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Mathematics

Stochastic Derivative of Poisson Polynomial Functionals and its Application

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In the theory of stochastic integration, in contrast to the standard integration theory, besides the fact that the integrand is the measurable function of two variables, it should be the adapted (nonanticipated) process. Skorokhod (1975) replaced this requirement with the requirement of smoothness in some sense of the integrand. Gaveau and Trauber (1982) proved that the Skorokhod operator of stochastic integration coincides with the conjugate operator of a stochastic derivative (with the so-called Malliavin) operator. Ocone (1984) proved that the integrand in the martingale representation theorem coincides with the predictable projection of the stochastic derivative of the functional. In the Weiner case there are two equivalent definitions of a stochastic derivative, but in general, for so called normal martingale classes these definitions are not equivalent. Ma, Protter and Martin (1998), built the corresponding example. In the present work, a new constructive definition of the stochastic derivative of the polynomial Poisson functional is introduced. It is shown that this definition is equivalent to a general definition based on a chaotic expansion of functional, and its properties are studied. The stochastic integral representation theorem with an explicit expression of the integrand is proved. *© 2020 Bull. Georg. Natl. Acad. Sci.*

Poisson functional, stochastic (Malliavin) derivative, Clark representation, Clark-Haussmann-Ocone representation

Since the 70s of the last century, many attempts are made to weaken the requirement that the integrand be adapted in Ito's stochastic integral. Skorokhod [1] suggested absolutely different method, which required for the integrand to be smooth in a certain sense. This idea was later on developed in the works of Gaveau and Trauber [2], Nualart and Zakai [3], Pardoux (1982), Protter and Malliavin (1979), etc. In particular, Gaveau and Trauber proved that the Skorokhod operator of stochastic integration coincides with the conjugate operator of a stochastic derivative (with the so-called Malliavin's) operator.

On the other hand, in the theory of random processes special place take the so-called martingale representation theorems which, for example, implies the representation of the Wiener and Poisson functionals in the form of stochastic integrals. According to the well-known result obtained by Clark [4], if

F is a $\Im_T^W = \sigma\{W_s : 0 \le s \le T\}$ -measurable random variable with $EF^2 < \infty$, then there exists the adapted process $\varphi(\cdot,\cdot) \in L_2([0,T]\times\Omega)$, such that the integral representation: $F = EF + \int_0^T \varphi(t,\omega)dW(t)$ holds. Due to the so-called Clark-Ocone's formula [5] $\varphi(t,\omega) = E[D_t^W F \mid \Im_t^W](\omega)$, where $D_t^W F$ is the stochastic derivative of the functional F. Application of the above expression needs as a rule, on the one hand, essential efforts, and, on the other hand, in the cases if the functional ξ has no stochastic derivative, its application is impossible. For "maximal" type functionals another distinct method of finding an integrand belongs to Shiryaev and Yor (2003).

Our approach within the classical Ito's calculus allows one to construct $\varphi(t,\omega)$ explicitly by using both the standard L_2 theory and the theories of weighted Sobolev spaces, if the functional ξ has no stochastic derivative [6]. It turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (with prof. O. Glonti, 2014) generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is smooth [7].

The subsequent generalization of the Clark-Ocone formula to the class of normal martingales is due to Ma, Protter and Martin [8]: if $F \in D_{2,1}$, then the Clark-Haussmann-Ocone representation $F = EF + \int_0^T {}^p(D_t F)dM(t)$ is valid, where ${}^p(D_t F)$ is the predictable projection of the stochastic derivative $D_t F$ of the functional F. As is seen, this representation likewise needs the existence of a stochastic derivative. On the other hand, in this case, unlike the Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev space $D_{2,1}$. In his work, Purtukhia [9] defined the space $D_{q,1}\,(1 < q < 2)$ for the normal martingales and generalized the Clark-Haussmann-Ocone formula.

We proposed [10, 11] a new approach to the determination of the stochastic derivative of the Poisson functionals and obtained an explicit form of the integrand in the integral representation. Here a more convenient and practical form for finding the explicit expression of the integrand in the Clark-Haussmann-Ocone representation of functionals of the Poisson process N will be found. In particular, in the conditional mathematical expectation of the above-mentioned integrand the σ-algebra $\Im_{t-}^N$ can be replaced by a more natural σ-algebra $\Im_t^N$, which, in turn, allows us to more effectively use the well-known properties of the Poisson process.

Auxiliary Concepts and Results

Definition 1 [8]. For any $F \in L_2(\Omega)$ we have by the CRP that there exists a sequence of symmetrical functions $f_n \in L_2\left([0,T]^n\right)$, $n = 1,2,\ldots$, such that $F = \sum_{n=0}^{\infty} I_n\left(f_n\right)$. Denote $D_{2,1} := \left\{ F = \sum_{n=0}^{\infty} I_n\left(f_n\right) : \sum_{n=1}^{\infty} nn! \| f_n \|^2_{L_2\left([0,T]^n\right)} < \infty \right\}$. The derivative operator is analogous to what is often called the Malliavin derivative in Wiener case, and it is defined as a linear operator $D_\cdot : D_{2,1} \to L_2([0,T]\times\Omega)$, by $D_t F = \sum_{n=1}^{\infty} n I_{n-1}\left(f_n\left(\cdot,t\right)\right)$.

Ma, Protter and Martin [8] gave an example showing that two possible ways of determination of a stochastic derivative coincide if and only if the quadratic martingale characteristic $[M,M]$ is the deterministic function (as, for example, in the Wiener case $[W,W]_t = t$). Consequently, the Clark-Haussmann-Ocone formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process, saying nothing on the construction of its predictable projection.

Example 1 [8]. Consider a symmetric function $f(s,t) = I_{(a,b]}(s) I_{(a,b]}(t)$. The second chaos $I_2(f)$ can be computed as

$$I_2(f) = 2! \int\limits_{0<s<t\le 1} f(s,t) dM_s dM_t = 2\int\limits_a^b \int\limits_a^{t-} dM_s dM_t = 2\int\limits_a^b (M_{t-} - M_a)^2 dM_t =$$

$$= (M_b - M_a)^2 - \{[M,M]_b - [M,M]_a\}. \tag{1}$$

Here, the last equality is due to Ito's formula. Now consider the function $F(x,y) = (y-x)^2$ and define a smooth functional $G = F(M_a, M_b)$. If we define the derivative of G in a manner analogous to one of the equivalent definitions in the Wiener case, we get:

$$D_t G = D_t\left[F(M_a, M_b)\right] = 2(M_b - M_a) I_{(a,b]}(t). \tag{2}$$

However, by definition based on decomposition in the form of a series of multiple Ito integrals, we have $D_t I_2(f) = 2 I_1(f(\cdot,t)) = 2(M_b - M_a) I_{(a,b]}(t)$.

We can substitute this into (2) and compare it with (1) to see that the two definitions coincide if and only if $D_t\{[M,M]_b - [M,M]_a\} = 0$, for all $t \in [0,1]$. This means that $[M,M]_b - [M,M]_a$ must be constant. If we look at the structure equation $[M,M]_t = t + \int\limits_0^t \varphi_s dM_s$, this amounts to saying that $\varphi_s \equiv 0$, therefore the two definitions are in contradiction and cannot hold simultaneously unless $M = W$, Wiener process.

Let $(\Omega, \Im, (\Im_t)_{0\le t\le T}, P)$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it: $P_k := P(N_t = k) = e^{-t} t^k / k!, k = 0,1,2,...$ and that $\Im_t$ is generated by N $(\Im_t = \Im_t^N)$, $\Im = \Im_T$. Let $M_t := N_t - t$ (the Compensated Poisson process) and $\Delta M_t = M_t - M_{t-}$.

Let $Z^+ = \{0,1,2,...\}$; $\Delta_- f(k) = f(k) - f(k-1)$ $(f(k) = 0,\ k < 0)$; $\Delta_-^n := (\Delta_-)^n$ and define the Poisson-Charlier polynomials: $\Pi_n(k) = (-1)^n \Delta_-^n P_k / P_k$, $n \ge 1$; $\Pi_0 = 1$. It is known that the system of normalized Poisson-Charlier polynomials is a basis in $L_2(Z^+) := \left\{ f : \sum\limits_{k=0}^{\infty} f^2(k) < \infty \right\}$. Let $L_2^T := \left\{ f : e^{-T} \sum\limits_{k=0}^{\infty} f^2(k) T^k / k! < \infty \right\}$. This is a Banach space with the basis $k^n e^{-T} T^k / k!$. Denote $\Delta_+^x f(x) = f(x+1) - f(x)$.

Proposition 1. The moment of the n-th order $\nu_n(t) = \sum\limits_{k=0}^{\infty} (k-t)^n \frac{t^k}{k!} e^{-t}$ $(n \ge 2)$ of the Compensated Poisson process satisfies the equation $\frac{d\nu_n(t)}{dt} = \sum\limits_{k=0}^{n-2} C_n^k \nu_k(t)$.

Proposition 2. For the n-th degree of the Compensated Poisson process the following integral representation takes place:

$$M_t^n = n\int_0^t M_{s-}^{n-1} dM_s + \sum_{i=2}^{n}\int_0^t C_n^i M_{s-}^{n-i} dN_s \text{ . (P- a.s.).}$$

Theorem 1 [11]. Let $f \in L_2^T$ and for some $\varepsilon > 0$: $\Delta_+^x f(\cdot - T) \in L_2^{(1+\varepsilon)T}$, then the stochastic integral below is well defined and the following representation is valid:

$$f(M_T) = E[f(M_T)] + \int_{(0,T]} E[\Delta_+^x f(M_T) \mid \Im_{t-}] dM_t \quad (P\text{-a.s.}).$$

New Construction of Derivative

Definition 2. The class of smooth Poisson functionals S^M is the class of a random variables which has the form $F = f(M_{t_1},...,M_{t_n})$, $f \in C_p^\infty(R^n)$, $t_i \in [0,T]$, $n \ge 1$, where f and all of its partial derivatives have polynomial growth.

Definition 3. The stochastic (Malliavin) derivative of a smooth random variable $F \in S^M$ is the stochastic process $D_t^M F$ given by

$$D_t^M F = \sum_{k=1}^{n}\sum_{i_1<\cdots<i_k} \Delta_+^{i_1}(\cdots(\Delta_+^{i_k} f(M_{t_1},...,M_{t_n})))I_{[0,t_{i_1}]}(t)\cdots I_{[0,t_{i_k}]}(t),$$

where $\Delta_+^i f(x_1,...,x_i,...,x_n) = f(x_1,...,x_i+1,...,x_n) - f(x_1,...,x_i,...,x_n)$.

Remark 1. Because $M_s = \int_0^T I_{[0.s]}(\theta)dM_\theta = I_1(I_{[0.s]}(\cdot))$ and, hence, $D_t^M M_s = I_{[0.s]}(t)$, we have the following definition

$$D_t^M F = \sum_{k=1}^{n}\sum_{i_1<\cdots<i_k} \Delta_+^{i_1}(\cdots(\Delta_+^{i_k} f(M_{t_1},...,M_{t_n})))D_t^M M_{t_1}\cdots D_t^M M_{t_n} \text{ .}$$

Lemma 1. The stochastic (Malliavin) derivative of the multiple Ito integral $I_m(f_m)$ is the same as in Ma, Protter and Martin [8]:

$$D_t^M I_m(f_m) = mI_{m-1}(f_m(\cdot,t)).$$

Proof. Fix $m \ge 1$. Let f_m elementary function of the form $f_m(t_1,...,t_m) = \sum_{i_1,...,i_m}^{n} a_{i_1,...,i_m} I_{A_{i_1}\times\cdots\times A_{i_m}}(t_1,...,t_m)$, where $A_1, A_2, ..., A_n$ are pairwise disjoint sets from $B([0,T])$ and $a_{i_1,...,i_m} = 0$ if any two of the indices $i_1,...,i_m$ are equal. For this function we define multiple integral as $I_m(f_m) = \sum_{i_1,...,i_m}^{n} a_{i_1,...,i_m} M(A_{i_1})\cdots M(A_{i_m})$, where $M(A) = M(I_A) = \int_0^T I_A dM_s$. According to the Definition 3, using the relation $\Delta_+^{x_1}(\cdots(\Delta_+^{x_k}(x_1\cdot x_2\cdots x_n))) = x_{k+1}\cdot x_{k+2}\cdots x_n$, we have

$$D_t^M\left[M(A_{i_1})\cdots M(A_{i_m})\right] = \sum_{k=1}^{m}\sum_{j_1<\cdots<j_k} \Delta_+^{x_{j_1}}\left(\cdots\left(\Delta_+^{x_{j_k}}\left[M(A_{i_1})\cdots M(A_{i_m})\right]\right)\right)I_{A_{j_1}}(t)\cdots I_{A_{j_k}}(t) =$$

$$=\sum_{k=1}^{m} M(A_1)\cdots M(A_{k-1})M(A_{k+1})\cdots M(A_m)I_{A_k}(t).$$

Therefore, due to the linearity of the operator I_m, the assertion of the lemma is true for elementary functions. After this, the lemma can be completed using the standard technique of approximating of any function $f_m \in L_2\left([0,T]^m\right)$ by a sequence of elementary functions. □

Example 2. If $F = f(M_a, M_b)$, then our definition gives

$$D_t^M f(M_a,M_b) = \Delta_+^x\left(\Delta_+^y f(M_a,M_b)\right)I_{[0,a]}(t)\cdot I_{[0,b]}(t) + \Delta_+^x f(M_a,M_b)I_{[0,a]}(t) + \Delta_+^y f(M_a,M_b)I_{[0,b]}(t).$$

If we take into account the relations: $\Delta_+^x\left(\Delta_+^y (y-x)^2\right) = -2$; $\Delta_+^x (y-x)^2 = -2(y-x)+1$ and $\Delta_+^y (y-x)^2 = 2(y-x)+1$ it is easy to see that for the functional from the example of Ma, Protter and Martin [8] we have $D_t^M (M_b - M_a)^2 = -2I_{[0,a]}(t) +$

$$+\left[-2(M_b - M_a)+1\right]I_{[0,a]}(t) + \left[2(M_b - M_a)+1\right]I_{[0,b]}(t) = 2(M_b - M_a)I_{(a,b]}(t) + I_{(a,b]}(t).$$

If we recall now that $[M,M]_s = M_s - s$ and $D_t^M M_s = I_{[0,s]}(t)$, then it is easy to see that these two definitions are equivalent in this case. Indeed, on the one hand, we have

$$I_2(f) = (M_b - M_a)^2 - \{[M,M]_b - [M,M]_a\} = (M_b - M_a)^2 - \{M_b - b - M_a + a\}$$

and therefore

$$D_t^M I_2(f) = 2(M_b - M_a)I_{(a,b]}(t) + I_{(a,b]}(t) - I_{[0,b]}(t) + I_{[0,a]}(t) = 2(M_b - M_a)I_{(a,b]}(t).$$

On the other hand, due to the Lemma 1, we can write

$$D_t^M I_2(f) = 2I_1(f(\cdot,t)) = 2\int_a^b I_{(a,b]}(s)\,dM_s \cdot I_{(a,b]}(t) = 2(M_b - M_a)I_{(a,b]}(t). \ \square$$

Lemma 2. Let F and G the smooth Poisson functionals, then $F\cdot G$ is also smooth and the following relaion is valid:

$$D_t^M(F\cdot G) = D_t^M F\cdot G + F\cdot D_t^M G + D_t^M F\cdot D_t^M G.$$

Proof. For simplicity, we consider the case $F = f(M_a)$ and $G = g(M_b)$, where $f, g \in C_p^\infty(R^n)$. It is clear that $\Delta(f\cdot g) = \Delta(f)\cdot g + f\cdot\Delta(g) + \Delta(f)\cdot\Delta(g)$. Hence, due to the Definition 3, we have

$$D_t^M[f(M_a)\cdot g(M_b)] = \Delta_+^x(\Delta_+^y[f(M_a)\cdot g(M_b)])I_{[0,a]}(t)\cdot I_{[0,b]}(t) +$$
$$+\Delta_+^y[f(M_a)\cdot g(M_b)]I_{[0,b]}(t) + \Delta_+^x[f(M_a)\cdot g(M_b)]I_{[0,a]}(t) =$$
$$= \Delta_+^x[f(M_a)]I_{[0,a]}(t)\cdot\Delta_+^y[g(M_b)]I_{[0,b]}(t) +$$
$$= f(M_a)\cdot\Delta_+^y[g(M_b)]I_{[0,b]}(t) + g(M_b)\cdot\Delta_+^x[f(M_a)]I_{[0,a]}(t) =$$
$$= D_t^M F\cdot D_t^M G + F\cdot D_t^M G + D_t^M F\cdot G. \ \square$$

Lemma 3. Let F be a smooth Poisson functional, then F^n $(n \geq 1)$ is also smooth and the following relaion is valid: $D_t^M\left(F^n\right) = \sum_{i=1}^{n} C_n^i F^{n-i}\left(D_t^M F\right)^i$.

Proof. For $n=1$ it is evident. For $n=2$ from the Lemma 2 we have

$$D_t^M\left(F^2\right)=D_t^M\left(F\cdot F\right)=D_t^M F\cdot F+F\cdot D_t^M F+D_t^M F\cdot D_t^M F=\sum_{i=1}^{2}C_2^i F^{2-i}\left(D_t^M F\right)^i .$$

Let the lemma be true for n and verify its validity for $n+1$. Indeed, due to the Lemma 2 and using the relation $C_n^i+C_n^{i-1}=C_{n+1}^i$, we easily obtain that

$$D_t^M(F^{n+1})=D_t^M(F^n\cdot F)=D_t^M(F^n)\cdot F+F^n\cdot D_t^M F+D_t^M(F^n)\cdot D_t^M F=$$

$$=nF^n\cdot D_t^M F+\sum_{i=2}^{n}C_n^i F^{n+1-i}(D_t^M F)^i+F^n\cdot D_t^M F+\sum_{i=1}^{n-1}C_n^i F^{n-i}(D_t^M F)^{i+1}+(D_t^M F)^{n+1}=$$

$$=C_{n+1}^1 F^{n+1-1}\cdot(D_t^M F)+\sum_{i=2}^{n}(C_n^i+C_n^{i-1})F^{n+1-i}(D_t^M F)^i+C_{n+1}^{n+1}F^{n+1-(n+1)}(D_t^M F)^{n+1}=$$

$$=\sum_{i=1}^{n+1}C_{n+1}^i F^{n+1-i}(D_t^M F)^i .$$

Verification ends using mathematical induction. □

Corollary 1. Let F be a smooth Poisson functional, then F^n $(n\geq 1)$ is also smooth and the following relaion is valid: $D_t^M\left(F^n\right)=\left(F+D_t^M F\right)^n-F^n$.

Corollary 2. Let F be a smooth Poisson functional and $P_n(x)=\sum_{k=0}^{n}a_k x^k$ is a polynomial function then $P_n(F)$ is stochastic differentiable and we have:

$$D_t^M P_n\left(F\right)=\sum_{k=0}^{n}a_k\sum_{i=1}^{k}C_k^i F^{k-i}\left(D_t^M F\right)^i=\sum_{k=0}^{n}a_k\left[\left(F+D_t^M F\right)^k-F^k\right].$$

Stochastic integral representation of Poisson polynomial functionals

Example 3. The following representation holds:

$$M_T^2=E\left(M_T^2\right)+\int_{(0,T]}\left(1+2M_{s-}\right)dM_s.\ (P\text{ - a.s.}).$$

Consider the random process: $X_t:=E[M_T^2|\Im_t]$. Due to the independence and stationarity of the increments of the Poisson process from its past, taking into account the values of the moments of the Poisson process, we can write that:

$$X_t=E[(M_T-M_t+M_t)^2|\Im_t]=E[(M_T-M_t)^2]+$$

$$+2M_T E(M_T-M_t)+M_t^2=T-t+M_t^2 .$$

Based on Proposition 2, if we substitute in the last representation the integral representation of M_t^2: $M_t^2=\int_{(0,t]}2M_{s-}dM_s+\int_{(0,t]}dN_s$, obtain that:

$$X_t=T-t+\int_{(0,t]}2M_{s-}dM_s+N_t=T+M_t+\int_{(0,t]}2M_{s-}dM_s .$$

But $E[M_T^2]=T$ and $X_T=M_T^2$, and thus the verification is complete. □

Example 4. Similarly, we get a representation for the third degree of the Compensated Poisson process:

$$M_T^3 = E\left(M_T^3\right) + \int_{(0,T]} \left[1 + 3\left(T-s\right) + 3M_{s-} + 3M_{s-}^2\right] dM_s \ .$$

Theorem 2. For every natural number $n \geq 1$ the representation is true:

$$M_T^n = E\left(M_T^n\right) + \sum_{k=1}^{n} C_n^k \sum_{i=1}^{k} C_k^i \int_{(0,T]} M_{s-}^{k-i} \nu_{n-k}\left(T-s\right) dM_s \ \ (P\text{ - a.s.}). \tag{3}$$

Proof. In fact, in the case $n=1$ the representation (3) is trivial, and for cases $n=2$ and $n=3$ we have Examples 3 and 4. So consider the case $n \geq 4$. Calculate the conditional mathematical expectation $X_t := E[M_T^n | \Im_t]$. According to Newton's binomial formula we can write:

$$X_t := E[M_T^n | \Im_t] = E[(M_T - M_t + M_t)^n | \Im_t] = E[\sum_{k=0}^{n} C_n^i (M_T - M_t)^{n-k} M_t^k | \Im_t] \cdot$$

Based on the known properties of the Poisson process and the conditional mathematical expectation, we get:

$$X_t = \sum_{k=0}^{n} C_n^i \nu_{n-k}\left(T-t\right) M_t^k = \nu_n\left(T-t\right) + \sum_{k=1}^{n-2} C_n^k M_t^k \nu_{n-k}\left(T-t\right) + M_t^n \ .$$

Calculate now the stochastic differential of the process X_t. We write the Ito formula and use the statements from Proposition 1 and Proposition 2. We get that:

$$X_t = X_0 - \sum_{k=0}^{n-2} C_n^k \int_{(0,t]} \nu_k (T-s) ds - \sum_{k=1}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int_{(0,t]} M_{s-}^k \nu_i (T-s) ds +$$

$$+ \sum_{k=2}^{n} C_n^k \int_{(0,t]} M_{s-}^{n-k} dN_s + \sum_{k=1}^{n-2} C_n^k \sum_{i=2}^{k} C_k^i \int_{(0,t]} M_{s-}^{k-i} \nu_{n-k} (T-s) dN_s +$$

$$+ \sum_{k=1}^{n-2} C_n^k k \int_{(0,t]} M_{s-}^{k-1} \nu_{n-k} (T-s) dM_s + n \int_{(0,t]} M_{s-}^{n-1} dM_s \ .$$

In this representation, taking into account the equality $\nu_1(T) = 0$, it is possible to combine the second and third terms, as well as the fourth and fifth terms. As a result we get:

$$X_t = X_0 - \sum_{k=0}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int_{(0,t]} M_{s-}^k \nu_i \left(T-s\right) ds + \sum_{p=1}^{n} C_n^p \sum_{j=2}^{p} C_p^j \int_{(0,t]} M_{s-}^{p-j} \nu_{n-p}\left(T-s\right) dN_s +$$

$$+ \sum_{k=1}^{n-2} C_n^k k \int_{(0,t]} M_{s-}^{k-1} \nu_{n-k} (T-s) dM_s + n \int_{(0,t]} M_{s-}^{n-1} dM_s =$$

$$= \nu_n(T) - I_s + I_{N_S} + \sum_{k=1}^{n} C_n^k k \int_{(0,t]} M_{s-}^{k-1} \nu_{n-k} (T-s) dM_s \ ,$$

Where

$$I_s := \sum_{k=0}^{n-2} C_n^k \sum_{i=0}^{n-k-2} C_{n-k}^i \int_{(0,t]} M_{s-}^k \nu_i (T-s) ds \ , \ I_{N_S} := \sum_{p=1}^{n} C_n^p \sum_{j=2}^{p} C_p^j \int_{(0,t]} M_{s-}^{p-j} \nu_{n-p} (T-s) dN_s \ .$$

It is not difficult to see (as in Examples 3 and 4) that for each integral with respect to dN_s one can found the corresponding integral with respect to ds, whose integrands are equal under in modulus and opposite in sign. To do this, we must compare the same degrees of M_{s-} with each other and use the relation $C_n^m C_{n-m}^{n-m-2-j} = C_n^{m+2+j} C_{m+2+j}^{2+j}$.

Hence, taking into account the equalities $X_T = M_T^n$ and $X_0 = E(M_T^n)$, we get:

$$M_T^n = E[M_T^n] + \sum_{k=1}^{n} C_n^k \int_{(0,T]} M_{s-}^{n-k} dM_s +$$

$$+ \sum_{k=2}^{n-2} C_n^k \sum_{i=1}^{k} C_k^i \int_{(0,T]} M_{s-}^{k-i} \nu_{n-k}(T-s) dM_s + n \int_{(0,T]} \nu_{n-1}(T-s) dM_s =$$

$$= E(M_T^n) + \sum_{k=1}^{n} C_n^k \sum_{i=1}^{k} C_k^i \int_{(0,T]} M_{s-}^{k-i} \nu_{n-k}(T-s) dM_s \,. \ \square$$

Corollary 3. For each natural number $n \geq 1$ the following representation is true:

$$M_T^n = E(M_T^n) + \int_{(0,T]} E[\Delta_+^x (M_{s-} + M_{T-s})^n \,|\, \Im_s] dM_s \quad (P\text{ - a.s.}). \tag{4}$$

Proof. Due to the Newton's binomial formula and taking into account well-known properties of Compensated Poisson's process, we obtain:

$$E\left[(1 + M_{s-} + M_{T-s})^n \,|\, \Im_s\right] = \sum_{k=0}^{n} C_n^k (1 + M_{s-})^k E\left[M_{T-s}^{n-k} \,|\, \Im_s\right] =$$

$$= \sum_{k=0}^{n} C_n^k (1 + M_{s-})^k E\left[(M_T - M_s)^{n-k} \,|\, \Im_s\right] =$$

$$= \sum_{k=0}^{n} C_n^k (1 + M_{s-})^k E M_{T-s}^{n-k} = \sum_{k=0}^{n} C_n^k (1 + M_{s-})^k \nu_{n-k}(T-s) \,.$$

Similarly, we have

$$E\left[(M_{s-} + M_{T-s})^n \,|\, \Im_s\right] = \sum_{k=0}^{n} C_n^k M_{s-}^k \nu_{n-k}(T-s) \,.$$

Therefore, due to the Theorem 2, using again the Newton's binomial formula $(M_{s-} + 1)^k - M_{s-}^k = \sum_{i=1}^{k} C_k^i M_{s-}^{k-i}$, we finish the proof of corollary. □

Corollary 4. For the polynomial function $P_n(x) = \sum_{k=0}^{n} a_k x^k$ the following representation is valid:

$$P_n(M_T) = E[P_n(M_T)] + \int_{(0,T]} E[\Delta_+^x P_n(M_{s-} + M_{T-s}) \,|\, \Im_s] dM_s \quad (P\text{ - a.s.}).$$

Corollary 5. Since for fixed T $P_n(x-T)$ is again polynomial $\overline{P}_n(x)$ of order n with respect to x, but with other coefficients, therefore it is clear that the representation of type (4) we will have for the polynomial functional with respect to N_T:

$$P_n(N_T) = E[P_n(N_T)] + \int_{(0,T]} E[\Delta_+^x \overline{P}_n(N_{s-} + N_{T-s}) \,|\, \Im_s] dM_s \quad (P\text{ - a.s.}).$$

Theorem 3. For any polynomial function $P_n(x)$, the relation is true where

$$ {}^{p}[D_t^M P_n(M_T)] = E[\Delta_+^x P_n(M_{t-} + M_{T-t}) | \Im_t] \ \ (dP \otimes dt \text{-a.s.}). \tag{5} $$

Proof. According to the well-known result of Ma, Protter and Martin [8], there is the following Clark-Haussmann-Ocone representation formula

$$ P_n(M_T) = E[P_n(M_T)] + \int_{(0,T]} \{{}^{p}[D_t^M P_n(M_T)]\} dM_t \ \ (P\text{- a.s.}). $$

Consider the difference:

$$ \xi_T := \int_{(0,T]} \{E[\Delta_+^x P_n(M_{t-} + M_{T-t}) | \Im_t] - {}^{p}[D_t^M P_n(M_T)]\} dM_t := \int_{(0,T]} \eta_t dM_t \ . $$

According to Theorem 2: $\xi_T = 0$ (P- a.s.). Further, due to the Ito formula:

$$ \xi_T^2 = 2 \int_{(0,T]} \xi_{t-} \eta_t dM_t + \int_{(0,T]} \eta_t^2 d[M,M]_t \ . $$

Taking the mathematical expectation of both sides of the last relation and using the properties of the quadratic and predictable characteristics of the martingale, we get:

$$ 0 = E \int_{(0,T]} \eta_t^2 d[M,M]_t = E \int_{(0,T]} \eta_t^2 d\langle M,M \rangle_t = E \int_{(0,T]} \eta_t^2 dt \ , $$

from which we conclude the validity of relation (5).

მათემატიკა

პუასონის პოლინომიალური ფუნქციონალების სტოქასტური წარმოებული და მისი გამოყენება

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