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FUBINI TYPE THEOREMS FOR ORDINARY AND STOCHASTIC INTEGRALS

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Abstract. Stochastic Fubini type theorems on the change of the integration order of stochastic (in both Ito's and Skorohod's sense) and ordinary (in Lebesgue's sense) integrals are proved under natural conditions on integrands. Moreover, the interchangeability of Sobolev's averaging operator and stochastic integrals is considered.

Introduction

As is well-known, the main result of Lebesgue's multiple integral theory is the Fubini theorem which reduces the double (or, in general, multiple) integrals to iterated integrals. Theorems of this type play a significant role in modern stochastic analysis as well (in particular, in the stochastic differential equation theory). The aim of this paper is to study the change of the integration order of Lebesgue and stochastic (Ito's and Skorohod's) integrals, which cannot be considered in the framework of the ordinary Fubini theorem.

In the ordinary integration theory the measurability requirement on the integrand is essentially less restrictive than the integrability condition, which imposes a certain bound on its absolute value. As for Ito's stochastic integral $\int_0^T f_t(\omega)dw_t$, the situation is in some sense the opposite one to the abovesaid. On the one hand, the integrand $f_t(\omega)$, besides being a measurable function of two variables, must also be an adapted (non-anticipative)

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process (for each $t \in [0,T]$ the function $f_t(\cdot)$ must be measurable with respect to the σ-algebra, $\mathcal{F}_t^w = \sigma\{w_s, s \le t\}$ i.e., it must be independent of an increment of the Wiener process). On the other hand, Ito's stochastic integral $\int\limits_0^T f_t(\omega)dw_t$ must not be understood as the Lebesgue-Stieltjes integral since Wiener process trajectories have an infinite variation (here the limit in integral sums is understood in the mean-square sense). Therefore the stochastic integral obtained by a formal change of the integration order may happen to be not well defined. It is true that in Skorohod's stochastic integral $\int\limits_0^T f_t(\omega)\delta w_t$ there is no requirement for the integrand to be adapted, but instead it becomes necessary to impose a condition of smoothness in some weak sense (in particular, the so-called stochastic differentiability). In the case of adaptive integrands Skorohod's integral coincides with Ito's stochastic integral.

The Fubini theorem for Lebesgue's and Ito's stochastic integrals was obtained by Kallianpur and Striebel in 1969 (see [1]) and was subsequently generalized by Ershov in 1972 (see [2]). Using a different method (in particular, the so-called martingale representation theorem), a Fubuni theorem of this type was proved by Liptser and Shiryaev in 1974 (see [3]). We consider, on the one hand, the case, where the integrand depends on an additional parameter (see Theorems 1.1 and 2.2), and, on the other hand, we present the most natural (from the viewpoint of the theory of random processes) form of the Fubini theorem for Lebesgue's and Ito's stochastic integrals (see Theorem 1.2 and Corollary 1.1; also Theorem 2.1 in the anticipative case). In the particular case, where the integrand is a conditional mathematical expectation, the Fubini theorem of this type was obtained by Korn in 1992 (see [4]), using financial mathematical methods. In this paper the Fubini theorem is proved for Lebesgue's and Skorohod's stochastic integrals as well. Also, the interchangeability of the so-called Sobolev's averaging operator and stochastic integrals is considered (see Theorems 1.3 and 2.3). The same results are obtained in [6] and [7] for the particular cases.

1. Fubini Type Theorems for Lebesgue's and Ito's Integrals

Let us fix $T \in R_+$. Assume that the measurable space $(X, \mathcal{X}, \lambda)$ (with $(X) < \infty$) is given. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a standard probability space with a given Wiener process $(w_t, \mathcal{F}_t)$, $t \in [0,T]$. For simplicity, we consider only the one-dimensional case. Let a function $f \in L_2(\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{X})$ be such that for each $x \in X$ it is adapted. Consider a sequence of partitions $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T$ of the interval $[0,T]$ such that

$$\sup_{0\le k\le n-1} \big(t_{k+1,n} - t_{k,n}\big) \to 0, \ \text{ as } \ n \to \infty.$$

We introduce the following sequence of step functions:

$$f_n(t,\omega,x) := \sum_{k=0}^{n-1} f_{k,n}(\omega,x) I_{[t_{k,n},t_{k-1,n}[}(t),$$

where

$$f_{0,n}(\omega,x) := f(0,\omega,x),$$
$$f_{k,n}(\omega,x) := \frac{1}{t_{k,n}-t_{k-1,n}} \int_{t_{k-1,n}}^{t_{k,n}} f(t,\omega,x)\,dt, \quad k \geq 1.$$

It is obvious that $f_n \in L_2(\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{X})$ and for each $x \in X$ the random variable $f_{k,n}(\cdot,x)$ is measurable with respect to the σ-algebra $\mathcal{F}t_{k,n}$. It is not difficult to verify that the following relations are valid:

$$\int_0^T \big|f_n(t,\omega,x)\big|^2\,dt \leq \int_0^T \big|f(t,\omega,x)\big|^2\,dt \quad (P \times \lambda - \text{a.s.})$$

and

$$\lim_{n\to\infty} \int_0^T \big|f_n(t,\omega,x) - f(t,\omega,x)\big|^2\,dt = 0 \quad (P \times \lambda - \text{a.s.}). \tag{1.1}$$

Hence, by virtue of the Lebesgue dominated convergence theorem (LDCT), we have

$$\lim_{n\to\infty} E\int_0^T \big|f_n(t,\omega,x) - f(t,\omega,x)\big|^2\,dt = 0 \quad (\lambda - \text{a.s.}), \tag{1.2}$$

$$\lim_{n\to\infty} E\int_X \int_0^T \big|f_n(t,\omega,x) - f(t,\omega,x)\big|^2\,dt\lambda(dx) = 0. \tag{1.3}$$

Theorem 1.1. *If $f \in L_2(\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{X})$, then the integrals below are well defined and the relation*

$$\int_X \int_0^T f(t,\omega,x) dw_t \lambda(dx) = \int_0^T \int_X f(t,\omega,x)\lambda(dx)\,dw_t \tag{1.4}$$

is valid P-a.s.

Proof. By the definition of Ito's stochastic integral, it is easy to see that for each n P-a.s., satisfies the equality

$$\int_X \int_0^T f_n(t,\omega,x)dw_t\lambda(dx) = \int_0^T \int_X f_n(t,\omega,x)\lambda(dx)dw_t. \tag{1.5}$$

By the isometricity property of a stochastic integral, relation (1.2) implies that λ-a.s.,:

$$\lim_{n\to\infty} E\Big|\int_0^T \big(f_n(t,\omega,x) - f(t,\omega,x)\big)dw_t\Big|^2 =$$
$$= \lim_{n\to\infty} E\int_0^T \big|f_n(t,\omega,x) - f(t,\omega,x)\big|^2\, dt = 0.$$

Using the Cauchy-Bunyakovski inequality, ordinary Fubuni theorem, isometricity property of a stochastic integral and LDCT, from relations (1.1) and (1.3) we obtain

$$\lim_{n\to\infty} E\Big|\int_X \int_0^T f_n(t,\omega,x)dw_t\lambda(dx) - \int_X \int_0^T f(t,\omega,x)dw_t\lambda(dx)\Big|^2 = 0.$$

Hence there exists a subsequence f_{n_k} of the sequence f_n such that the relation

$$\lim_{n\to\infty} \int_X \int_0^T f_n(t,\omega,x)\, dw_t\lambda(dx) = \int_X \int_0^T f(t,\omega,x)\, dw_t\lambda(dx) \tag{1.6}$$

holds P-a.s.

Analogously, using the Lyapunov inequality, isometricity property of a stochastic integral, the Cauchy-Bunyakovski inequality and ordinary Fubuni theorem, from (1.3) we conclude that

$$\lim_{n\to\infty} E\Big|\int_0^T \int_X f_n(t,\omega,x)\lambda(dx)dw_t - \int_0^T \int_X f(t,\omega,x)\lambda(dx)dw_t\Big| = 0. \tag{1.7}$$

If we now pass to the limit in relation (1.5), written for the subsequence f_{n_k}, as $k\to\infty$, then due to relations (1.6) and (1.7) we obtain equality (1.4). $\square$

Theorem 1.2. *If a function $f(s,\omega,t) : [0,T]^2 \times R_1$ is such that $f \in L_2(\mathcal{B}([0,T]^2) \otimes \mathcal{F})$ and, for each $s \in [0,T]$, $f(s,\cdot,t)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$, then the following equality is P-a.s. valid:*

$$\int_0^T \int_t^T f(s,\omega,t)\,ds\,dw_t = \int_0^T \int_0^t f(t,\omega,s)\,dw_s\,dt. \tag{1.8}$$

Proof. Let us introduce the notations:

$$\begin{aligned}
f_{0,n}(s,\omega) &:= f(s,\omega,0),\\
f_{k,n}(s,\omega) &:= \frac{1}{t_{k+1,n}-t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} f(s,\omega,t)\,dt, \quad k=0,1,\dots,n-1,\\
f_n(s,\omega,t) &:= \sum_{k=0}^{n-1} f_{k,n}(s,\omega) I_{[t_{k,n},t_{k+1,n}[}(t),\\
h_n(\omega,t) &:= \int_t^T f_n(s,\omega,t)\,ds,\\
h_{k,n}(\omega,t) &:= \int_t^{k+1,n} f_{k,n}(s,\omega)\,ds + \sum_{j=k+1}^{n-1} \int_{t_{j,n}}^{t_{j+1,n}} f_{k,n}(s,\omega)\,ds,\\
&\quad k=0,1,\dots,n-1.
\end{aligned}$$

Then it is obvious that

$$h_n(\omega,t) = \sum_{k=0}^{n-1} f_{k,n}(\omega,t) I_{[t_{k,n},t_{k+1,n}[}(t)$$

and

$$\begin{aligned}
\int_0^T h_n(\omega,t) &= \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} h_{k,n}(\omega,t)\,dw_t =\\
&= \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} \Big(\int_t^{t_{k+1,n}} f_{k,n}(s,\omega)\,ds \Big) dw_t +\\
&+ \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} \Big(\sum_{j=k+1}^{n-1} \int_{t_{j,n}}^{t_{j+1,n}} f_{k,n}(s,\omega)\,ds \Big) dw_t :=\\
&= H_n^R(\omega) + \sum_{k=0}^{n-1} \sum_{j=k+1}^{n-1} \int_{t_{j,n}}^{t_{j+1,n}} f_{k,n}(s,\omega)\,ds \big(w_{t_{k+1,n}} - w_{t_{k,n}}\big) :=
\end{aligned}$$

$$= H_n^R(\omega) + H(\omega).$$

Let us also introduce the notations:

$$g_n(t,\omega) := \int_0^t f_n(t,\omega,s)\,dw_s,$$

$$g_{k,n}(t,\omega) := \sum_{j=0}^{k-1} f_{j,k}(t,\omega)\big(w_{t_{j+1,n}} - w_{t_{j,n}}\big) + f_{k,n}(t,\omega)\big(w_t - w_{t_{k,n}}\big).$$

Since in our case

$$f_n(t,\omega,s) = \sum_{k=0}^{n-1} f_{k,n}(t,\omega) I_{[t_{k,n},t_{k+1,n}[}(s),$$

we obtain

$$g_n(t,\omega) = \sum_{k=1}^{n} g_{k,n}(t,\omega) I_{[t_{k,n},t_{k+1,n}[}(t),$$

and

$$\int_0^T g_n(t,\omega)\,dt = \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} g_{k,n}(t,\omega)\,dt =$$

$$= \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} \sum_{j=0}^{k-1} f_{j,n}(t,\omega)\big(w_{t_{j+1,n}} - w_{t_{j,n}}\big)\,dt +$$

$$+ \sum_{k=0}^{n-1} \int_{t_{k,n}}^{t_{k+1,n}} f_{k,n}(t,\omega)\cdot(w_t - w_{t_{k,n}})dt =$$

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{k-1} \big(w_{t_{j+1,n}} - w_{t_{j,n}}\big) \int_{t_{k,n}}^{t_{k+1,n}} f_{j,n}(t,\omega)\,dt + G_n^R(\omega) =$$

$$:= G_n(\omega) + G - n^R(\omega).$$

Using the properties of double sums, we conclude that $H_n(\omega) = G_n(\omega)$ P-a.s., for each $n \geq 1$.

On the other hand, using Ito's formula and the differentiation rule of the Lebesgue integral with a variable lower boundary, we obtain

$$\int_{t_{k,n}}^{t_{k+1,n}} \bigg(\int_t^{t_{k+1,n}} f_{k,n}(s,\omega)\,ds\bigg)\,dw_t =$$

$$= \int_t^{t_{k+1,n}} f_{k,n}(s,\omega)\,ds w_t\Big|_{t_{k,n}}^{t_{k+1,n}} - \int_{t_{k,n}}^{t_{k+1,n}} w_t d\Big(\int_t^{t_{k+1,n}} f_{k,n}(s,\omega) ds\Big) =$$

$$= -\int_{t_{k,n}}^{t_{k+1,n}} f_{k,n}(s,\omega)\,ds w_{t_{k,n}} + \int_{t_{k,n}}^{t_{k+1,n}} w_t f_{k,n}(t,\omega)\,dt =$$

$$= \int_{t_{k,n}}^{t_{k+1,n}} f_{k,n}(t,\omega)\big(w_t - w_{t_{k,n}}\big)\,dt.$$

Hence it is obvious that for each $n \geq 1$ P-a.s.: $H_n^R(\omega) = G_n^R(\omega)$.

Finally, we ascertain that equality (1.8) is true for functions f_n. Further, using arguments similar to those presented above and following the scheme of the proof of Theorem 1.1, it can be easily shown that relation (1.8) is fulfilled. □

Corollary 1.1. *If $f(t,\omega) : [0,T] \times \Omega \to R^1$ is a non-anticipative process adapted to the flow of σ-algebras $\mathcal{F}_t$ and belonging to the class $L_2(\mathcal{B}([0,T]) \otimes \mathcal{F})$, then the equality*

$$\int_0^T \int_t^T f(s,\omega)\,ds\,dw_t = \int_0^T \int_0^t f(t,\omega)\,dw_s dt$$

takes place P-a.s.

Before formulating the next result, we will recall the definition of Sobolev's averaging operator T_ε. Let

$$\zeta(x) := \exp\big\{|x|^2(|x|^2-1)^{-1}\big\}, \quad \textit{for} \ \ |x| \leq 1$$

and

$$\zeta(x) := 0, \quad \textit{for} \quad |x| > 1.$$

Then, by definition, for any locally integrable function $\varphi : R^1 \to R^1$, we have

$$T_\varepsilon \varphi(x) = \varkappa \int_{-1}^{1} \varphi(x+\varepsilon y)\zeta(y)\,dy = \varkappa\varepsilon^{-1} \int_{-\infty}^{+\infty} \zeta\Big(\frac{y-x}{\varepsilon}\Big)\varphi(y)dy,$$

where

$$\varkappa = \Big(\int_{-1}^{1} \zeta(x)dx\Big)^{-1}.$$

Theorem 1.3. *Let $f(\cdot,\cdot,\cdot):[0,T]\times\Omega\times X\to R^1$ be a $\mathcal{B}[0,T]\otimes\mathcal{F}\otimes\mathcal{X}$–measurable function adapted for each $x\in R^1$ and belonging to $L_2(\mathcal{B}[0,T]\otimes\mathcal{F})$; for almost all (t,ω) it is locally integrable with respect to x. Then P-a.s. we have the relation*

$$T_\varepsilon\Big[\int_0^T f(t,\omega,x)dw_t\Big]=\int_0^T T_\varepsilon\big[f(t,\omega,x)\big]\,dw_t. \tag{1.9}$$

Proof. From the linearity of the stochastic entegral and Sobolev's averaging operator it readily follows that for any $n\geq 1$ P-a.s.

$$T_\varepsilon\Big[\int_0^T f_n(t,\omega,x)dw_t\Big]=\sum_{k=0}^{n-1}T_\varepsilon\Big[\int_{t_{k,n}}^{t_{k+1,n}} f_{k,n}(\omega,x)\,dw_t\Big]=$$
$$=\sum_{k=0}^{n-1}T_\varepsilon\big[f_{k,n}(\omega,x)\big]\big(w_{t_{k+1,n}}-w_{t_{k,n}}\big)=$$
$$=\sum_{k=0}^{n-1}\int_{t_{k,n}}^{t_{k+1,n}}T_\varepsilon\big[f_{k,n}(\omega,x)\big]dw_t=\int_0^T T_\varepsilon\big[f(t,\omega,x)\big]\,dw_t. \tag{1.10}$$

By arguments similar to those used in proving Theorem 1.1 and by the well-known properties of Sobolev's averaging operator (see, for example, [8] and [9]), passing to the limit in (1.10) as $n\to\infty$, we ascertain that relation (1.9) is valid. □

Remark 1.1. If instead of the requirement that all the above-discussed processes belong to $L_2([0,T]\times\Omega)$, we assume that for almost all ω they belong to $L_2([0,T])$, then it is not difficult to see that all the results obtained above can be easily extended to the latter case.

Remark 1.2. The results given above can be easily extended to the case, where the stochastic integral with respect to the Wiener process is replaced by the stochastic integral with respect to a square integrable continuous martingale.

2. Fubini Type Theorems for Lebesgue ang Skorohod's Integrals

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a given standard Wiener process w_t, $t\in[0,T]$. Assume $\mathcal{F}:=\mathcal{F}_T=\sigma\{w_t,\,t\in[0,T]\}$. In what follows we will use the same notation as in [5], in particular, Skorohod's stochastic integral is denoted by the same symbol as Ito's stochastic integral. Recall now the definition of a stochastic derivative operator and introduce the associated Sobolev spaces.

Let $C_b^\infty(R^k)$ be the set of C^∞ functions $f:R^k\to R^1$ which are bounded and have bounded derivatives of all orders. A smooth functional will be a random variable $F : \Omega \to R^1$ of the form $F = f(w_{t_1}, \dots, w_{t_m})$, where the function $f(x^1, \dots, x^m)$ belongs to $C_b^\infty(R^m)$ and $t_1, \dots, t_m \in [0, T]$. The class of smooth functionals is denoted by φ.

The derivative of a smooth functional F can be defined as the stochastic process given by

$$(DF)_t := \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}\big(w_{t_1}, \dots, w_{t_m}\big) I_{[0,t_i]}(t)$$

for $t \in [0, T]$.

The derivative DF can be regarded as a random variable taking values in the Hilbert space $H = L_2([0, T]; R^1)$. We also write $D_t F$ for $(DF)_t$. We introduce the norm on vf

$$\|F\|_{2,1} := \|F\|_2 + \|\|DF\|_H\|_2.$$

Then $\mathbf{D_{2,1}}$ denotes the Hilbert space which is the completion of *varnothing* with respect to the norm $\| \cdot \|_{2,1}$.

Let $u \in L_2([0, T] \times R^1; R^1)$. Then (see Proposition 3.1 in [5]) u is Skorohod integrable if and only if there exists a constant c such that

$$\left| E\bigg(\int\limits_0^T u_t \cdot D_t F dt \bigg) \right| \le c \|F\|_2$$

for any $F \in \mathbf{D_{2,1}}$ and, in this case, we have

$$E\bigg(\int\limits_0^T u_t \cdot D_t F dt \bigg) = E\big(F\delta(u)\big).$$

Notice that the Skorohod integrable operator δ is a closed operator because δ is the adjoint of D and $\mathbf{D_{2,1}}$ is dense in $L_2(\Omega)$.

Let $\mathbf{L^{2,1}}$ denote (see Definition 3.3 in [5]) the class of scalar processes $u \in L_2([0, T] \times \Omega)$ such that $u \in \mathbf{D_{2,1}}$ for a.a. t and there exists a measurable version of $D_s u_t$ verifying

$$E \int\limits_0^T \int\limits_0^T \big|D_s u_t\big|^2 ds\, dt < \infty.$$

Then $\mathbf{L^{2,1}} \subset \operatorname{Dom} \delta$ and $\mathbf{L^{2,1}}$ is a Hilbert space with the norm

$$\|u\| := \bigg(E \int\limits_0^T |u_t|^2 dt \bigg)^{1/2} + \bigg(E \int\limits_0^T \int\limits_0^T \|D_s u_t\|^2 ds\, dt \bigg)^{1/2}.$$

Theorem 2.1. *If $f \in \mathbf{L^{2,1}}$, then the integrals below are well defined and P-a.s. the following relation is valid:*

$$\int_0^T \int_t^T f(s,\omega)\,ds\,dw_t = \int_0^T \int_0^t f(t,\omega)\,dw_s\,dt. \tag{2.1}$$

The proof of Theorem 2.1 *will be given in several stages.*

Proposition 2.1. *If $f \in L^{2,1}$, then all integrals from relation* (2.1) *are well defined.*

Proof. At first, we verify that the process $g(t,\omega) := \int_t^T f(s,\omega)ds$ belongs to $\mathbf{L^{2,1}}$. By the definition of $\mathbf{L^{2,1}}$ we have

(i) $f \in L_2([0,T] \times \Omega)$;

(ii) $f(t,\cdot) \in \mathbf{D_{2,1}}$ for a.s. t;

(iii) there exists a measurable version of $D_s f_t$, such that

$$Df(\cdot,\cdot) \in L_2([0,T]^2 \times \Omega).$$

Hence, using the Cauchy-Bunyakovski inequality, it is easy to see that g satisfies the conditions (i) and (ii).

On the other hand, using the requirement (iii), we conclude that there exists a measurable version of the function

$$D_s g(t) = \int_t^T D_s f(\nu)\,d\nu$$

such that

$$\big\|D_s g(t)\big\|_{L_2([0,T]^2\times\Omega)} \le T^2 \cdot \big\|D_s f(\nu)\big\|_{L_2([0,T]^2\times\Omega)} < \infty,$$

hence g satisfies the condition (iii) as well. That is why $g \in \mathbf{L^{2,1}} \subset \operatorname{Dom}\delta$ and therefore the integrals from the left-hand side of relation (2.1) are well defined.

Further, it is obvious that the process $h(t,\omega) := \int_0^t f(t,\omega)dw_s$ exists. Using the ordinary Fubuni theorem and the estimation 3.7 from [5], we ascertain that:

$$\big\|h\big\|_{L_2([0,T]\times\Omega)} \le c_2(T) \cdot \big\|f\big\|_{2,1}.$$

Hence, P-a.s. $h(\cdot,\omega) \in L_2([0,T])$ and therefore the integrals on the right-hand side of (2.1) are well defined. $\square$

Proposition 2.2. *If $f(s,\omega) \equiv \xi(\omega) \in \operatorname{Dom}\delta$, then relation* (2.1) *is valid.*

Proof. Using Theorem 3.2 from [5] and Ito's formula, we can P-a.s. write

$$\int_0^T \int_t^T \xi \, ds \, dw_t = \int_0^T \xi(T-t)\, dw_t = \xi \int_0^T (T-t)\, dw_t - \int_0^T (T-t) D_t \xi \, dt =$$

$$= \xi \cdot T \cdot w_T - \xi \cdot T \cdot w_T + \xi \int_0^T w_t \, dt - \int_0^T (T-t) D_t \xi \, dt =$$

$$\xi \int_0^T w_t \, dt - \int_0^T (T-t) d_t \xi \, dt.$$

On the other hand, again using Theorem 3.2 [5] and applying the formula of integration by parts and differentiation rule of a Lebesgue integral with the variable upper boundary, we obtain that P-a.s.

$$\int_0^T \int_0^t \xi \, dw_s \, dt = \int_0^T \left[\xi \int_0^t dw_s - \int_0^t D_s \xi \, ds \right] dt = \xi \int_0^T w_t \, dt -$$

$$- \int_0^T \left(\int_0^t D_s \xi \, ds \right) dt = \xi \int_0^T w_t \, dt - \int_0^t D_s \xi \, ds \cdot t \Big|_0^T + \int_0^T t \left(\frac{pa}{\partial t} \int_0^t D_s \xi \, ds \right) =$$

$$\xi \int_0^T w_t \, dt - T \int_0^T D_s \xi \, ds + \int_0^T t \cdot D_t \xi \, dt = \xi \int_0^T w_t \, dt - \int_0^T (T-t) D_t \xi \, dt.$$

Thus in our case relation (2.1) is true. □

Proof of Theorem 2.1. Let $\{\Pi_n, n \in N\}$ be a sequence of partitions of $[0,T]$ of the form: $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n=T}$ such that

$$|\Pi_n| = \sup_{0 \le k \le n-1} (t_{k+1,n} - t_{k,n}) \to 0 \ \text{ as } \ n \to \infty.$$

Given $u \in L_2([0,T] \times \Omega; R^1)$, we define

$$\overline{u}_{k,n} := \frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} u_s \, ds \ \text{ for } \ 0 \le k \le n-1$$

and

$$\overline{u}_{-1,n} = \overline{u}_{n,n} := 0;$$

$$u^n : = \sum_{k=0}^{n-1} \overline{u}_{k,n} I_{[t_{k,n}, t_{k+1,n}[}.$$

It is not difficult to see that $f^n \in \mathbf{L^{2,1}}$ and, by the linearity of Skorohod's and ordinary integrals, using Proposition 2.2, for each $n \geq 1$ P-a.s. we have

$$\int\limits_0^T \int\limits_t^T f^n(s,\omega)\,ds\,dw_t = \int\limits_0^T \int\limits_0^t f^n(t,\omega)\,dw_s\,dt. \tag{2.2}$$

By arguments similar to those used in the adaptive case to prove (1.2) and (1.3) and taking into consideration the properties of a stochastic derivative operator, we find that $f^n \to f$ in $\mathbf{L^{2,1}}$ as $n \to \infty$.

Let us denote

$$g^n(t,\omega) := \int\limits_t^T f^n(s,\omega)\,ds$$

and

$$h^n(t,\omega) := \int\limits_0^t f^n(t,\omega)\,dw_s.$$

By the Cauchy-Bunyakovski inequality we conclude that $g^n \to g$ in $\mathbf{L^{2,1}}$ as $n \to \infty$. Therefore the left-hand side of (2.2) tends to left-hand side of (2.1) as $n \to \infty$.

On the other hand, because $f^n \to f$ in $\mathbf{L^{2,1}}$ as $n \to \infty$, by Proposition 3.5 [5] we ascertain that

$$\delta(f^n) \to \delta(f) \ \text{ in } \ L_2(\Omega) \ \text{ as } \ n \to \infty.$$

Therefore

$$\delta(f^n \cdot I_{[0,t]}) \to \delta\big(f \cdot I_{[0,t]}\big) \ \text{ in } \ L_2(\Omega) \ \text{ as } \ n \to \infty,$$

hence, $h^n(t) \to h(t)$ in $L_2(\Omega)$ as $n \to \infty$. Further, using estimation 3.7 [5], we see that:

$$\big\|h^n - h\big\|^2_{L_2([0,T]\times\Omega)} \leq c_2(T)\big[\big\|f^n - f\big\|^2_{L_2([0,T]\times\Omega)} + \\ + \big\|D_s f^n(\nu) - D_s f(\nu)\big\|^2_{L_2([0,T]^2\times\Omega)}\big] \to 0,$$

as $n \to \infty$. Thus there exists a subsequence h^{n_k} of the sequence f^n such that P-a.s.

$$\lim_{k\to\infty} \int\limits_0^T \big[h^{n_k}(t,\omega) - h(t,\omega)\big]^2\,dt = 0.$$

Hence the right-hand side of (2.2), written for the subsequence h^{n_k}, tends to the right-hand side of (2.1) as $k \to \infty$. Combining the above-obtained limit expressions, we ascertain that relation (2.1) is valid. □

As in the non-anticipative case, let us now consider the integrand which depends an additional parameter. Assume that the measurable space $(X, \mathcal{X}, \lambda)$ (with $\lambda(X) < \infty$) is given.

Theorem 2.2. *Let $\varphi(\cdot,\cdot,\cdot):[0,T]\times X\times\Omega\to R^1$ be a $\mathcal{B}([0,T]\otimes\mathcal{X}\otimes\mathcal{F})$-measurable function belonging, for each $x\in X$, to $\mathbf{L^{2,1}}$ and for almost all (t,ω) the Lebesgue integral $\int\limits_X\varphi(t,x,\omega)\lambda(dx)$ be well defined. Then the integrals below are well defined and the following relation is P-a.s. valid:*

$$\int\limits_X\int\limits_0^T\varphi(t,x,\omega)\,dw_t\lambda(dx)=\int\limits_0^T\int\limits_X\varphi(t,x,\omega)\lambda(dx)\,dw_t. \tag{2.3}$$

Proof. Let the sequence of functions $\varphi^n(t,x,\omega)$ be defined for each $x\in X$ as in the proof of the theorem above. Then relation (2.3) is true for any function $\varphi^n(t,x,\omega)$ and in that case the integrals are well defined. Further, passing to the limit in (2.3), written for $\varphi^n(t,x,\omega)$, as $n\to\infty$, and using arguments like in the proof of Theorem 2.1, we conclude that the assertion of the theorem is true. □

Theorem 2.3. *Let $\varphi(\cdot,\cdot,\cdot):[0,T]\times X\times\Omega\to R^1$ be a $\mathcal{B}([0,T]\otimes\mathcal{X}\otimes\mathcal{F})$-measurable function belonging, for each $x\in X$, to $\mathbf{L^{2,1}}$ and for almost all (t,ω) the function $\varphi(t,\cdot,\omega)$ be locally integrable with respect to x. Then the integrals below are well defined and the following relation is valid P-a.s.:*

$$T_\varepsilon\Big[\int\limits_0^T\varphi(t,x,\omega)\,dw_t\Big]=\int\limits_0^T T_\varepsilon\big[\varphi(t,x,\omega)\big]\,dw_t. \tag{2.4}$$

Proof. The proof is analogous to that of the preceding theorems. First, using the linearity of Skorohod's stochastic integral and Sobolev's averaging operator, we verify that relation (2.4) is valid for each function $\varphi^n(t,x,\omega)$ and then, using the above-mentioned properties of Skorohod's integral and the well-known properties of Sobolev's averaging operator (see [8], [9]), we finish the proof of the theorem. □

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