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STOCHASTIC INTEGRAL REPRESENTATION OF SOME BROWNIAN FUNCTIONAL WITH EXPLICIT EXPRESSION OF INTEGRAND

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Abstract. In the present work we consider some path-dependent Brownian functionals that are interesting from the point of view of stochastic financial mathematics and derive formulas for a stochastic integral representation of the Clark-Ocone type with an explicit form of the integrands.

Keywords and phrases: Brownian functional, Malliavin stochastic derivative, stochastic integral representation, Clark-Ocone formula.

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1 Introduction. Let B_t be a Brownian motion on a standard filtered probability space $(\Omega, \Im, \Im_t, P)$ and let $\Im_t = \Im_t^B$ be the augmentation of the filtration generated by B. One of the important properties of Ito's stochastic integral is the following: if f is a square integrable adapted process, then the process $\xi_t = \int_0^t f(s,\omega)dB_s(\omega)$ is a martingale with respect to the filtration $\{\Im_t\}_{t\geq 0}$. On the other hand, according to the well-known Clark formula, the inverse statement (so-called stochastic integral representation theorem) is also true. Indeed, if F is a square integrable $\Im_T$-measurable random variable, then (see [1]) there exist a square integrable $\Im_t$-adapted random process $\varphi(t,\omega)$ such that

$$F = EF + \int_0^T \varphi(t,\omega)dB_t(\omega)$$

(taking the conditional mathematical expectation from the both sides of the last relation one can obtain the stochastic integral representation for the associated to F Levy's martingale $M_t = E[F|\Im_t]$. Therefore, statements of this type are also known as martingale representation theorems).

It should be noted that finding an explicit expression for $\varphi(t,\omega)$ is a very difficult problem. In this direction, it is known one general result, called Clark-Ocone formula (see [2]), according to which $\varphi(t,\omega) = E[D_tF|\Im_t]$, where D_t is the so called Malliavin stochastic derivative. A different method for finding the process $\varphi(t,\omega)$ (for so-called running maximum of Brownian motion) was proposed by Shiryaev, Yor and Graversen (2003, 2006), which was based on the Ito (generalized) formula and the Levy theorem for the Levy martingale $M_t = E[F|\Im_t]$, associated with F. Later on, using the Clark-Ocone formula, Renaud and Remillard (2006) have established explicit martingale representations for some path-dependent Wiener functionals (precisely for the smooth function of the Wiener process and its maximum and minimum). Jaoshvili and Purtukhia (see [3])in the frame of the classical Ito calculus constructed $\varphi(t,\omega)$ explicitly, by using both the standard L_2 theory and the theories of weighted Sobolev spaces, for some class of functionals F that do not have a stochastic derivative. Further, we generalized the Clark-Ocone formula in case, when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding the integrand (see [4]).

On the other hand, in spite of the fact that Clark-Ocone formula gives construction of integrand, there are problems with practical realizations. In particular, even in case of smoothness of F, calculation of its Malliavin derivative and then conditional mathematical expectation (or predictable projection in general case) of obtained expression are rather difficult. Here we consider some path-dependent Brownian functional that is interesting from the point of view of stochastic financial mathematics and derive formulas for a stochastic integral representation of the Clark-Ocone type with an explicit form of the integrands (see also [5, 6]).

2 Main results. We fix a positive integer n and consider the following path-dependent Brownian functional $F(n) = (\int_0^T B_s ds)^n$.

Theorem 1. *The functional $F(n)$ admits the following stochastic integral representation*

$$F(n) = EF(n) + n \sum_{r=0}^{[(n-1)/2]} \frac{(2r-1)!!}{3^r} C_{n-1}^{2r} \int_0^T (T-t)^{3r+1} \Big(\int_0^t (T-s)dB_s \Big)^{n-1-2r} dB_t$$

(here and below $(-1)!! := 1$).

Proof. According to the stochastic version of integration by parts, we have

$$\int_0^T B_s ds = B_s s|_0^T - \int_0^T s dB_s = \int_0^T (T-s)dB_s.$$

Therefore, it is easy to see that the random variable $\int_0^T B_s ds$ has a normal distribution with parameters zero and $T^3/3$. Hence,

$$\int_t^T (T-s)dB_s \sim N(0, (T-t)^3/3).$$

Based on the rule of stochastic differentiation of a composite function for the integrand in the Clark-Ocone representation, we have

$$\varphi(t,\omega) = E\Big[n \Big(\int_0^T B_s ds \Big)^{n-1} \cdot \int_0^T I_{[0,s]}(t) ds | \Im_t \Big]$$

$$= E\Big[n \Big(\int_0^T (T-s)dB_s \Big)^{n-1} (T-t) | \Im_t \Big]$$

$$= n(T-t)E\Big[\Big(\int_t^T (T-s)dB_s + \int_0^t (T-s)dB_s\Big)^{n-1} |\Im_t\Big].$$

Further, thanks to the well-known properties of conditional mathematical expectation, using the Newton binomial formula and the values of the moments of the normal distribution, it is not difficult to establish that

$$\varphi(t,\omega) = n(T-t)E\Big[\Big(\int_t^T (T-s)dB_s + x\Big)^{n-1}\Big]\Big|_{x=\int_0^t (T-s)dB_s}$$

$$= n(T-t)\sum_{r=0}^{[(n-1)/2]} \frac{(2r-1)!!}{3^r} C_{n-1}^{2r} (T-t)^{3r}\Big(\int_0^t (T-s)dB_s\Big)^{n-1-2r}.$$

This completes the proof of the theorem. □

Example.

$$\Big(\int_0^T B_s ds\Big)^2 = \frac{T^3}{3} + \int_0^T \Big[2(T-t)\int_0^t (T-s)dB_s\Big] dB_t.$$

We introduce the following notation:

$$\sigma^2 = (T-t)^3/3; \quad erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt; \quad \alpha(2i-1,x) = 0,$$

$$\alpha(2i,x) = (2i-1)!!\sqrt{\frac{\pi}{2}}\Big[erf\Big(\frac{x}{\sqrt{2}}\Big)+1\Big]; \quad \beta(2i-1)=0, \quad \beta(2i)=1;$$

$$\gamma(2i-1,x)=0, \quad \gamma(2i,x) = (2i-1)!!\sqrt{\frac{\pi}{2}}\Big[1-erf\Big(\frac{x}{\sqrt{2}}\Big)\Big];$$

$$\delta(i,x) = e^{-x^2/2}\cdot \sum_{r=1}^{[i/2]-\beta(i)+1} \frac{(i-1)!!}{(i-2r+1)!!} x^{i-(2r-1+\beta(i))}.$$

Using similar approaches, one can prove the following theorems.

Theorem 2. *For any odd number n the functional $(F(n))^+$ admits the integral representation*

$$(F(n))^+ = E(F(n))^+$$

$$+\frac{n}{\sqrt{2\pi}}\sum_{i=0}^{n-1} C_{n-1}^i \int_0^T \sigma^i (T-t)x^{n-1-i}\Big[\gamma\Big(i,-\frac{x}{\sigma}\Big) + \Big(-\frac{x}{\sigma}\Big)^{\beta(i)}\delta\Big(i,-\frac{x}{\sigma}\Big)\Big]\Big|_{x=\int_0^t (T-s)dB_s} dB_t.$$

Theorem 3. *For the functional $G(n,K) = (F(n)-K)^+$ with even n the following stochastic integral representation is fulfilled*

$$G(n,K) = EG(n,K) + \frac{n}{\sqrt{2\pi}}\sum_{i=0}^{n-1} C_{n-1}^i \int_0^T \sigma^i (T-t)[Q_1(n,K,t) + Q_2(n,K,t)]dB_t,$$

where

$$Q_1(n,K,t) = x^{n-1-i}\Big[\alpha\Big(i,\frac{-K^{1/n}-x}{\sigma}\Big) - \Big(\frac{-K^{1/n}-x}{\sigma}\Big)^{\beta(i)}\delta\Big(i,\frac{-K^{1/n}-x}{\sigma}\Big)\Big]\Big|_{x=\int_0^t (T-s)dB_s}$$

and

$$Q_2(n,K,t) = x^{n-1-i}\Big[\gamma\Big(i,\frac{K^{1/n}-x}{\sigma}\Big) + \Big(\frac{K^{1/n}-x}{\sigma}\Big)^{\beta(i)}\delta\Big(i,\frac{K^{1/n}-x}{\sigma}\Big)\Big]\Big|_{x=\int_0^t (T-s)dB_s}.$$

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