tag{3.48}$$

and $\alpha > 0$ *is the constant of the uniform ellipticity.*

Proof [Sketch of the Proof] Take a sequence D_k, $k = 1, 2, \ldots$ of subdomains of domain D, such that $D_k \in C^{2+\gamma}$ and

$$\overline{D}_k \subset D_{k+1} \subset \overline{D}_{k+1} \subset D, \quad D = \bigcup_{k=1}^{\infty} D_k. \tag{3.49}$$

Together with (3.45) consider the Dirichlet problem for each domain D_k, $k = 1, 2, \ldots$

$$\begin{cases} L^*h_k(x) = -1 & \text{in } D_k, \\ h_k(x) = 0 & \text{on } \partial D_k. \end{cases} \tag{3.50}$$

We have by Theorem 6.14 on the global regularity in Gilbarg, Trudinger [8, Chapter 6] that the Dirichlet problems (3.45), (3.50) have the unique solutions $h(x)$, $h_k(x)$, which are smooth up to the boundary, i.e.

$$h(x) \in C^{2+\gamma}(\overline{D}), \qquad h_k(x) \in C^{2+\gamma}(\overline{D}_k).$$

By the Hopf's strong maximum principle we obtain

$$h(x) > 0 \quad \text{in } D, \qquad h_k(x) > 0 \quad \text{in } D_k. \tag{3.51}$$

Hence $h(x)$ and $h_k(x)$ are smooth weight functions in corresponding domains. Let us extend each function $h_k(x)$ outside D_k trivially to be equal to zero. One can easily see that

$$\lim_{k\to\infty} h_k(x) = h(x) \quad \text{pointwise in } \overline{D}. \tag{3.52}$$

It is a straightforward task to check that all propositions and arguments valid for a ball B remain valid also for the bounded smooth domain D with only trivial changes if we consider domains D_k instead of balls B_k and the weight functions $h_k(x)$ as the solutions of the Dirichlet problem (3.50). □

4 Application to the Approximation Problem of the Gradient of a Solution of the Hamilton–Jacobi–Bellman Equation

In this section we consider the infinite horizon discounted stochastic optimal control problem (see Fleming, Soner [6, Chapter 3, Section 9])

$$u(x) = \inf_U E_x \int_0^\infty e^{-\int_0^t c(X_s, V_s)\,ds} f(X_t, V_t)\,dt, \tag{4.1}$$

where $f(x,\theta)$ is the running cost function, $c(x,\theta)$ is the nonnegative discount factor, θ is the control parameter belonging to the space of controls Θ. U is the family of admissible controls $(V_s)_{s\geq 0}$, $V_s \in \Theta$ and the pair $(X_t, V_t)_{t\geq 0}$ is the controlled Markov diffusion process with values in $\mathbb{R}^n$ and governed by a system of stochastic differential equations of the form

$$dX_t = b(X_t, V_t)\,dt + \sigma(X_t, V_t)\,dW_t, \quad t \geq 0, \tag{4.2}$$

with the initial condition $X_0 = x$, where $V_t \in \Theta$ is the control applied at time t and $(W_t)_{t\geq 0}$ is the n-dimensional brownian motion.

$u(x)$ is called the value function of the stochastic optimal control problem and it is well-known (see Fleming, Soner [6], Lions [19]) that $u(x)$ is a unique viscosity solution of the following Hamilton–Jacobi–Bellman equation

$$\inf_{\theta\in\Theta} \big\{ \mathrm{tr}\big[a(x,\theta)D^2u(x)\big] + b(x,\theta)Du(x) - c(x,\theta)u(x) + f(x,\theta) \big\} = 0, \tag{4.3}$$

where $x \in \mathbb{R}^n$.

Here for arbitrary $\theta \in \Theta$

$$a(x,\theta) = \frac{1}{2}\sigma(x,\theta)\sigma^T(x,\theta),$$

where $\sigma(x,\theta)$ is an $n \times n$ matrix-valued function, $b(x,\theta)$ is an n-dimensional vector-valued function, $c(x,\theta)$ and $f(x,\theta)$ are real-valued functions.

There exists no explicit analytic formula for the value function $u(x)$ and hence the methods of numerical calculation of it and its partial derivatives are of significant practical importance.

The first results on convergence rates of finite-difference approximations for second-order Hamilton–Jacobi–Bellman equations were obtained by Krylov for the constant coefficients case in [14] and for the variable coefficients case in [15]. These results were then extended by Barles and Jacobsen [2, 3], Krylov [16] and some other authors. The state of the art today is represented by papers [3, 16], where [3] and [16] represent two different directions of extensions.

Barles and Jakobsen established in [2] the rate of convergence of the approximation schemes to the solution $u(x)$ of the Hamilton–Jacobi–Bellman equation under the following assumptions (see (A1), (A2) in [2])

(A1) For any $\theta \in \Theta$, the functions $\sigma(x,\theta)$, $b(x,\theta)$, $c(x,\theta)$ and $f(x,\theta)$ are bounded and Lipschitz continuous in the whole space $\mathbb{R}^n$.

(A2) Let

$$\lambda_0 = \sup_{\substack{x\neq y\\ \theta\in\Theta}} \left\{ \frac{1}{2} \frac{\operatorname{tr}[(\sigma(x,\theta)-\sigma(y,\theta))(\sigma(x,\theta)-\sigma(y,\theta))^T]}{|x-y|^2} + \frac{(b(x,\theta)-b(y,\theta),x-y)}{|x-y|^2} \right\},$$

then there is $\lambda > \lambda_0$ such that $c(x,\theta) \ge \lambda$ for any $x \in \mathbb{R}^n$ and $\theta \in \Theta$.

Assumption (A2) clearly requires that the discount factor $c(x,\theta)$ be "sufficiently large".

Under these assumptions it is a classical fact that $u(x)$, $x \in \mathbb{R}^n$, is a Lipschitz continuous function (see Lions [19]), i.e.

$$u(x) \in W^{1,\infty}\big(\mathbb{R}^n\big). \tag{4.4}$$

To calculate the value function, Barles and Jakobsen considered two particular approximation schemes in [2] as an application of their general results.

The first one is the so-called control-scheme, it is defined in the following manner

$$u_h^{(1)}(x) = \inf_{\theta\in\Theta} \big\{ \big(1 - hc(x,\theta)\big)\Pi_{h,\theta} u_h^{(1)}(x) + hf(x,\theta) \big\}, \tag{4.5}$$

where h is a small parameter which typically measures the mesh size and $\Pi_{h,\theta}$ is the operator

$$\Pi_{h,\theta}\phi(x) = \frac{1}{2n} \sum_{i=1}^{n} \big[\phi\big(x + hb(x,\theta) + \sqrt{h}\sigma_i(x,\theta)\big) + \phi\big(x + hb(x,\theta) - \sqrt{h}\sigma_i(x,\theta)\big) \big] \tag{4.6}$$

and $\sigma_i(x,\theta)$ is the i-th column of $\sigma(x,\theta)$.

The second one is the finite difference scheme (the so-called Kushner scheme) which can be defined in a manner similar to the first one

$$u_h^{(2)}(x) = \inf_{\theta\in\Theta}\left\{\frac{1}{1+h^2c(x,\theta)} \times\left(\sum_{z\in h\mathbb{Z}^n} p_\theta(x,x+z)u_h^{(2)}(x+z) + h^2 f(x,\theta)\right)\right\}, \tag{4.7}$$

where $p_\theta(x,y)$ are the so-called "one step transition probabilities".

Barles and Jakobsen [2] proved that both functions $u_h^{(i)}(x)$, $i=1,2$, are Lipschitz continuous in $\mathbb{R}^n$, i.e.

$$u_h^{(i)}(x) \in W^{1,\infty}\big(\mathbb{R}^n\big), \quad i=1,2, \tag{4.8}$$

and established the following rates of convergence to the value function $u(x)$:

$$\begin{aligned}\|u_h^{(1)} - u\|_{L^\infty(\mathbb{R}^n)} &\le c^{(1)}h^{1/4},\\ \|u_h^{(2)} - u\|_{L^\infty(\mathbb{R}^n)} &\le c^{(2)}h^{1/2}.\end{aligned} \tag{4.9}$$

The second estimate is shown to be valid only if $a(x,\theta)$ is independent of the variable x. Note however that Krylov [16] treats equations with x-depending $a(\theta)$-coefficients and obtains error bounds of order $h^{1/2}$ for certain types of finite-difference schemes.

In this section we aim to propose a method of approximation of the unknown $\operatorname{grad} u(x)$ starting from any uniform approximation $u_h(x)$ (in particular $u_h^{(i)}(x)$, $i=1,2$) to the value function $u(x)$ provided that $u_h(x)$ is Lipschitz continuous

$$u_h(x) \in W^{1,\infty}\big(\mathbb{R}^n\big). \tag{4.10}$$

In what follows, we fix an arbitrary ball $B = B(x_0, R)$ in $\mathbb{R}^n$.

Consider a product space $H^1(B)\times H^1(B)$ and introduce on it the bilinear form

$$a_\theta(u,v) = \int_B\left[\sum_{i,j=1}^n a_{ij}(x,\theta)\frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i} + \sum_{i=1}^n b_i^*(x,\theta)\frac{\partial u(x)}{\partial x_i}v(x) + c(x,\theta)u(x)v(x)\right]dx, \tag{4.11}$$

where

$$b_i^*(x,\theta) = \sum_{j=1}^n \frac{\partial a_{ij}(x,\theta)}{\partial x_j} - b_i(x,\theta), \quad i=1,\dots,n. \tag{4.12}$$

Let us write the related elliptic differential operator

$$L_\theta u(x) = \sum_{i,j=1}^n a_{ij}(x,\theta)\frac{\partial^2 u(x)}{\partial x_i\partial x_j} + \sum_{i=1}^n b_i(x,\theta)\frac{\partial u(x)}{\partial x_i} - c(x,\theta)u(x) \tag{4.13}$$

and denote its adjoint operator by $L_\theta^* u(x)$.

Now we introduce our assumption (A3).

There exists $\theta \in \Theta$ such that:

(A3) (1) the operator $L_\theta u(x)$ is uniformly elliptic

$$\sum_{i,j=1}^{n} a_{ij}(x,\theta) y_i y_j \ge \alpha |y|^2, \; x \in B, \; y \in \mathbb{R}^n \tag{4.14}$$

for some $\alpha > 0$;

(2) the bilinear form $a_\theta(u, v)$ is coercive

$$a_\theta(v, v) \ge \delta \|v\|^2_{H_0^1(B)} \quad \text{for } \delta > 0; \tag{4.15}$$

(3) the coefficients $a_{ij}(x,\theta)$, $b_i(x,\theta)$ are smooth

$$\begin{aligned} &a_{ij}(x,\theta) \in C^{2+\gamma}(\overline{B}), \qquad b_i(x,\theta) \in C^{1+\gamma}(\overline{B}), \\ &i, j = 1, \dots, n \text{ for some exponent } \gamma, \; 0 < \gamma \le 1. \end{aligned} \tag{4.16}$$

We note that the coercivity requirement will be automatically satisfied if the constant λ in the assumption (A2) is large enough.

In the sequel, we assume that we have chosen some θ satisfying the assumption (A3).

Consider the Dirichlet problem in the ball B

$$L_\theta u_0(x) = -f(x,\theta), \quad x \in B, \qquad u_0(x)|_{\partial B} = 0. \tag{4.17}$$

We know from Gilbarg, Trudinger [8, Chapter 6] that it has a unique solution $u_0(x) \in C^2(\overline{B})$.

From Green's formulas (2.4) and (2.10) we get

$$\int_B v(x) L_\theta^* \rho(x)\, dx = -a_\theta(v, \rho),$$

where $v(x) \in C^2(\overline{B})$ and $\rho(x) \in C_0^2(B)$.

Any $v(x) \in H^1(B)$ can be approximated by $\widetilde{v}(x) \in C^\infty(\overline{B})$ (see [8, Section 7.6]) in the norm of $H^1(B)$, hence we obtain

$$\int_B v(x) L_\theta^* \rho(x)\, dx = -a_\theta(v, \rho) \tag{4.18}$$

for $v(x) \in H^1(B)$ and $\rho(x) \in C_0^2(B)$.

Proposition 4.1 *Suppose $v(x) \in H^1(B)$. Then the following two conditions are equivalent*

(1)

$$a_\theta(v, \rho) \le 0 \quad \textit{for any } \rho(x) \in H_0^1(B), \tag{4.19}$$

such that $\rho(x) \ge 0$ (a.e.);

(2) $v(x)$ *is a weak* L_θ*-subsolution, i.e.*

$$\int_B v(x) L_\theta^* \rho(x)\, dx \ge 0 \quad \textit{for any } \rho(x) \in C_0^2(B), \ \textit{with } \rho(x) \ge 0. \tag{4.20}$$

Proof Suppose (4.20) holds, then from (4.18) we get

$$a_\theta(v, \rho) \le 0 \quad \text{for } \rho(x) \ge 0, \ \rho(x) \in C_0^2(B). \tag{4.21}$$

Let us check that the latter inequality remains valid for arbitrary Lipschitz continuous $\rho(x)$, $\rho(x) \ge 0$, with compact support in B. Indeed, if we consider the mollified function $\rho_\delta(x)$, then it is obvious that $\rho_\delta(x) \ge 0$, $\rho_\delta(x) \in C_0^\infty(B)$, $\rho_\delta(x)$ is with compact support in B and $\rho_\delta(x)$ tends to $\rho(x)$ in $H^1(B)$. Hence, applying the inequality (4.21) to $\rho_\delta(x)$ and then passing to the limit as $\delta \to 0$, we check its validity for Lipschitz continuous $\rho(x)$, $\rho(x) \ge 0$ with compact support.

Now take any $\rho(x) \in H_0^1(B)$, $\rho(x) \ge 0$ (a.e.). By the definition of the space $H_0^1(B)$ there exists a sequence of $\rho_m(x)$, $\rho_m(x) \in C_0^\infty(B)$, such that

$$\|\rho_m - \rho\|_{H^1(B)} \underset{m\to\infty}{\longrightarrow} 0. \tag{4.22}$$

Take now $\rho_m^+(x) = \max(\rho_m(x), 0)$. Clearly, $\rho_m^+(x)$ are Lipschitz continuous with compact support in B, hence

$$a_\theta(v, \rho_m^+) \le 0, \quad m = 1, 2, \ldots.$$

Now we apply the well-known Stampacchia's theorem, which states that

$$\rho_m^+(x) \quad \text{tends to} \quad \rho^+(x) = \rho(x) \quad \text{in } H^1(B), \tag{4.23}$$

we pass to the limit as $m \to \infty$ and obtain (4.19).

Suppose now that (4.19) is valid. Take in it any $\rho(x) \in C_0^2(B)$, $\rho(x) \ge 0$, and after that recall the equality (4.18), then we get the inequality (4.20). □

Consider again the value function $u(x)$ of the optimal stochastic control problem (4.1). We know from Krylov [17, Chapter 4, Section 1, Lemma 5] that for any $\rho(x) \in C_0^\infty(B)$ with $\rho(x) \ge 0$ we have

$$\int_B \big(u(x) L_\theta^* \rho(x) + f(x, \theta)\rho(x)\big)\, dx \ge 0, \tag{4.24}$$

but this means that

$$\int_B \big(u(x) - u_0(x)\big) L_\theta^* \rho(x)\, dx \ge 0 \tag{4.25}$$

for arbitrary $\rho(x) \in C_0^2(B)$, $\rho(x) \ge 0$, i.e. we get the important fact that the function $(u(x) - u_0(x))$ is a weak L_θ-subsolution in the ball B. From Proposition 4.1 we deduce

$$\begin{aligned}&a_\theta(u-u_0,\rho)\le 0 \quad \text{for any } \rho(x)\in H_0^1(B)\\&\text{such that } \rho(x)\ge 0 \text{ (a.e.).}\end{aligned} \tag{4.26}$$

Take any nonnegative constant $c \ge 0$, then we have

$$a_\theta(u-u_0-c,\rho)=a_\theta(u-u_0,\rho)-a_\theta(c,\rho),$$

but

$$a_\theta(c,\rho)=\int_B c(x,\theta)\,c\,\rho(x)\,dx\ge 0,$$

therefore we have as well

$$a_\theta(u-u_0-c,\rho)\le 0 \quad \text{for } c\ge 0, \quad \rho(x)\ge 0 \text{ (a.e.)}, \ \rho(x)\in H_0^1(B). \tag{4.27}$$

Consider now any Lipschitz continuous uniform approximation $u_h(x)$ to the value function $u(x)$, and define the corresponding convex subset K of the Sobolev space $H^1(B)$

$$\begin{aligned}K=\big\{v\in H^1(B): v(x)\le u_h(x) \text{ a.e. in } B\\ \text{and } v(x)-u_h(x)\in H_0^1(B)\big\}.\end{aligned} \tag{4.28}$$

Let us introduce the obstacle problem (see e.g. Kinderlehrer, Stampacchia [13], Bensoussan [4], or Baiocchi, Capelo [1]).

Find $v_h(x)\in K\cap L^\infty(B)$ such that

$$a_\theta(v_h,v-v_h)\ge (f_\theta,v-v_h) \quad \text{for any } v(x)\in K. \tag{4.29}$$

Here $f_\theta(x)$ denotes the function $f(x,\theta)$. From Bensoussan [4, Chapter 7] we know that the obstacle problem (4.28)–(4.29) has a unique solution $v_h(x)$ such that

$$v_h(x)\in C(\overline{B}), \tag{4.30}$$

and hence $v_h(x)=u_h(x)$ on the boundary ∂B. Take arbitrary $\rho(x)\in H_0^1(B)$ such that $\rho(x)\ge 0$ (a.e.) and define

$$v(x)=v_h(x)-\rho(x),$$

put $v(x)$ in the inequality (4.29), we have

$$a_\theta(v_h,-\rho)\ge (f_\theta,-\rho), \tag{4.31}$$

i.e.

$$a_\theta(v_h-u_0,\rho)\le 0 \quad \text{for any } \rho(x)\in H_0^1(B), \quad \rho(x)\ge 0 \text{ (a.e. in } B).$$

By Proposition 4.1 this means that the function $(v_h(x)-u_0(x))$ is a weak L_θ-subsolution in the ball B.

Proposition 4.2 *Under the assumptions* (A1)–(A3) *the following inequality holds*

$$\|v_h - u\|_{L^\infty(B)} \le \|u_h - u\|_{L^\infty(B)}. \tag{4.32}$$

Proof Denote

$$c_h = \|u_h - u\|_{L^\infty(B)}, \tag{4.33}$$

then we have

$$v_h(x) \le u_h(x) \le u(x) + c_h, \quad x \in B,$$

hence

$$v_h(x) - u(x) \le c_h \quad \text{for } x \in B. \tag{4.34}$$

Define

$$\begin{cases} \widehat{u}_h(x) = u(x) - c_h, & x \in \overline{B}, \\ v(x) = \max(\widehat{u}_h(x), v_h(x)), & x \in \overline{B}. \end{cases} \tag{4.35}$$

It is obvious that $\widehat{u}_h(x) \le u_h(x)$ and hence

$$v(x) \le u_h(x) \quad \text{for } x \in B.$$

We also have the following properties of $v(x)$

$$v(x) \in H^1(B) \quad \text{and} \quad v(x) - u_h(x) \in H_0^1(B).$$

Put $v(x)$ in the inequality (4.29); we have

$$a_\theta(v_h, v - v_h) \ge (f_\theta, v - v_h),$$

i.e.

$$a_\theta(v_h - u_0, v - v_h) \ge 0. \tag{4.36}$$

Take $c = c_h$ and $\rho(x) = v(x) - v_h(x)$ in (4.27) (note that $\rho(x) \ge 0$ and $\rho(x) \in H_0^1(B)$)

$$a_\theta(u - u_0 - c_h, v - v_h) \le 0,$$

i.e.

$$a_\theta\big(u_0 - (u - c_h), v - v_h\big) \ge 0. \tag{4.37}$$

We add the inequalities (4.36) and (4.37) and get

$$a_\theta(v_h - \widehat{u}_h, v - v_h) \ge 0,$$

i.e.

$$a_\theta(\widehat{u}_h - v_h, v - v_h) \le 0. \tag{4.38}$$

Let us show that

$$a_\theta(v - \widehat{u}_h, v - v_h) = 0. \tag{4.39}$$

Divide B into two subsets

$$D = \{x \in B : \widehat{u}_h(x) \le v_h(x)\}, \qquad \overline{D} = \{x \in B : \widehat{u}_h(x) > v_h(x)\}.$$

We have

$$\begin{aligned} v(x) - v_h(x) &= 0 \quad \text{on } D, \text{ hence} \\ \frac{\partial(v - v_h)}{\partial x_i} &= 0, \quad i = 1, \dots, n \text{ (a.e. on } D), \\ \widehat{u}_h(x) - v(x) &= 0 \quad \text{on } \overline{D}, \text{ hence} \\ \frac{\partial(\widehat{u}_h - v)}{\partial x_i} &= 0, \quad i = 1, \dots, n \text{ (a.e. on } \overline{D}). \end{aligned} \tag{4.40}$$

Therefore by the definition of the bilinear form $a_\theta(u, v)$ the equality (4.39) is satisfied.

After adding the relations (4.38)–(4.39) we get

$$a_\theta(v - v_h, v - v_h) \le 0. \tag{4.41}$$

By the coercivity assumption (4.15) we conclude that

$$v(x) - v_h(x) = 0, \quad x \in \overline{B},$$

i.e.

$$u(x) - c_h \le v_h(x) \quad \text{for } x \in \overline{B}. \tag{4.42}$$

From the estimates (4.34) and (4.42) we get the inequality (4.32). □

We know already that both functions $(u(x) - u_0(x))$ and $(v_h(x) - u_0(x))$ are weak L_θ-subsolutons in B and they belong to the spaces $H^1(B) \cap C(\overline{B})$ and $H^1(B; \widehat{h}(\beta)) \cap C(\overline{B})$. Therefore we can apply the reverse Poincaré inequality (2.15) (or its variant (3.39)) and get the following assertion.

Proposition 4.3 *Let the assumptions* (A1)–(A3) *be satisfied. Then the following estimate of* $\operatorname{grad} u(x)$ *through* $\operatorname{grad} v_h(x)$ *is valid*

$$\begin{aligned} &\|\operatorname{grad} v_h - \operatorname{grad} u\|^2_{L^2(B;\widehat{h}(\beta))} \\ &\quad \le \frac{c}{\alpha}\big[4\|u_h - u\|_{L^\infty(B)}(\|u\|_{L^\infty(B)} + \|u_0\|_{L^\infty(B)}) + 3\|u_h - u\|^2_{L^\infty(B)}\big], \end{aligned} \tag{4.43}$$

where

$$c = \int_B \big(|L_\theta^* \overline{h}(x)| + c(x, \theta)\overline{h}(x)\big)\, dx. \tag{4.44}$$

Proof We apply the inequality (3.39) to the weak L_θ-subsolutons $(u(x) - u_0(x))$ and $(v_h(x) - u_0(x))$ in the ball B, we get

$$
\begin{aligned}
&\int_B |\operatorname{grad} v_h - \operatorname{grad} u|^2 \widehat{h}(\beta; x)\,dx \\
&\quad \le \frac{c}{\alpha}\big[2\|v_h - u\|_{L^\infty(B)}(\|v_h - u_0\|_{L^\infty(B)} + \|u - u_0\|_{L^\infty(B)}) \\
&\qquad + \|v_h - u\|^2_{L^\infty(B)}\big]. \qquad (4.45)
\end{aligned}
$$

But

$$\|v_h - u_0\|_{L^\infty(B)} \le \|v_h - u\|_{L^\infty(B)} + \|u - u_0\|_{L^\infty(B)},$$

hence taking into account Proposition 4.2 we obtain the final result—the inequality (4.43). □

Thus the problem of the numerical approximation of the gradient of the unknown value function $u(x)$ is reduced to the numerical calculation of $\operatorname{grad} v_h(x)$, which is a well-studied mathematical problem and resolved in Glowinski, Lions and Tremolieres [9].

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