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STOCHASTIC INTEGRAL REPRESENTATION OF MULTIDIMENSIONAL POISSON FUNCTIONALS

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Abstract. We suggest the method which allows to construct explicit expressions for integrands taking part in the stochastic integral representation of multidimensional Poisson functionals and the formula for calculation of the predictable projection of stochastic derivatives for these functionals are given.

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In the last quarter of the 20th century, in stochastic analysis and the measure theory the extended stochastic integral (the so-called Skorokhod integral) was constructed, where the independence of the integrand of the future is replaced by its smoothness in a certain sense. The construction of an extended stochastic integral requires the integrand smoothness, i.e. the fulfillment of the condition of stochastic differentiability. It turned out (as it was shown by Gaveau and Trauber in 1982) that the operator of Skorokhod stochastic integration coincides with the conjugate operator of stochastic differentiation in the sense of Malliavin. As is known, the original aim of Malliavin's infinite-dimensional stochastic investigation was to study the density smoothness of a solution of a stochastic differential equation. The situation changed in 1991 when Karatsas and Ocone showed how to apply in financial mathematics Ocone's theorem of stochastic integral representation for the functional of diffusion processes. This theorem was subsequently called the Ocone-Haussmann-Clark formula. It is used in constructing hedging strategies at full financial markets driven by Brownian motion. Due to this result, the interest in Malliavin calculus on the part of mathematicians and financial researchers grew essentially. Since that time Malliavin's theory has been actively developing. Also, an active search for new areas of its application is being carried out. Malliavin's methods for jump processes (in particular, for Levy processes) were developed by many outhors. Despite the fact that in the general case, financial markets driven by Levy processes are not full, the Ocone-Haussmann-Clark formula nevertheless plays an important role in financial applications.

According to the well-known formula due to Clark (see Clark 1972): if F is a $\mathcal{F}_T$-measurable random variable, $EF^2 < \infty$, then there exists an adapted process $\varphi_t(\omega) \in L_2([0,T] \times \Omega)$ such that P-almost everywhere there exists an integral representation

$$F = EF + \int_{(0,T]} \varphi_t(\omega) dw_t.$$

However this result says nothing of the explicit form of the process $\varphi_t(\omega)$. One sufficiently general result, the so-called Ocone-Clark formula is known in this direction.

According to this formula, in the Wiener case

$$\varphi_t(\omega) = E[D_t^w F|\mathcal{F}_t^w],$$

where $D_t^w F$ is the stochastic derivative (in the Malliavin sense) of the functional F. As a rule the application of this formula demands, on the one hand, much effort and, on other hand, if the functional F does not have a stochastic derivative, then it cannot be used altogether. Another approach to finding the integrand $\varphi_t(\omega)$ belongs to Graversen, Shiryaev, Yor (2006) in the case where F is a maximum type functional. They related the associated Levy martingale to this functional and applied the generalized Ito's formula. Using the standard theory of L_2 and the theory of weighted Sobolev spaces, we managed in 2005 to construct, within the framework of classical Ito's calculus, $\varphi_t(\omega)$ in the explicit form even in the cases where the functional F does not have a stochastic derivative.

A further generalization of the Ocone-Clark formula belongs to Ma, Potter and Martin [1] for the so-called normal martingale classes (a martingale is called normal if $\langle M, M\rangle_t = t$ holds) according to which if $F \in D_{2,1}^M$, then the Ocone-Haussmann-Clark representation

$$F = EF + \int\limits_{(0,T]} {}^p(D_t^M F)dM_t \ \ (P - a.s.) \tag{1}$$

is valid, where $D_{2,1}^M$ denotes the space of square integrable functionals having a stochastic derivative of first order, while ${}^p(D_t^M F)$ denotes the predictable projection of the stochastic derivative $D_t^M F$ of the functional F. As seen, this functional also demands that the functional F would have the stochastic derivative. On the other hand, in that case, as different from the Wiener case, it is impossible to define the stochastic differentiation operator so that the Sobolev structure of the space $D_{2,1}^M$ could be obtained. Here the construction of the stochastic derivative is based on the expansion of a functional into a series of multiple stochastic integrals, whereas in the Wiener approach, in addition to this approach, use is also made of the Sobolev structure of a space.

The definition of a space $D_{p,1}^M$ $(1 < p < 2)$ in a usual manner (i.e., by the closure of the class of smooth functionals with respect to the corresponding norm) is impossible for classes of normal martingales. In [2] the Sobolev type spaces $D_{p,1}^M$, where $1 < p < 2$, were introduced and a generalization of the Ocone-Clark representation was obtained for functions from these spaces.

Definition 1. (cf. Definition 2.1 [2]). Fix $1 < p < 2$ and introduce the norm

$$\| F \|_{p,1} := E(\| F \|)_{L_p} + \| D.F \|_{L_2(0,T)}$$

on $D_{2,1}^M$ and denote by $D_{p,1}^M$ the Banach space which is the closure of $D_{2,1}^M$ under the norm $\| \cdot \|_{p,1}$.

Theorem 1. (cf. Theorem 2.1 [2]). *Let M be a normal martingale with the chaos representation property and $F \in D_{p,1}^M$ $(1 < p < 2)$, then the representation (1) is true.*

In [3], [4] we constructed the explicit expression for the integrand of a stochastic integral in the theorem of martingale representation for one and two-dimensional Poisson functionals and derived the formula which makes it possible to construct explicitly the predictable projection of the stochastic derivative for these functionals. Here we consider the multidimensional case and generalized the Ocone-Haussmann-Clark formula for multidimensional Poisson functionals.

Let $(\Omega, \Im, \mathrm{P}, (\Im_t)_{0\leq t\leq\infty})$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it ($P(N_t = k) = \frac{t^k}{k!}\times e^{-t}$, $n = 0, 1, 2, ...$) and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Denote $M_t := N_t - t$.

Let $Z^+ = 0, 1, 2, ...$ and $P = P_1, P_2, P_3, ...$ be the Poisson distribution: $P_x = e^{-T}T^x/x!, x = 0, 1, 2, ...$ Let us denote $\Delta f(x) = f(x) - f(x-1)$ ($f(x) := 0$, if $x < 0$) and define the Poisson-Charlier polynomials: $\Pi_n(x) = [(-1)^n \Delta^n P_x]/P_x, n \geq 1; \Pi_0 = 1$. It is wellknown from the course of Functional Analysis that the sequence $\pi_n(x)_{n\geq 0}$ ($\pi_n(x) = \Pi_n(x)/c_n$) is a basis in the space $L_2(Z^+)$, where

$$L_2(Z^+) = f : \sum_{x=0}^{\infty} f^2(x) < \infty.$$

Let us denote:

$$\rho(x_1, x_2, ..., x_n, T_1, T_2, ..., T_n) = \frac{T_1^{x_1}}{x_1!}e^{-T_1} \times \frac{(T_2 - T_1)^{x_2 - x_1}}{(x_2 - x_1)!}e^{-(T_2 - T_1)} \times \cdots$$

$$\times \frac{(T_n - T_{n-1})^{x_n - x_{n-1}}}{(x_n - x_{n-1})!}e^{-(T_n - T_{n-1})} \quad (x_1 \leq x_2 \leq \cdots \leq x_n)$$

and for each fixed $T_1 \leq T_2 \leq \cdots \leq T_n$ denote by

$$L_2^{T_1,T_2,...,T_n} := L_2(Z^{+n}; \rho(x_1, x_2, ..., x_n, T_1, T_2, ..., T_n))$$

the functional space on Z^{+n} of measurable functions with the finite norm:

$$\|g\|_{2,T_1,T_2,...,T_n} = \|g(x_1, x_2, ..., x_n)\rho(x_1, x_2, ..., x_n, T_1, T_2, ..., T_n)\|_2.$$

Proposition 1. *The space $L_2^{T_1,T_2,...,T_n}$ is a Banach space with basis*

$$\{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \rho(x_1, x_2, ..., x_n, T_1, T_2, ..., T_n), m_i \in Z^+, i = 1, 2, ..., n\}.$$

Let us denote:

$$\nabla_x h(x) := h(x+1) - h(x), \nabla_x h(M_T) := h(M_T + 1) - h(M_T),$$

$$\nabla^n f(x_1, x_2, ..., x_n) := \sum_{k=1}^{n} \sum_{i_1,i_2,...,i_k=1}^{n} \nabla_{x_{i_1}} \nabla_{x_{i_2}} \cdots \nabla_{x_{i_k}} f(x_1, x_2, ..., x_n),$$

$$\nabla_t^n f(M_{T_1}, M_{T_2}, ..., M_{T_n}) := \sum_{k=1}^{n} \sum_{i_1,i_2,...,i_k=1}^{n} \nabla_{x_{i_1}} \nabla_{x_{i_2}} \cdots \nabla_{x_{i_k}} f(M_{T_1}, M_{T_2}, ..., M_{T_n})$$

$$\times I_{[0,t_{i_1}]}(t) I_{[0,t_{i_2}]}(t) \cdots I_{[0,t_{i_k}]}(t).$$

Theorem 2. *If* $F \in L_2^{T_1,T_2,...,T_n}$ *and for some number* $0 < \alpha < 1 : \nabla^n F \in L_2^{T_1/\alpha,T_2/\alpha,...,T_n/\alpha}$, *then the stochastic integral below is well defined and the following stochastic integral representation is valid:*

$$F(M_{T_1}, M_{T_2}, ..., M_{T_n}) = E[F(M_{T_1}, M_{T_2}, ..., M_{T_n})]$$

$$+ \int\limits_{(0,T_1\vee T_2\vee\cdots\vee T_n]} E[\nabla_t^n F(M_{T_1}, M_{T_2}, ..., M_{T_n})|\Im_{t-}^M]dM_t \quad (P-a.s.).$$

Theorem 3. *In the conditions of the Theorem 2 the following relation holds:*

$${}^p[D_t^M F(M_{T_1}, M_{T_2}, ..., M_{T_n})] = E[\nabla_t^n F(M_{T_1}, M_{T_2}, ..., M_{T_n})|\Im_{t-}^M] \quad (dP \otimes d\lambda - a.s.),$$

where ${}^p[D_t^M F(M_{T_1}, M_{T_2}, ..., M_{T_n})]$ *denotes the predictable projection of the stochastic derivative (with respect to the compensated Poisson process)* $D_t^M F(M_{T_1}, M_{T_2}, ..., M_{T_n})$ *of the functional* $F(M_{T_1}, M_{T_2}, ..., M_{T_n})$.

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