Proceedings of A. Razmadze
Mathematical Institute
Vol. **130** (2002), 115–124

ON INTEGRAL EQUATIONS WITH FIXED SINGULARITIES

A. SAGINASHVILI

Abstract. Boundedness conditions and the norm of integral operators with fixed singularities in the kernel are established in weighted Lebesgue spaces. Second kind integral equations containing the abovementioned operators are investigated. This equations have important applications in the theory of automatized design of complex systems.

1°. Boundedness. Let V_α and W_α be the integral operators defined by the equalities

$$(V_\alpha\varphi)(x)=\int_x^1 \frac{x^{\alpha-1}}{y^\alpha}\,\varphi(y)\,dy,\quad x\in(0,1),\quad \alpha\in\mathbb{R}, \tag{1}$$

$$(W_\alpha\varphi)(x)=\int_x^1 \frac{(y-x)^{\alpha-1}}{y^\alpha}\,\varphi(y)\,dy,\quad x\in(0,1),\quad \alpha>0. \tag{2}$$

Let us denote by $L_{p,\beta}$, $p\geq 1$, $\beta\in\mathbb{R}$ the Banach space of functions measurable on the interval $(0,1)$ and having a finite norm

$$\|\varphi\|_{p,\beta}:=\Big(\int_0^1 |x^\beta\varphi(x)|^p\,dx\Big)^{1/p}<+\infty.$$

Lemma 1. *The operator V_α is bounded in the space $L_{p,\beta}$ if and only if $\alpha+\beta+\frac1p>1$. In this case*

$$\|V_\alpha\|_{p,\beta}=\frac{1}{\alpha+\beta+\frac1p-1}\,. \tag{3}$$

2000 *Mathematics Subject Classification.* 45E05, 45F15.
Key words and phrases. Integral equations with fixed singularities, norms of integral operators, weights.

Proof. Necessity: let χ be a characteristic function of the interval $(\frac{1}{2},1)$. It is obvious that $\chi\in L_{p,\beta}$ for all $p\ge 1$, $\beta\in\mathbb{R}$. However, if $\alpha+\beta+\frac{1}{p}\le 1$, it is easy to see that the function $V_\alpha\chi$ does not belong to the space $L_{p,\beta}$.

Sufficiency: let us introduce the operators

$$(\mathbb{Z}\varphi)(t):=e^{-(\beta+\frac{1}{p})t}\varphi(e^{-t}),$$
$$(\mathbb{Z}^{-1}\psi)(x):=x^{-(\beta+\frac{1}{p})}\psi(-\ln x),\quad t>0,\quad 0<x<1.$$

These are isometric mutually inverse operators acting from the space $L_{p,\beta}$ into the space $L_p(\mathbb{R}^+)$, $\mathbb{R}_+=(0,+\infty)$, and from $L_p(\mathbb{R}^+)$ into $L_{p,\beta}$, respectively.

Let $\alpha+\beta+\frac{1}{p}>1$ and

$$\widetilde{V}_\alpha=\mathbb{Z}V_\alpha\mathbb{Z}^{-1}. \tag{4}$$

After easy transformations we obtain

$$(\widetilde{V}_\alpha\psi)(t)=\int\limits_0^t e^{(1-\alpha-\beta-\frac{1}{p})(t-\tau)}\psi(\tau)\,d\tau=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}v_\alpha(t-\tau)\psi_+(\tau)\,d\tau, \tag{5}$$

where

$$\begin{aligned} v_\alpha(t)&=\begin{cases}\sqrt{2\pi}e^{(1-\alpha-\beta-\frac{1}{p})t} & \text{for } t>0,\\ 0 & \text{for } t\le 0,\end{cases}\\ \psi_+(t)&=\begin{cases}\psi(t) & \text{for } t>0,\\ 0 & \text{for } t<0.\end{cases}\end{aligned} \tag{6}$$

Hence $\widetilde{V}_\alpha$ is a convolution one with the kernel $v_\alpha\in L_1(\mathbb{R})$. Therefore it is bounded in the space $L_p(\mathbb{R}^+)$ and

$$\|\mathcal{F}v_\alpha\|_\infty\le\|\widetilde{V}_\alpha\|_p\le\frac{1}{\sqrt{2\pi}}\,\|v_\alpha\|_1, \tag{7}$$

where $\mathcal{F}$ is the Fourier transform

$$(\mathcal{F}v_\alpha)(t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{it\tau}v_\alpha(\tau)\,d\tau=\frac{i}{t-i(1-\alpha-\beta-\frac{1}{p})},\quad t\in\mathbb{R}. \tag{8}$$

On the other hand, we have $v_\alpha(t)\ge 0$, $t\in\mathbb{R}$ and therefore

$$\frac{1}{\sqrt{2\pi}}\,\|v_\alpha\|_1=(\mathcal{F}v_\alpha)(0)\le\|\mathcal{F}v_\alpha\|_\infty. \tag{9}$$

Inequalities (7) and (9) imply

$$\|\widetilde{V}_\alpha\|_p=(\mathcal{F}v)(0),$$

which together with (4) and (8) gives (3). □

Lemma 2. *The operator W_α is bounded in $L_{p,\beta}$ if and only if $\beta > -\frac{1}{p}$. Then*

$$\|W_\alpha\|_{p,\beta} = B\Big(\frac{1}{p} + \beta, \alpha\Big), \tag{10}$$

where $B(a,b)$ is the Beta-function.

Remark 0.1. If $b = n$ is a natural number, then (see 12.4,[6])

$$B(a,n) = \frac{(n-1)!}{a(a+1)\cdots(a+n-1)}.$$

Thus if α is natural, from we obtain (10)

$$\|W_\alpha\|_{p,\beta} = \frac{(\alpha-1)!}{(\frac{1}{p}+\beta)(1+\frac{1}{p}+\beta)\cdots(\alpha-1+\frac{1}{p}+\beta)}\,.$$

The proof is similar to that we used for the operator V_α. We only to note that now

$$\widetilde{W}_\alpha = \mathbb{Z} W_\alpha \mathbb{Z}^{-1}, \tag{11}$$

$$\begin{aligned}(\widetilde{W}_\alpha \psi)(t) &= \int\limits_0^t e^{(1-\frac{1}{p}-\alpha-\beta)(t-\tau)}(e^{t-\tau}-1)^{\alpha-1}\psi(\tau)\,d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} w_\alpha(t-\tau)\psi_+(\tau)\,d\tau,\end{aligned}$$

where

$$w_\alpha(t) = \begin{cases} \sqrt{2\pi} e^{(1-\frac{1}{p}-\alpha-\beta)t}(e^t-1)^{\alpha-1} & \text{for } \ t>0, \\ 0 & \text{for } \ t \le 0. \end{cases}$$

It is obvious that $w_\alpha(t) \ge 0$, and the conditions $\alpha > 0$, $\beta > -\frac{1}{p}$ imply $w_\alpha \in L_1(\mathbb{R})$. Therefore

$$(\mathcal{F} w_\alpha)(0) \le \|\mathcal{F} w_\alpha\|_\infty \le \|\widetilde{W}_\alpha\|_p \le \frac{1}{\sqrt{2\pi}}\,\|w_\alpha\|_1 = (\mathcal{F} w)(0),$$

i.e.,

$$\|\widetilde{W}_\alpha\|_p = (\mathcal{F} w)(0), \tag{12}$$

but (see 12.4, [6])

$$\begin{aligned}(\mathcal{F} w_\alpha)(t) &= \int\limits_0^\infty e^{(1-\frac{1}{p}-\alpha-\beta+it)\tau}(e^\tau_{-1})^{\alpha-1}\,d\tau = \int\limits_0^1 y^{\frac{1}{p}+\beta-it-1}(1-y)^{\alpha-1}\,dy \\ &= B\Big(\frac{1}{p}+\beta-it, \alpha\Big), \quad t \in \mathbb{R}.\end{aligned} \tag{13}$$

Equality (10) follows from (11), (12), (13).

2°. Equations.

Theorem 1. *Let $p \geq 1$, $\alpha+\beta+\frac{1}{p}>1$. Then the integral equation*

$$\varphi - \alpha V_\alpha \varphi = f \tag{14}$$

a) has a unique solution $\varphi \in L_{p,\beta}$ for any right-hand side $f \in L_{p,\beta}$ when $\beta+\frac{1}{p}>1$, and

$$\varphi(x) = f(x) + \frac{\alpha}{x}\int\limits_x^1 f(y)\,dy, \quad 0<x<1; \tag{15}$$

b) is solvable if and only if

$$\int\limits_0^1 f(x)\,dx = 0 \tag{16}$$

when $\beta+\frac{1}{p}<1$. In that case a solution is given by

$$\varphi(x) = f(x) - \frac{\alpha}{x}\int\limits_0^x f(y)\,dy; \tag{17}$$

c) is not normally solvable in the space $L_{p,\beta}$ when $\beta+\frac{1}{p}=1$.

Proof. By applying the operator $\mathbb{Z}$ to both sides of equation (14) we obtain an equivalent equation in the space $L_p(\mathbb{R}^+)$

$$\psi(t) - \alpha(\widetilde{V}_\alpha \psi)(t) = g(t), \quad t>0, \tag{18}$$

where the operator $\widetilde{V}_\alpha$ is defined by equality (4),

$$\psi = \mathbb{Z}\varphi, \quad g = \mathbb{Z}f.$$

Equation (18) is a Winner–Hopf one. If we extend it to a convolution equation on the whole axis $\mathbb{R}$, then we have

$$\psi_+(t) - \alpha(\widetilde{V}_\alpha \psi_+)(t) = g_+(t), \quad t \in \mathbb{R}, \tag{19}$$

where ψ_+ and g_+ are the unknown and the given function, respectively, on $\mathbb{R}$, which are defined by (6). (Note that we have used the obvious equality $(\widetilde{V}_\alpha \psi_+)(t)=0$, at $t<0$; (see (5)).) On rewriting equation (19) as

$$(\mathcal{F}^{-1}\sigma\mathcal{F})\psi_+ = g_+,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform and σ is the symbol of (19) (see (8)),

$$\sigma(t) = 1 - \alpha(\mathcal{F}v)(t) = \frac{t - i(1-\frac{1}{p}-\beta)}{t - i(1-\frac{1}{p}-\alpha-\beta)}\,,$$

we conclude that a unique solution in the class $L_p(\mathbb{R})$ has the form

$$\psi(t) = \mathcal{F}^{-1}\sigma^{-1}\mathcal{F}g_+, \tag{20}$$

To have $\psi = \psi_+$, i.e.,$\psi(t) = 0$ a. e. for $t < 0$, it is necessary and sufficient that the function $\sigma^{-1}\mathcal{F}g_+$ be analytic in the upper half-plane. The latter condition is automatically fulfilled for $\beta + \frac{1}{p} > 1$ and, for $\beta + \frac{1}{p} < 1$, is equivalent to the condition $(\mathcal{F}g_+)(i(1 - \frac{1}{p} - \beta)) = 0$ which is the same as condition (16).

Representing $\sigma^{-1} = 1 + \sigma_0$, where

$$\sigma_0(t) = \frac{i\alpha}{t - i(1 - \frac{1}{p} - \beta)}\,,$$

we obtain from (20)

$$\psi_+(t) = g_+(t) + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} (\mathcal{F}^{-1}\sigma_0)(t-\tau)g_+(\tau)\,d\tau.$$

On the other hand, when $\beta + \frac{1}{p} > 1$ we have

$$(\mathcal{F}^{-1}\sigma_0)(t) = \frac{i\alpha}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{e^{-it\tau}}{\tau - i(1 - \frac{1}{p} - \beta)}\,d\tau = \begin{cases} \sqrt{2\pi}\alpha e^{(1-\frac{1}{p}-\beta)t} & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

Therefore

$$\psi_+(t) = g_+(t) + \alpha \int\limits_0^t e^{(1-\frac{1}{p}-\beta)(t-\tau)} g_+(\tau)\,d\tau.$$

Finally,

$$\begin{aligned} \varphi(x) &= (\mathbb{Z}^{-1}\psi_+)(x) \\ &= f(x) + \alpha x^{-(\beta+\frac{1}{p})} \int\limits_0^{-\ln x} e^{(\beta+\frac{1}{p}-1)(\ln x+\tau)} \cdot e^{-(\beta+\frac{1}{p})\tau} f(e^{-\tau})\,d\tau \\ &= f(x) + \frac{\alpha}{x} \int\limits_x^1 f(y)\,dy. \end{aligned}$$

When $\beta + \frac{1}{p} < 1$ we have

$$(\mathcal{F}^{-1}\sigma_0)(t) = \begin{cases} 0 & \text{for } t > 0, \\ -\sqrt{2\pi}\alpha e^{(1-\frac{1}{p}-\beta)t} & \text{for } t < 0, \end{cases}$$

which gives

$$\psi_+(t) = g_+(t) - \alpha \int\limits_t^{+\infty} e^{(1-\beta-\frac{1}{p})(t-\tau)} g_+(\tau)\, d\tau.$$

By restoring the solution $\varphi = \mathbb{Z}^{-1}\psi_+$ we obtain (17).

For $\beta + \frac{1}{p} = 1$ we have $\sigma(0) = 0$. Hence equation (18) is not normally solvable (see, e.g. §3.2, [5]) and therefore neither is (14). □

Let $p \geq 1$, $\beta + \frac{1}{p} > 0$. In the space $L_{p,\beta}$ consider the equation

$$\varphi - \alpha W_\alpha \varphi = f, \tag{21}$$

where W_α is the operator defined by (2).

Using the same arguments as in the previous case, equation (21) can be reduced to a convolution equation on the whole axis

$$(\mathcal{F}^{-1} s \mathcal{F})\psi_+ = g_+, \tag{22}$$

where the symbol $s = 1 - \alpha\mathcal{F}\widetilde{W_\alpha}$ is a function from the Winner algebra $W(\mathbb{R})$, which can be analytically continued into the upper half-plane (see (13)). Thus for $s(t) \neq 0$, $t \in \overline{\mathbb{R}}$, we obtain $s^{-1} \in W(\mathbb{R})$ and (22) has a unique solution $\psi \in L_p(\mathbb{R})$,

$$\psi = \mathcal{F}^{-1} s^{-1} \mathcal{F} g_+. \tag{23}$$

To have $\psi(t) = \psi_+(t) = 0$ for $t < 0$, it is necessary and sufficient that the function $s^{-1}\widetilde{\mathcal{F}} g_+$ be analytic in the upper half-plane.

Like for equation (14), we consider three cases:

a) $\frac{1}{p} + \beta > 1$; by (10)

$$\|W_\alpha\|_{p,\beta} = \int\limits_0^1 x^{\beta+\frac{1}{p}-1}(1-x)^{\alpha-1}\, dx < \int\limits_0^1 (1-x)^{\alpha-1} dx = \frac{1}{\alpha}. \tag{24}$$

Therefore the operator $I - \alpha W_\alpha$ is invertible in the space $L_{p,\beta}$, i. e., for any $f \in L_{p,\beta}$ equation (21) has a unique solution $\varphi \in L_{p,\beta}$ which by virtue of (23) is calculated by the formula

$$\varphi = \mathbb{Z}^{-1}\mathcal{F}^{-1} s^{-1} \mathcal{F}\mathbb{Z} f. \tag{25}$$

b) $\beta + \frac{1}{p} < 0$; $s(i(1 - \beta - \frac{1}{p})) = 0$ by virtue of (13) and, since the point $i(1 - \frac{1}{p} - \beta)$ is in the upper half-plane, for the function $s^{-1}\mathcal{F} g_+$ to be analytical, it is necessary that the condition $(\mathcal{F} g_+)(i(1 - \beta - \frac{1}{p})) = 0$ be fulfilled. The latter condition is equivalent to condition (16). In this case, too, the solution φ is calculated by formula (25).

c) $\beta + \frac{1}{p} = 1$; then $s(0) = 0$ (see (13)), i. e., the symbol of the equation has a real zero and (21) is not normally solvable.

As an example let us consider equation (21) for $\alpha = 2$. Then

$$\begin{aligned} s(t) &= \frac{(t - i(1-\beta-\frac{1}{p}))(t + i(\beta+\frac{1}{p}+2))}{(t + i(\beta+\frac{1}{p}))(t + i(\beta+\frac{1}{p}+1))}, \\ s^{-1}(t) &= 1 + s_0(t), \end{aligned} \tag{26}$$

where

$$s_0(t) = -\frac{2}{(t - i(1-\beta-\frac{1}{p}))(t + i(\beta+\frac{1}{p}+2))}.$$

Therefore for $\beta + \frac{1}{p} > 1$ we have

$$(\mathcal{F}^{-1}s_0)(t) = \begin{cases} \frac{2\sqrt{2\pi}}{3}\left(e^{(1-\beta-\frac{1}{p})t} - e^{-(\beta+\frac{1}{p}+2)t}\right) & \text{if } t > 0, \\ 0 & \text{if } t < 0 \end{cases}$$

and from (25) we obtain

$$\varphi(x) = f(x) + \frac{2}{3x}\int_x^1 f(y)\,dy - \frac{2x^2}{3}\int_x^1 \frac{f(y)}{y^3}\,dy. \tag{27}$$

For $\beta + \frac{1}{p} < 0$ the function $s(z)$ has a unique zero in the upper half-plane at $z = i(1-\beta-\frac{1}{p})$, which means that condition (16) is necessary and sufficient for the solvability. Taking into account that

$$(\mathcal{F}^{-1}s_0)(t) = \begin{cases} -\frac{2\sqrt{2\pi}}{3}\,e^{-(\beta+\frac{1}{p}+2)t} & \text{if } t > 0 \\ -\frac{2\sqrt{2\pi}}{3}\,e^{(1-\beta-\frac{1}{p})t} & \text{if } t < 0, \end{cases}$$

we obtain from (25)

$$\varphi(x) = f(x) - \frac{2}{3x}\int_0^x f(y)\,dy - \frac{2x^2}{3}\int_x^1 \frac{f(y)}{y^3}\,dy. \tag{28}$$

As a result of the above, we obtain

Theorem 2. *Let $p \geq 1$, $\beta + \frac{1}{p} > 0$. Then:*

a) if $\beta+\frac{1}{p} > 1$, the integral equation (21) *has a unique solution calculated by the formula* (25) (*by the formula* (27) *if $\alpha = 2$);*

b) if $\beta + \frac{1}{p} < 1$, for (21) *to be solvable it is necessary* (*and for $\alpha = 2$ it is also sufficient*) *that condition* (16) *be fulfilled. In this case a unique solution, is calculated by* (25) (*by* (28) *if $\alpha = 2$).*

c) if $\beta + \frac{1}{p} = 1$, equation (21) *is not normally solvable.*

Remark 0.2. When $\beta+\frac{1}{p}<1$, condition (16) is not sufficient for arbitrary α. For example, if α is a natural number, the equation $s(z)=0$ is equivalent to the equation (see Remark 1)

$$P_\alpha(\zeta)=\zeta(\zeta+1)(\zeta+2)\cdots(\zeta+\alpha-1)=\alpha!\,, \tag{29}$$

where $\zeta=\beta+\frac{1}{p}-iz$.

Let γ_α be an arc lying in the strip $0\le\operatorname{Re}\zeta\le1$, satisfying the equation

$$|P_\alpha(\zeta)|=\alpha!$$

and having the end points $\zeta=1$ and $\zeta=iy_\alpha$, $y_\alpha>1$. When ζ runs along the arc from 1 to iy_α, $\arg P_\alpha(\zeta)$ continuously changes from 0 to $\Delta_\alpha=\frac{\pi}{2}+\sum\limits_{k=1}^{\alpha-1}\operatorname{arctg}\frac{y_0}{k}$. Since

$$\Delta_\alpha\ge\frac{\pi}{2}+\frac{\pi}{4}\sum_{k=1}^{\alpha-1}\frac{1}{k}>\frac{\pi}{2}+\frac{\pi}{4}\,\log\alpha,$$

we have $\Delta_\alpha>2\pi$ for $\log\alpha\ge6$, and $\arg P_\alpha(\zeta_0)=2\pi$ takes place at least at one point $\zeta_0\in\gamma_\alpha$. Hence equation (29) has a root in the strip $0<\operatorname{Re}\zeta<1$ (computer calculations performed by Mathematica 4.0 software show that such roots already appear for $\alpha\ge12$).

Let $\zeta_1,\zeta_2,\dots\zeta_m$ be the roots of equation (29) and

$$1=\operatorname{Re}\zeta_1>\operatorname{Re}\zeta_2\ge\cdots\ge\operatorname{Re}\zeta_m>0=\zeta_{m+1}.$$

Then if $\operatorname{Re}\zeta_{k+1}<\beta+\frac{1}{p}<\operatorname{Re}\zeta_k$, for some $k=1,\dots,m$, the equation $s(z)=0$ has roots at the points $z_j=-\operatorname{Im}\zeta_j+i(\operatorname{Re}\zeta_j-\beta-\frac{1}{p})$ and, since $\operatorname{Re}\zeta_j-\beta-\frac{1}{p}>0$, $j=\overline{1,k}$, for (21) to be solvable in the space $L_{p,\beta}$ it is necessary and sufficient that $(\mathcal{F}g_+)(z_j)=0$, $j=\overline{1,k}$, which is equivalent to the equalities

$$\int_0^1 x^{\zeta_j-1}f(x)\,dx=0,\quad j=\overline{1,k},$$

i.e., in this case equation (21) has an index $-k$.

If $\beta+\frac{1}{p}=\operatorname{Re}\zeta_k$ for some $k=1,\dots,m$, then $s(-\operatorname{Im}\zeta_k)=0$ so that equation (21) is not normally solvable.

Remark 0.3. Though the considered equations are second kind Volterra integral equations, because of the fact that their kernels have first order fixed singularities they have a nonzero (negative) index in the space $L_{p,\beta}$ when $\beta+\frac{1}{p}<1$, and the method of successive approximations converges only if $\beta+\frac{1}{p}>1$ (the latter statement follows from equalities (3) and (10)).

The considered equations are model equations in the theory of design and control of automatized design of cycles (of crushing and grinding processes). In the general linear case the equation has the form (see [1], Chapter 3, §1)

$$\varphi(x) - \int\limits_x^1 K(x,y)\varphi(y)\,dy = f(x), \quad 0 < x < 1,$$

where the functions φ and f are defined by the densities of mass distribution according to sizes of the input and output of the system, while the kernel $K(x,y)$ depends on the physical and technical parameters of the crushing equipment and has properties

$$K(x,y) \geq 0, \quad \int\limits_0^y K(x,y)\,dx = 1, \quad 0 < x \leq y < 1;$$

moreover, condition (16) is always fulfilled.

Integral convolution equations and reducible to them equations with fixed singularities were investigated by many authors (see [2], [3]) and the referenses therein). The boundedness and compactness criteria for a wide class of integral operators (including the operators investigated above) are given in [4].

Acknowledgements

The author extends his thanks to the Royal Society for giving him an opportunity to work at Sussex University and to Dr. E. Shargorodsky for constant attention and care during the work on this paper.

References

1. D. Basheleishvili, Software for automated design and control systems. (Russian) *Part IV, Tbilisi State University Publishing House,* 1993.
2. R. Duduchava, Integral equations with fixed singularities. *Teubner, Leipzig,* 1979.
3. I. C. Gohberg and I. A. Fel'dman, Convolution equations and projection method for their solution. *Transl. Math Monographs,* **41**, *Amer. Math. Soc. Providence,* TI, 1974.
4. V. Kokilashvili and A. Meskhi, The boundedness and compactness criteria for some classical integral operators. *Lecture notes in Pure and Appl. Math.,* **213**, *"Functional Spaces", Proceedings of the conference, Poznan, Poland, August* 28-*September* **3**(1998), 78–296, *New York, Basel, Marcel, Dekker,* 2000.
5. Prössdorf S., Eigine Klassen singulärer Gleichungen. *Berlin* 1974, *Engl. Transl. North-Holland,* 1978.

6. E. T. Whittaker and G. N. Watson, A work of modern analisis. *Cambridge University Press, Cambrige,* 1962.

(Received 13.09.2002)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., 380093
Georgia