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STOCHASTIC INTEGRAL REPRESENTATION OF TWO-DIMENSIONAL POISSON FUNCTIONALS

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Abstract. In this paper we suggest the method which allows to construct explicit expressions for integrands taking part in the stochastic integral representation of functionals of Poisson processes and for these functionals the formula for calculation of the predictable projection of their stochastic derivatives are given.

Keywords and phrases: Stochastic derivative, predictable projection, Compensated Poisson process, Ocone-Haussmann-Clark's formula.

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1. Introduction. In the 80th of past century, it turned out (Harison, Pliska (1981)) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. According to the well-known Clark's formula, if F is a $\mathcal{F}_T^w$-measurable square integrable random variable, then $F = EF + \int\limits_{(0,T]} \varphi_t(\omega)dw_t$ for some $\varphi_t(\omega) \in L_2([0,T]\times\Omega)$. Due to the so-called Ocone-Clark's formula: $\varphi_t(\omega) = E[D_t^w F|\mathcal{F}_t^w]$, where $D_t^w F$ is the stochastic derivative (so-called Malliavin's derivative) of the functional F. But, in the cases if the functional F has no stochastic derivative, its application is impossible. Another distinct method of finding an integrand $\varphi_t(\omega)$ for "maximal" type functional belongs to Shyryaev, Yor (2003). Our approach allows one to construct $\varphi_t(\omega)$ even when the functional F has no the stochastic derivative (Jaoshvili, Purtukhia [3]).

On the other hand, if $F \in D_{2,1}^M$, then the Ocone-Haussmann-Clark's representation $F = EF + \int\limits_{(0,T]} {}^p(D_t^M F)dM_t$ is valid (Ma, Protter, Martin [1]), where M is a normal martingale – i. e. $\langle M,M\rangle_t = t$, $D_{2,1}^M$ denotes the space of square integrable functionals having the first order stochastic derivative – $D_t^M F$, and ${}^p(D_t^M F)$ is the predictable projection of the $D_t^M F$.But, in this case (exactly, when the quadratic variation $[M,M]$ is not deterministic), unlike the Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev spaces, which allows one to construct explicitly the stochastic derivative operator in many cases (in particular, the space $D_{p,1}^M$ $(1 < p < 2)$ cannot be defined in the usual way – i.e., by closing the class of smooth functionals with respect to the corresponding norm. Later on, in work of Purtukhia [2] the space $D_{p,1}^M$ $(1 < p < 2)$ is proposed for a class of normal martingales and the integral representation formula of Ocone-Haussmann-Clark is established for functionals from this space). Consequently, the Ocone-Haussmann-Clark's formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process (which, obviously, belongs to a class of normal martingales $\langle M, M\rangle_t = t$, but its quadratic variation is not deterministic $[M, M]_t = N_t = M_t + t$), saying nothing on the construction of its predictable projection. Our approach within the framework of nonanticipative stochastic calculus of semimartingales allows one to construct explicitly the expression for the integrand for functionals of the Compensated Poisson process, and to derive the formula allowing one to construct explicitly predictable projections of their stochastic derivatives.

2. Auxiliary notations and results. Let $(\Omega, \Im, \mathrm{P}, (\Im_t)_{0\le t\le\infty})$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it ($P(N_t = k) = \frac{t^k}{k!} \times e^{-t}$, $n = 0, 1, 2, ...$) and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Denote $M_t := N_t - t$ and by $\nu_n(t)$– its n-th order moment ($\nu_n(t) := E[M_t^n], n \ge 1$).

Let us denote $\Delta_x f(x) := f(x+1)-f(x)$. In particular, $\Delta_x P_n(M_T) = \Delta_x P_n(x)|_{x=M_T}$.

Proposition 2.1 *(cf. Proposition 1.5 [4]). The n-th order moment of the Compensated Poisson process satisfies the following differential equation:*

$$d\nu_n(t) = \left[\sum_{i=0}^{n-2} C_n^i \nu_i(t)\right] dt. \tag{2.1}$$

Proposition 2.2 *(cf. Proposition 2.1 [4]). For any natural power n of the Compensated Poisson process M_t the following representation is valid:*

$$M_t^n = \int\limits_{(0,t]} nM_{s-}^{n-1} dM_s + \sum_{i=2}^{n} \int\limits_{(0,t]} C_n^i M_{s-}^{n-i} dN_s \ (P-a.s.). \tag{2.2}$$

Theorem 2.1 *(cf. Theorem 2.2 [4]). For any polynomial function of one variable $P_n(x)$ $(n \ge 1)$ the following stochastic integral representation holds:*

$$P_n(M_T) = E[P_n(M_T)] + \int\limits_{(0,T]} E[\Delta_x P_n(M_T)|\Im_{t-}] dM_t \ \ (P-a.s.). \tag{2.3}$$

Theorem 2.2 *(cf. Theorem 2.3 [4]). For any polynomial function $P_n(x)$ the following relation is valid:*

$$^p[D_t^M P_n(M_T)] = E[\Delta_x P_n(M_T)|\Im_{t-}] \ (dP \otimes ds - a.s.), \tag{2.4}$$

where $^p[D_t^M P_n(M_T)]$ denotes the predictable projection of the stochastic derivative (with respect to the Compensated Poisson process) of functional $P_n(M_T)$.

3. Main results. Let us denote

$$\Delta_2^t g(M_T, M_S) := \Delta_x(\Delta_y g(M_T, M_S)) I_{[0,T]}(t) I_{[0,S]}(t)$$
$$+\Delta_x g(M_T, M_S) I_{[0,T]}(t) + \Delta_y g(M_T, M_S) I_{[0,S]}(t).$$

Theorem 3.1. *For any polynomial function of two variables $P_n(x,y)$ the following stochastic integral representation holds:*

$$P_n(M_T,M_S)=E[P_n(M_T,M_S)]+\int\limits_{(0,T\vee S]}E[\Delta_2^t P_n(M_T,M_S)|\Im_{t-}]dM_t\ (P-a.s.). \quad (3.1)$$

Proof. Fix $T\geq S\geq l$ and consider the power function $M_S^n\cdot M_T^m$. It is clear that:

$$E[M_T^m|\Im_S^M]=E[(M_T-M_S+M_S)^m|\Im_S^M]=\sum_{i=0}^m C_m^i\nu_{m-i}(T-S)M_S^i.$$

Therefore,

$$X_l:=E[M_S^n\cdot M_T^m|\Im_l^M]=E[M_S^nE\{M_T^m|\Im_S^M\}|\Im_l^M]=\sum_{i=0}^m C_m^i\nu_{m-i}(T-S)$$
$$\times\sum_{j=0}^{n+i}C_{n+i}^j\nu_{n+i-j}(S-l)M_l^j=\sum_{i=0}^m\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)\nu_{n+i-j}(S-l)M_l^j.$$

Then, according to the Ito's formula, taking into account the Propositions 2.1 and 2.2, it is not difficult to see that:

$$X_l=\sum_{i=0}^m\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)\{\int\limits_{(0,l]}j\nu_{n+i-j}(S-t)M_{t-}^{j-1}dM_t$$
$$+\int\limits_{(0,l]}-M_{t-}^j\sum_{k=0}^{n+i-j-2}C_{n+i-j-2}^k\nu_k(S-t)dt+\int\limits_{(0,l]}\nu_{n+i-j}(S-t)\sum_{k=2}^jC_j^kM_{t-}^{j-k}dN_t\}.$$

Studying carefully the last relation, one can easily notice that for any Lebesgue integral with respect to dt there exists the corresponding stochastic integral with respect to dN_t with the same integrand of opposite sign. Therefore, due to equality $dM_t=dN_t-dt$, we have:

$$X_l=\sum_{i=0}^m\sum_{j=0}^{n+i}C_m^iC_{n+i}^j\nu_{m-i}(T-S)\times\{\int\limits_{(0,l]}j\nu_{n+i-j}(S-t)M_{t-}^{j-1}dM_t$$
$$+\int\limits_{(0,l]}\nu_{n+i-j}(S-t)\sum_{k=2}^jC_j^kM_{t-}^{j-k}dM_t\}.$$

Furthermore, if $S<l\leq T$, analogously, one can write:

$$X_l:=E[M_S^n\cdot M_T^m|\Im_l^M]=M_S^nE[M_T^m|\Im_l^M]=M_S^n\sum_{i=0}^mC_m^i\nu_{m-i}(T-l)M_l^i$$
$$=M_S^n\sum_{i=0}^mC_m^i\{\int\limits_{(0,l]}i\nu_{m-i}(T-t)M_{t-}^{i-1}dM_t+\int\limits_{(0,l]}\nu_{m-i}(T-t)\sum_{j=2}^iC_i^jM_{t-}^{i-j}dN_t$$
$$+\int\limits_{(0,l]}[-M_{t-}^i\sum_{k=0}^{m-i-2}C_{m-i}^k\nu_k(T-t)]dt\}=\sum_{i=0}^mC_m^iM_S^n\int\limits_{(0,l]}[\nu_{m-i}(T-t)\sum_{j=1}^iC_i^jM_{t-}^{i-j}]dM_t.$$

Summing up the above results, using the equality $X_T = M_S^n \cdot M_T^m$, it is not difficult to see that the following representation is valid:

$$M_S^n \cdot M_T^m = E[M_S^n \cdot M_T^m] + \sum_{i=0}^{m}\sum_{j=0}^{n+i} C_m^i C_{n+i}^j \nu_{m-i}(T-S) \times \{ \int_{(0,S]} \nu_{n+i-j}(S-t)[jM_{t-}^{j-1} + \sum_{k=2}^{j} C_j^k M_{t-}^{j-k}]dM_t\} + \sum_{i=0}^{m} C_m^i M_S^n \int_{(S,T]} [\nu_{m-i}(T-t) \sum_{j=1}^{i} C_i^j M_{t-}^{i-j}]dM_t.$$

On the other hand, we have:

$$\Delta_2^t[M_S^n \cdot M_T^m] = \{[(M_S+1)^n(M_T+1)^m - M_S^n(M_T+1)^m]$$
$$-[(M_S+1)^n M_T^m - M_S^n \cdot M_T^m]\}I_{(0,S]}(t) + [(M_S+1)^n M_T^m - M_S^n \cdot M_T^m]I_{(0,S]}(t)$$
$$+[M_S^n(M_T+1)^m - M_S^n \cdot M_T^m]I_{(0,T]}(t) = \{[(M_S+1)^n - M_S^n][(M_T+1)^m - M_T^m]\}I_{(0,S]}(t)$$
$$+\{[(M_S+1)^n - M_S^n]M_T^m\}I_{(0,S]}(t) + \{[(M_T+1)^m - M_T^m]M_S^n\}I_{(0,T]}(t)$$
$$= \sum_{i=0}^{n-1} C_n^i M_S^i \sum_{j=0}^{m} C_m^j M_T^j I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(0,T]}(t)$$
$$= \{\sum_{i=0}^{n-1} C_n^i M_S^i \sum_{j=0}^{m} C_m^j M_T^j + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n\}I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(S,T]}(t)$$
$$= \{\sum_{i=0}^{n-1}\sum_{j=0}^{m} C_n^i C_m^j M_T^j M_S^i + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n\}I_{(0,S]}(t) + \sum_{j=0}^{m-1} C_m^j M_T^j M_S^n I_{(S,T]}(t).$$

Therefore, using the arguments similar to those presented above, we obtain that:

$$\int_{(0,T\vee S]} E\{\Delta_2^t[M_S^n \cdot M_T^m]|\Im_{t-}\}dM_t = \sum_{i=0}^{m}\sum_{j=0}^{n+i} C_m^i C_{n+i}^j \nu_{m-i}(T-S)\{\int_{(0,S]} \nu_{n+i-j}(S-t)$$
$$\times[jM_{t-}^{j-1} + \sum_{k=2}^{j} C_j^k M_{t-}^{j-k}]dM_t\} + \sum_{i=0}^{m} C_m^i M_S^n \int_{(S,T]} [\nu_{m-i}(T-t) \sum_{j=1}^{i} C_i^j M_{t-}^{i-j}]dM_t.$$

Hence, the representation is true for the power function $M_S^n \cdot M_T^m$. Further, taking into account the linearity of the mathematical expectation, of the conditional mathematical expectation,of the operator Δ_2^t and of the stochastic integral, we complete the proof of theorem. □

Theorem 3.2. *For any polynomial function of two variables $P_n(x,y)$ the following relation is valid:*

$$^p[D_t^M P_n(M_T, M_S)] = E[\Delta_2^t P_n(M_T, M_S)|\Im_{t-}](dP \otimes d\lambda - a.s.), \tag{3.2}$$

where $^p[D_t^M P_n(M_T, M_S)]$ denotes the predictable projection of the stochastic derivative (with respect to the Compensated Poisson process) of functional $P_n(M_T, M_S)$, $S \le T$.

Proof. According to the well-known result of Ma, Protter and Martin [1] we have:

$$P_n(M_T, M_S) = E[P_n(M_T, M_S)] + \int_{(0,T]} \{^p[D_t^M P_n(M_T, M_S)]\}dM_t \ \ (P - a.s.).$$

Consider the difference:

$$y_T := \int\limits_{(0,T]} \{E[\Delta_2^t P_n(M_T, M_S)|\Im_{t-}] - {}^p[D_t^M P_n(M_T, M_S)]\} dM_t := \int\limits_{(0,T]} \eta_t dM_t.$$

In the one hand, it is clear, due to the Theorem 3.1, that $y_T = 0$ (P-a.s.)
On the other hand, according to the Ito's formula, we can write:

$$y_T^2 = 2 \int\limits_{(0,T]} y_{t-}\eta_t dM_t + \int\limits_{(0,T]} \eta_t^2 d[M,M]_t.$$

If now we take the mathematical expectation from the both sides of the last relation, using the well-known properties of the square and predictable characteristics of the martingale, we ascertain that:

$$0 = E \int\limits_{(0,T]} \eta_t^2 d[M,M]_t = E \int\limits_{(0,T]} \eta_t^2 d\langle M,M\rangle_t = E \int\limits_{(0,T]} \eta_t^2 dt,$$

whence we conclude that the relation (3.2) is true. □

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