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ON THE OBLIQUE DERIVATIVE PROBLEM

Let D be a simply connected domain bounded by a simple piecewise smooth curve Γ. $E_p(D)$, $p>1$ is the Smirnov class of analytic in D functions.

$e'_p(D)$, $p>1$ will stand for the spaces of harmonic functions with the following property:

$$\sup_{0<r<1}\int\limits_{\Gamma_r}\Big(\Big|\frac{\partial u}{\partial x}\Big|^p+\Big|\frac{\partial u}{\partial x}\Big|^p\Big)|dz|<\infty, \tag{1}$$

where Γ_r is the image of the circumference $|\omega|=r$ under the conformal mapping of the unit disk U onto D.

The space $e'_p(D)$ coincides with the space of harmonic functions represented as the real part of the analytic function Φ from $E'_p(D)$, where $E'_p(D)=\{\Phi:\Phi'\in E_p(D)\}$.

Let l_t be the given vector at the point $t\in\Gamma$, and $\alpha(t)$ be the angle between the vector l_t and the real axis. The oblique derivative problem is formulated as follows: find a harmonic in D function $u\in e'_p(D)$, whose derivative, with respect to the vector l_t, $t\in\Gamma$, angular boundary values coincide almost everywhere on the boundary Γ with the given real function f from $L_p(\Gamma)$. Thus u satisfies the conditions

$$\begin{cases}\Delta u=0, & u\in e'_p(D)\\ \frac{\partial u}{\partial l_t}\Big|_\Gamma^+=f, & f\in \operatorname{Re}L_p(\Gamma).\end{cases} \tag{2}$$

Let $u=\operatorname{Re}\Phi$, $\Phi'\in E_p(D)$ be a solution of the problem (2). Since

$$\Phi'=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y},\quad \frac{\partial u}{\partial l_t}\Big|^+(t)=\frac{\partial u}{\partial x}\Big|^+(t)\cos\alpha(t)+\frac{\partial u}{\partial y}\Big|^+(t)\sin\alpha(t),$$

we can write the boundary condition from (1.2) in the form:

$$\operatorname{Re}(\exp i\alpha(t)(\Phi')^+(t))=f(t),\quad t\in\Gamma,\text{a.e.}(3) \tag{3}$$

Let $z:U\to D$ be the conformal map from the unit disk U onto D. Then we write (3) as

$$\operatorname{Re}\Big(\frac{\exp i\alpha(z(\tau))}{(z'(\tau))^{\frac1p}}\Psi^+(\tau)\Big)=\varphi(\tau),\quad |\tau|=1, \tag{4}$$

where

$$\Psi\in E_p(U),\ \Psi(\omega)=(z'(\omega))^{\frac1p}\Phi'(z(\omega)),\ \omega\in U,\ \varphi=\operatorname{Re}L_p(\Gamma_0),\ \varphi(\tau)=f(z(\tau)),\ |\tau|=1.$$

The problem (4) is equivalent to the following Rieman-Hilbert problem:

$$\begin{cases}\Omega^+(\tau)=G(\tau)\Omega^-(\tau)+g(\tau), & |\tau|=1,\\ \Omega(\omega)=\Omega_*(\omega), & |\omega|\neq 1.\end{cases} \tag{5}$$

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where

$$\Omega(\omega) = \begin{cases} \Psi(\omega), & |\omega| < 1, \\ \overline{\Psi}(\omega), & |\omega| > 1. \end{cases} \qquad F_*(\omega) = \overline{F}\Big(\frac{1}{\omega}\Big);$$

$$G(\tau) = \frac{2\exp(-2i\alpha(z(\tau))\sqrt[p]{z'(\tau)})}{\sqrt[p]{\overline{z}'(\tau)}}, \quad g(\tau) = 2f(z(\tau))\sqrt[p]{z'(\tau)}\exp(-i\alpha(z(\tau))).$$

The problem (2) is equivalent to the problem (5) in the following statement (see, [2] Chapter IV): any solution of (2) generates the function Ω which satisfies the conditions (5), and vice versa, if Ω satisfies (5), then

$$u(z) = \operatorname{Re}\int_{z_0}^{z} \frac{\Omega(\omega(\zeta))d\zeta}{(z'(\omega(\zeta)))^{\frac{1}{p}}} + \text{constant} \tag{6}$$

is a solution of (2).

As is proven in [1]

$$\lim_{\omega\to\exp i\theta} \arg(z'(\omega)) = \beta(\theta) - \theta - \frac{\pi}{2}, \tag{7}$$

where $\beta(\theta)$ is the angle between the oriented tangent at the point $z(e^{i\theta})$ and the real axis. The problem of linear conjugation from (5) takes the form

$$\Omega^+(\tau) = e^{\frac{\pi i}{p}} \exp\Big(-2i\alpha(\theta) - \frac{\beta(\theta)}{p} + \frac{\theta}{p}\Big)\Omega^-(\tau) + g(\tau). \tag{8}$$

Assume that $\alpha(t)$ is the piecewise continuous function on Γ. Since Γ is a piecewise smooth curve, the function $\beta(\theta)$ will be the piecewise continuous function on the unit circle γ_0. Therefore the coefficient of the problem (8)

$$G(\tau) = e^{\frac{\pi i}{p}} \exp\Big(-2i\alpha(\theta) - \frac{\beta(\theta)}{p} + \frac{\theta}{p}\Big), \quad \tau = e^{i\theta}$$

is the piecewise continuous uniocular function. Thus B. Khvedelidze's theory is applicable.

Reasoning just as in [2], (Ch. IV), we can get a complete picture of solvability of the problem (2). Under the above-mentioned conditions, the problem (2) is the problem with a finite index. As an example, let us consider the problem (2) with an infinite index.

Let $\Gamma = R$, $\alpha(t) = at$ where a is an arbitrarily fixed real number and the unknown function u is from the Hardy class of analytic functions in the upper half-plane H_p, $p > 1$. In this case the oblique derivative problem has the form:

$$\begin{cases} \Delta u = 0, & u \in \operatorname{Re} H_p, \\ \Big(\frac{\partial u}{\partial x}\Big)'(t)\cos at + \Big(\frac{\partial u}{\partial y}\Big)^+(t)\sin at = f(t), & t \in R, \quad f \in \operatorname{Re} L_p\Big) \end{cases} \tag{9}$$

Theorem.

I. *For $a > 0$, the homogeneous problem ($f(t) = 0$) has only the constant solution, while the inhomogeneous problem is, in general, unsolvable. The solvability is equal to the condition*

$$f(t) = 0, \quad -a < t < a, \quad a.\ e.,$$

and in this case

$$\frac{\partial u_0(x,y)}{\partial x} = \frac{e^{ay}}{\pi}\Big(\cos ax\int_{-\infty}^{+\infty}\frac{yf(t)dt}{(t-x)^2+y^2} - \sin ax\int_{-\infty}^{+\infty}\frac{(t-x)f(t)dt}{(t-x)^2+y^2}\Big). \tag{10}$$

II. *For $a < 0$ the homogeneous problem has an infinite-dimensional space of solutions*

$$\frac{\partial u(x,y)}{\partial x} = \frac{e^{(2a+\varepsilon)y}(x\cos(2a+\varepsilon)x + y\sin(2a+\varepsilon)x - e^{-xy}(x\cos\varepsilon x + y\sin\varepsilon x))}{x^2+y^2}, \tag{11}$$

where ε is an arbitrary number from the $(0; -a)$ interval.

The inhomogeneous problem is solvable for all $f \in \operatorname{Re} L_p$, *and the solution* $u + u_0$ *is given by* (10) *and* (11).

Singular integral equations with an infinite index, appearing in solving the problem (9), have been studied in [6].

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