SOBOLEV AND LOGARITHMIC SOBOLEV TYPE INEQUALITIES

O. Purtukhia

Ivane Javakhishvili Tbilisi State University, Faculty of Exact an Natural Sciences, Department of Mathematics;
University Street 2, Tbilisi, 0186, Georgia
A. Razmadze mathematical institute

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Abstract

Using the notation of stochastic derivative for a class of normal martingale and for Poisson functionals the Sobolev type Hilbert spaces for normal martingale and stochastic differentiable Poisson functionals are introduced and the Sobolev, Sobolev-Poincare and logarithmic Sobolev type inequalities for random variables from this spaces are proved.

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1 Introduction

The classical Sobolev inequalities play a fundamental role in an analysis in Euclidean spaces and in the research of partial differential equations. Sobolev type inequalities provide some of the very basic tools in the study the existence, regularity and uniqueness of the solutions of all sorts of partial differential equations, linear and nonlinear, elliptic, parabolic, and hyperbolic. On the other hand, as we will partially see below, the inequalities introduced by S.L.Sobolev and their many modifications have turned out to be extremely useful flexible tools in surprisingly various settings.

Classical Sobolev inequalities usually state that if f a function defined on R^n and its first (weak) derivatives belong to class $L_p(R^n)$, then f also belongs to $L_q(R^n)$ for some $q > p$, particularly for $q = (p^{-1} - n^{-1})^{-1}$ if $q < \infty$. On the other hand, Sobolev-type inequalities explains how can one control the size of a function in terms of the size of its gradient. On the real line, the answer is given by a simple and yet extremely useful calculus inequality (see [2]): for any smooth function f on the line with compact support

$$|f(x)| \leq \frac{1}{2} \int_{R^1} |f^{'}(x)|dx \tag{1}$$

(if f is smooth, but no other restriction is imposed, the inequality above may fail). As concerns to a multidimensional case a question of the existence of such estimates was first studied by Sobolev in [1]. In this case one has:

$$\forall f \in C_0^\infty(R^n): \ ||f||_q \leq ||\nabla f||_p \tag{2}$$

for any integer $n \geq 2$ and a real p, $1 \leq p < n$, where $q = np/(n-p)$ and $C = C(n,p)$ is a constant.

This inequality is called the Sobolev inequality although the case $p = 1$ is not contained in [1]. As it turned out, when $p = 1$, (2) has a very simple proof based on (1) and Holder's inequality, which was independently discovered by E. Gagliardo and L. Nirenberg. Moreover, the case $p > 1$ follows from the case $p = 1$ by a simple trick.

Note that, if (2) holds for all $f \in C_0^\infty(R^n)$, it obviously also holds for a larger class of functions including for instance all C^1 functions with compact support or even Lipschitz functions vanishing at infinity. In fact, (2) holds for all functions vanishing at infinity whose gradient in the sense of distributions is in $L_p(R^n)$. On the other side, (2) restricted to non-negative functions in $C_0^\infty(R^n)$ suffices to prove (2) in its full generality. Indeed, the correctness of (2) for such functions implies that it also holds for non-negative Lipschitz functions with compact support and, if $f \in C_0^\infty(R^n)$, $|f|$ is Lipschitz and satisfies $|\bigtriangledown |f|| \leq |\bigtriangledown f|$ almost everywhere. It then follows that (2) holds for $f \in C_0^\infty(R^n)$.

Further, for the Gaussian measure ν on R^n, it is well known the so called logarithmic Sobolev inequality:

$$\int_{R^n} |f(x)|^2 ln|f(x)| d\nu(x) \leq \int_{R^n} |gradf(x)|^2 d\nu(x) + ||f||_2^2 ln||f||_2, \tag{3}$$

where $||f||_p$ denotes the $L_p(\nu)$ norm of f.

Logarithmic Sobolev (or Log-Sobolev) inequalities were introduced by L. Gross [9] in 1975 as an attempt of isolating smoothing properties of Markov semigroups in infinite-dimensional settings. It can be used to obtain quantitative bounds on the convergence of finite Markov chains to stationary. Given an irreducible finite Markov chain K with invariant probability π, consider the Dirichlet form

$$\aleph(f,g) = \langle (I - Kf, g) \rangle.$$

In general, a logarithmic Sobolev inequalities are inequalities of the type $\Re(f) \leq C\aleph(f,f)$ holding for all functions f, where the entropy-like quantity $\Re(f)$ is defined by

$$\Re(f) = \sum_{x \in X} |f(x)|^2 ln(\frac{|f(x)|^2}{||f||_2^2})\pi(x).$$

Log-Sobolev inequalities are one of the essential tools for proving concentration phenomena, not only because they require in some sense less understanding about the underlying geometry of the measured space, but also because they yield sharper results for concentration, i.e., Gaussian rather than exponential. They are particularly well-suited for infinite-dimensional analysis.

Let (X, d, μ) be a metric measure space in which the log-Sobolev inequality holds. Then (see [3]) every K-Lipschits function is integrable and if $F : X \to R$ is such a function, we have:

$$\mu\{x : F(x) \geq \int F d\mu + r\} \leq exp\{-r^2/(2CK^2)\}.$$

Moreover, if μ satisfies the Log-Sobolev inequality, then the Poincare inequality is satisfied:

$$\forall f \in C_0^\infty(R^n) : Var_\mu f \leq C \int |\nabla f|^2 d\mu.$$

Next we present a weighted Poincare inequality (see [4]) which looks similar to but is weaker than the Sobolev inequality. It seems weaker because there is no gain in the integrability of a function over the integrability of its gradient. However, even a non-weighted Poincare inequality actually implies a Sobolev inequality under the doubling condition on the metric balls of the ambient space. Let g be a non-negative, continuous function in R^n, with compact support D,

$$\int_D u(x)g(x)dx = 1$$

and the super level set $\{g \geq k\}$ is convex for all k. Write r as the diameter of D and

$$L = \int_D g(x)dx = 1.$$

Then for all $u \in W^{1,p}(D)$, $p \geq 1$, there exists $C = C(n) > 0$ such that

$$\int_{R^n} |u(x) - L|^p g(x)dx \leq C(n)||g||_\infty r^{n+p} \int_{R^n} |\nabla u(x)|^p g(x)dx.$$

A more technical but very important fact is the equivalence between strong forms and weak forms of Sobolev inequalities. An example of this phenomenon is that it is enough to have the weak Sobolev inequality

$$\forall f \in C_0^\infty(D) : \quad sup_{x>0}\{s\mu(\{x : |f(x)| > s\})^{1/q}\} \leq C||\nabla f||_p$$

with $1 \leq p < q$ to conclude that the strong inequality (2) holds. Another example is the equivalence between the Nash inequality

$$\forall f \in C_0^\infty(D): \quad ||f||_2^{1+2/r} \leq C||\nabla f||_2 \cdot ||f||_1^2/r$$

and the Sobolev inequality

$$\forall f \in C_0^\infty(D): \quad ||f||_{2r/(r-2)} \leq C||\nabla f||_2$$

when $r > 2$.The Nash inequality is weaker in the sense that it is easily deduced from the Sobolev inequality above and Holder's inequality. The equivalence between weak and strong forms of Sobolev-type inequalities turns out to be extremely useful when it comes to prove that a certain manifold satisfies a Sobolev inequality. A basic tool used here is the notion of pseudo-Poincare inequality. Given a smooth function f, let $f_r(x)$ denote the mean off over the ball with center x and radius r. One says that D satisfies an L_p-pseudo-Poincare inequality if for all $\forall f \in C_0^\infty(D)$ and all $r > 0$:

$$||f - f_r||_p \leq C|| \bigtriangledown f||_p.$$

In the case when $p > n$ (see [2]) there exists a constant $C = C(n, p)$ such that for any set D of finite volume we have

$$\forall f \in C_0^\infty(D): \; ||f||_\infty \leq C\{\mu(D)\}^{1/n-1/p}|| \bigtriangledown f||_p. \tag{4}$$

One crucial difference between the last statement and Sobolev inequality (2) for $1 \leq p < n$ is that the right-side of (4) depends on the set D on which the function f is supported. As the measure of D tends to infinity, the term $\{\mu(D)\}^{1/n-1/p}$ also tends to infinity since $p > n$. In fact, when $n \leq p$, there is no way to control the size of f purely in terms $|| \bigtriangledown f||_p$.

In mathematical analysis a class of Sobolev inequalities, is relating norms including those of Sobolev spaces. These are used the Sobolev embedding theorem, giving inclusions between certain Sobolev spaces, and the Rellich Kondrachov theorem showing that under slightly stronger conditions some Sobolev spaces are compactly embedded in others. They are named after Sergei Lvovich Sobolev. On the other hand, the fundamental role that Sobolev inequalities have played in the study of elliptic differential operators is well known.

The theory of partial differential equations provides a most of important applications of Sobolev inequalities. Consider, for instance, divergence form, uniformly elliptic equation in R^n:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\{a_{i,j}\frac{\partial}{\partial x_j}u(x)\} = 0,$$

where the coefficient $a_{i,j}$ are real measurable functions such that $||a_{i,j}||_\infty < c$ and

$$\forall x \in R^n,\ \forall \xi \in R^n: \ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \epsilon \sum_{i=1}^{n} \xi_i^2.$$

Moser's elliptic Harnak inequality (which a striking application of Sobolev inequalities) states that any positive weak solution u of this equation in an Euclidian ball B satisfies

$$\sup_{B/2}\{u\} \leq C \inf_{B/2}\{u\},$$

where C depends neither on u nor on B but only on the constants c, ϵ above and the dimension n.

One might approach the problem of finding infinite dimensional versions of Sobolev's inequalities from an intrinsic point of view. In this case one should note that the equation $q = (p^{-1} - n^{-1})^{-1}$ implies $q \to p$ as the dimension goes to infinity, and consequently there is a loss of information in the usual form of Sobolev's inequality as the dimension gets larger. Moreover, Lebesgue measure in infinite dimensional space is meaning-less. The inequality (3), on the other hand, has a simple meaning in infinite dimensions and is valid there because the coefficients in (3) are independent of dimension.

Because, Sobolev inequalities relate the size of ∇f to the size of f, in order to prove such inequalities, one may try to express f in terms of its gradient. Let E and F be separable Banach spaces, and let $f : E \to F$ be a given, possibly nonlinear, function. There are several ways to approach the notion of derivative . The first notion of a derivative of a function on a vector space is that of the Frechet deriva-tive. The second and weaker notion of derivative is the Gateaux derivative. But these two notions of differentiability are too strong for many purposes. Many pathologies arise when dealing with infinite dimensional spaces that are not present in finite dimensional ones. If we suppose that Banach space E supports a Gaussian measure μ, and $H_\mu \subset E$ is the associated reproducing kernel Hilbert space, then the Malliavin calculus concerns functions on E that are differentiable in the directions of H_μ. It turns out that a function may be differentiable in this weak sense, and yet not even be continuous on E!

2 Notation and preliminaries

The Malliavin derivative is a linear map from a space of random variables to a space of processses indexed be a Hilbert space. Being a derivative, it is not surprising that this operator is unbounded. If the random variable is differentiable (in Malliavin sense), the Clark-Ocone formula allows one to explicitly compute the integrand in the martingale representation in terms of the Malliavin derivative of ξ (see [5]). In turn, the Clark-Ocone formula allow one to prove the Sobolev-Poincare type inequalities in Wiener case. A further generalization of Clark-Ocone formula belongs to Ma, Protter and Martin (see [6]) for the so-called normal martingales classes (i.e. $\langle M, M\rangle_t = t$) according to which if $\xi \in D^M_{1,2}$, then the Clark-Haussmann-Ocone representation

$$\xi = E\xi + \int_{(0.T]} {}^p(D^M_t \xi) dM_t \quad (P - a.s.) \tag{5}$$

is valid, where $D^M_{1,2}$ denotes the space of square integrable functional having a stochastic derivative of the first order, while ${}^p(D^M_t\xi)$ denotes the predictable projection of the stochastic derivative $D^M_t\xi$ of the functional ξ.

Let $\sum_n$ be an increasing simplex of R^n_+: $\sum_n = \{(t_1, ..., t_n) \in R^n_+ : 0 < t_1 < \cdots < t_n\}$, and extend a function f defined on $\sum_n$ by making f symmetric on R^n_+. Let $\Re = \sigma\{M_t : t \geq 0\}$ be the σ-algebra generated by a normal martingale M. Let H_n be the n-th homogeneous chaos, $H_n = I_n(f)$, where f ranges over all $L_2(\sum_n)$ and $I_n(f)$ denotes the multiple stochastic integral:

$$I_n(f) := n! \int_{\sum_n} f(t_1, ..., t_n) dM_{t_1} \cdots dM_{t_n}.$$

If $L_2(\Re, P) = \bigoplus_{n=0}^{\infty} H_n$, then we say that M possesses the chaos representation property (CRP).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}), P$ be a filtered probability space satisfying the usual conditions. We assume that a normal martingale M with the CRP is given on it and that $\Im$ is generated by M. Thus, for any random variable $\xi \in L_2(\Re, P)$, we have by virtue of the CRP that there exists a sequence of functions $f_n \in L^2_s([0,1]^n)$ ($=h \in L_2([0,1]^n)$: h is symmetric in all variables), $n = 1, 2, ...$, such that $\xi = \sum_{n=0}^{\infty} I_n(f_n)$. Consider the following subset $D^M_{1,2} \subset L_2(\Re, P)$:

$$D^M_{1,2} = \{\xi = \sum_{n=0}^{\infty} I_n(f_n) : \xi = \sum_{n=1}^{\infty} nn! ||f_n||^2_{L_2([0,1]^n)} < \infty\}.$$

Definition 1 (see [6]). The derivative operator is defined as a linear operator D^M from space $D^M_{1,2}$ into the space $L_2([0,1] \times \Omega)$ by the relation:

$$D^M_t \xi := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0,1],$$

whenever $\xi = \sum_{n=0}^{\infty} nI_n(f_n)$.

As is well-known there are two ways to describe the variational derivative (also known as the Malliavin derivative in the Brownian case), and they are equivalent in the Brownian case but not in the martngale case. In the martingale case one cannot define the derivative operator in the usual way to obtain the Sobolev space structure for the space $D_{1,2}^M$ (in [4] an example is given, which shows that the two definitions – Sobolev space and chaos expansion – are compatible if and only if $[M, M]_t$ is deterministic). Therefore in martingale case the space $D_{1,q}^M$ $(1 < q < 2)$ cannot be defined in the usual way (i.e., by closing the class of smooth functional with respect to the corresponding norm).

Later, in [7] the Sobolev type spaces $D_{1,q}^M$, where $1 < q < 2$, were introduced and a generalization of Clark-Haussmann-Ocone representation was obtained for functionals from these spaces.

Definition 2 (see [7]). Fix $1 < q < 2$ and introduce the norm

$$||\xi||_{1,q} := ||\xi||_q + ||||D^M\xi||_{L_2([0,T])}||_q$$

on $D_{1,2}^M$, and denote by $D_{1,q}^M$ $(1 < q < 2)$ the Banach space which is the closure of $D_{1,2}^M$ under the norm $|| \cdot ||_{1,q}$.

Note that the stochastic derivative $D^M\xi$ is well-defined on $D_{1,q}^M$ $(1 < q < 2)$ by the closure. Given $\xi \in D_{1,q}^M$ $(1 < q < 2)$ we can find a measurable stochastic process $(t, \omega) \longmapsto D_t^M\xi(\omega)$ such that for a.e. $\omega \in \Omega$, the equality $D_t^M\xi = D^M\xi(\omega)(t)$ holds for almost all $t \in [0, T]$ (more precisely, $t \longmapsto D_t^M\xi(\omega)$ is in the equivalence class from $L_2([0, T])$ defined by $D^M\xi(\omega)$). $D_t^M\xi(\omega)$ is defined uniquely on $[0, T] \times \Omega$ up to sets of measure zero (in general, if $\eta : \Omega \to L_2[0, T]$ is measurable random element, then there exists a $(\mathcal{B})([\prime, \mathcal{T}])\otimes\Im$-measurable stochastic process, $\{\overline{\eta}(t, \omega) : (t, \omega) \in [0, T]\times\Omega\}$, such that $\overline{\eta}(\cdot, \omega) = \eta(\omega)$ holds almost surely. In this case, we shall identify $\eta(\omega)(t)$ with $\overline{\eta}(t, \omega)$).

If now M is a normal martingale with the chaos representation property and $\xi \in L_q(\Omega) \bigcap D_{1,q}^M$ $(1 < q < 2)$, then the representation (5) is true (see Theorem 2.1 [7]).

On the other hand, an explicit construction of the stochastic derivative operator for compensated Poisson functionals, which was introduced by us in [8] and which is not based on the chaos expansion of func-tionals as it is in Ma, Protter and Martin's work, allow us to receive the Clark-Haussmann-Ocone explicit formula in Poisson cases. Our aims, using the above-mentioned Clark-Haussmann-Ocone representations on the one hand to prove the Sobolev-Poincare type inequalities in general case for a class of normal martingales and on the other hand to give more explicit estimations in special case for Poisson functionals [10].

3 Auxiliary results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Let N_t be the standard Poisson process ($P(N_t = k) = t^k e^{-t}/k!$, $k = 0, 1, 2, ...$) and $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_t$. Let M_t be the compensated Poisson process ($M_t = N_t - t$). Let us denote:

$$\nabla_x f(x) := f(x+1) - f(x);$$

$$\nabla_x f(M_T) := \nabla_x f(x)|_{x=M_T};$$

$$\overline{D}_t^M[(M_s)^n] := [\nabla_x(x^n)]|_{x=M_t} \cdot \overline{D}_t^M[M_s] := [\nabla_x(x^n)]|_{x=M_t} \cdot I_{[0,s]}(t)$$

and

$$\overline{D}_t^M[P_m(M_{t_1}, M_{t_2}, ..., M_{t_n})] :=$$

$$:= \sum_{k=1}^{n} \sum_{1\le i_1<i_2<\cdots<i_k\le n} \nabla_{x_{i_1}} \nabla_{x_{i_2}} \cdots \nabla_{x_{i_k}} [P_m(M_{t_1}, M_{t_2}, ..., M_{t_n})]\times$$

$$\times I_{[0,t_1]}(t) I_{[0,t_2]}(t) \cdots I_{[0,t_k]}(t)$$

for any polynomial function $P_m(x_1, x_2, ..., x_n)$ (as we see, if $n = 1$, then the stochastic derivative for Wiener and Poisson processes formally are the same. The difference begins from $n = 2$. Indeed, if we take here $n = 2$, we obtain that

$$\overline{D}_t^M[(M_s)^2] := [\nabla_x(x^2)]|_{x=M_t} \cdot \overline{D}_t^M[M_s] = (2M_s + 1) I_{[0,s]}(t),$$

whereas in the Wiener process cases

$$D_t^w[(w_s)^2] = \frac{\partial}{\partial x}(x^2)|_{x=w_s} \cdot D_t^w[w_s] = 2w_s I_{[0,s]}(t).$$

This fact can be explained as follows: in the Wiener case in the definition of stochastic derivative the main component is a usual (classical) derivative, whereas in the Poisson case the main component is the operator ∇ and if $n = 1$ we have $x' = \nabla x = 1$, while if $n = 2$, then $2x = (x^2)' \neq \nabla x^2 = 2x + 1$).

We denote by $W_{1,2}^M$ the Hilbert space of real random variables ξ in the domain of $\overline{D}^M$ such that

$$E(\xi^2) + E(||\overline{D}_\cdot^M \xi||^2_{L_2([0,T])}) < +\infty.$$

with corresponding scalar product.

Proposition 1. *For every random variable ξ from the space $W_{1,2}^M$ we have the differentiation rule*

$$\overline{D}_\cdot^M \xi^2 = (2\xi + 1)\overline{D}_\cdot^M \xi.$$

before we formulate the logarithmic Sobolev type estimation, let's make the following remark and verifying the Cauchy-Bunyakovsky type inequality for the conditional mathematical expectation.

Remark. Note that, due to the classical Cauchy-Bunyakovsky inequality if $g \in L_2$ and $h \in L_2$, one can only state that $gh \in L_1$. Hence, if $g \in L_2$, then the truth it is yet not known that lng and $ln(g^2)$ belongs or not to L_2, but also in this case we can only state that $glng \in L_1$ and $gln(g^2) \in L_1$ and not say anything about square integrability of the function $g^2ln(g^2)$.

Proposition 2. *Let ξ and η are square integrable random variables and G is sub-σ-algebra of $\mathcal{F}$. Then P-a.s is fulfilled the relation*

$$[E(|\xi\eta||G)]^2 \leq [E(\xi^2|G)]^{1/2}[E(\eta^2|G)]^{1/2}. \tag{6}$$

Proof. Note at firs that the inequality is true on the set where the any random variable from the right side of (6) P-a.s. is equal to zero. Indeed, on the set $A = \{\omega : E(\xi^2|G)(\omega) = 0\}$, it is obvious that

$$E(I_A\xi^2) = E[E(I_A\xi^2|G)] = E[I_AE(\xi^2|G)] = 0.$$

Hence, using the well-known properties of the conditional expectation and the classical Cauchy-Bunyakovsky inequality, one can easily ascertain that

$$0 \leq E[I_AE(|\xi\eta||G)] = E[E(I_A|\xi\eta||G)] = E[I_A|\xi\eta|] =$$

$$= E[(I_A|\xi|)(I_A|\eta|)] \leq [E(I_A|\xi^2)]^{1/2}[E(I_A|\eta^2)]^{1/2} = 0,$$

i.e., $E[I_AE(|\xi\eta||G)] = 0$. Therefore, on the set A we have

$$E(|\xi\eta||G) = 0 \quad (P - a.s.)$$

(if the mathematical expectation from the nonnegative random variable is zero, then this random variable P-a.s. is zero) and, hence, $[E(|\xi\eta||G)]^2 = 0$. Analogously, it is obvious, that the (6) is true on the set $\{\omega : E(\eta^2|G)(\omega) = 0\}$.

Further, without community restriction we can suppose that P-a.s.: $E(\xi^2|G) > 0$ and $E(\eta^2|G) > 0$. Let's enter the following designations

$$\overline{\xi} = \frac{\xi}{[E(\xi^2|G)]^{1/2}}; \quad \overline{\eta}^2 = \frac{\eta}{[E(\eta^2|G)]^{1/2}}.$$

According to the well-known properties of the conditional expectation, one can easily see that:

$$E(\overline{\xi}^2) = E[E(\overline{\xi}^2|G)] = E[E(\frac{\xi^2}{E(\xi^2|G)}|G)] = E[\frac{E(\xi^2|G)}{E(\xi^2|G)}] = 1,$$

$$E(\overline{\eta}^2) = E[E(\overline{\eta}^2|G)] = E[E(\frac{\eta^2}{E(\eta^2|G)}|G)] = E[\frac{E(\eta^2|G)}{E(\eta^2|G)}] = 1.$$

Therefore, using the elementary inequality $2|ab| \leq a^2 + b^2$, due to the well-known properties of the conditional expectation, it is not difficult to see that P-a.s.:

$$2E(|\overline{\xi} \cdot \overline{\eta}||G) = 2E(\frac{|\xi|}{[E(\xi^2|G)]^{1/2}} \cdot \frac{|\eta|}{[E(\eta^2|G)]^{1/2}}|G) \leq$$

$$\leq E[\frac{\xi^2}{E(\xi^2|G)}|G] + E[\frac{\eta^2}{E(\eta^2|G)}|G] = 2,$$

i.e.,

$$E(\frac{|\xi|}{[E(\xi^2|G)]^{1/2}} \cdot \frac{|\eta|}{[E(\eta^2|G)]^{1/2}}|G) \leq 1.$$

On the other hand, because

$$E(\frac{|\xi|}{[E(\xi^2|G)]^{1/2}} \cdot \frac{|\eta|}{[E(\eta^2|G)]^{1/2}}|G) = \frac{E(|\xi\eta||G)}{[E(\xi^2|G)]^{1/2}[E(\eta^2|G)]^{1/2}},$$

we have

$$\frac{E(|\xi\eta||G)}{[E(\xi^2|G)]^{1/2}[E(\eta^2|G)]^{1/2}} \leq 1,$$

hence,

$$E(|\xi\eta||G) \leq [E(\xi^2|G)]^{1/2}[E(\eta^2|G)]^{1/2},$$

that is equivalent to the statement of a proposition.

4 Main results

Theorem 1. *Let M be a normal martingale with the chaos representation property and $\xi \in L_q(\Omega) \bigcap D^M_{1,q}$ $(1 < q \leq 2)$. Then the following estimation is fulfilled*

$$||\xi - E\xi||_q \leq ||\{||^p(D^M_{\cdot}\xi)||^2_{L_2([0,T])}\}||_2. \tag{7}$$

Proof. Due to the theorem 2.1 from [7] (in the case of $q = 2$ see [6]), the representation (5) is fulfilled for the functionals ξ from the space $L_q(\Omega) \bigcap D^M_{1,q}$ $(1 < q < 2)$. Hence, $(P - a.s.)$ we have

$$\xi - E\xi = \int_{(0.T]} {}^p(D^M_t\xi)dM_t.$$

Then, according to Lyapunov inequality, using the well-known properties of stochastic integral with respect to the normal martingale, we can write

$$\{E|\xi - E\xi|^q\}^{2/q} \le E|\int_{(0.T]} {}^p(D_t^M \xi) dM_t|^2 = E\int_{(0.T]} |{}^p(D_t^M \xi)|^2 dt. \quad (8)$$

By the definition of predictable projection one can conclude that if $X \ge 0$ then ${}^pX \ge 0$ and ${}^p(X - Y) = {}^pX - {}^pY$. Therefore, if $X \le Y$, then ${}^pX \le^p Y$. Due to the elementary inequality $2ab \le a^2 + b^2$ we can write that $a^2 \ge 2ab - b^2 = 2b(a-b) + b^2$ for any $a, b \in R^1$. Taking here $a = |X_t|$ and $b = {}^p|Y_t|$ we see that

$$|X_t|^2 \ge 2^p(|X|)[|X| - {}^p(|X|)] + ({}^p|X|)^2.$$

Thus, taking the predictable projection from the both side of above inequality, we conclude that

$${}^p(|X_t|^2) \ge 2^p(|X|)[{}^p(|X|) - {}^p(|X|)] + ({}^p|X|)^2 = ({}^p|X|)^2. \quad (9)$$

Therefore, summing up the above-mentioned relations, using the Fubini theorem and the well-known properties of predictable projection, from the inequality (8) we ascertain that

$$\{E|\xi - E\xi|^q\}^{2/q} \le E\int_{(0.T]} [{}^p(D_t^M \xi)]^2 dt \le \int_{(0.T]} E\{{}^p[(D_t^M \xi)^2]\} dt =$$

$$= \int_{(0.T]} E[(D_t^M \xi)^2] dt = E\int_{(0.T]} (D_t^M \xi)^2 dt = ||||D_t^M \xi||^2_{L_2([0,T])}||_2^2.$$

It is obvious that the last relation is equivalent to an inequality (7).

Theorem 2. *For every random variable ξ from the space $W_{1,2}^M$ we have the relation*

$$E(\xi^2) \le (E\xi)^2 + E(||\overline{D}_\cdot^M \xi||^2_{L_2([0,T])}).$$

Proof. According to the results from [8], for every random variable ξ from the space $W_{1,2}^M$ we have the representation

$$\xi = E\xi + \int_{(0.T]} {}^p(\overline{D}_t^M \xi) dM_t \quad (P - a.s.)$$

Taking the mathematical expectation from the second degree of the both side of above relation, using the well-known properties of the stochastic integral, we conclude that

$$E(\xi^2) = (E\xi)^2 + E[\int_{(0.T]} {}^p(\overline{D}_t^M \xi) dM_t]^2.$$

Farther, Due to the well-known properties of stochastic integral and normal martingale, we can write

$$E[\int_{(0.T]} {}^p(\overline{D}_t^M \xi) dM_t]^2 = E\int_{(0.T]} [{}^p(\overline{D}_t^M \xi)]^2 d[M,M]_t = E\int_{(0.T]} [{}^p(\overline{D}_t^M \xi)]^2 dt.$$

On the other hand, according to the relation (8), using the Fubini theorem and the well-known properties of predictable projection, we ascertain that

$$E\int_{(0.T]} [{}^p(\overline{D}_t^M \xi)]^2 dt \le E\int_{(0.T]} {}^p[(\overline{D}_t^M \xi)^2] dt = E\int_{(0.T]} [(\overline{D}_t^M \xi)^2] dt.$$

Summing up the above-obtained relations, we complete the proof of the theorem.

Theorem 3. *For every random variable $\xi \in W_{1,2}^M$, with $\xi^2(\omega) \ge \epsilon > 0$ (for some $\epsilon > 0$), the following inequality holds*

$$E[\xi^2 ln(\xi^2)] \le (E\xi)^2 ln[E(\xi^2)] + 2(4 + \frac{1}{\epsilon})E(||\overline{D}^M_{\cdot} \xi||^2_{L_2([0,T])}).$$

Proof. Suppose that the random variable $\xi \in W_{1,2}^M$ is bounded. It is not difficult to see that in this case $(\xi)^2 \in W_{1,2}^M$. Therefore, if we denote $Q_t := E(\xi^2|\Im_{t-})$, then due to the results from [8], (P-a.s.) we have the representation

$$Q_T = \xi^2 = E(\xi^2) + \int_{(0.T]} {}^p[\overline{D}_t^M (\xi)^2] dM_t.$$

Hence,

$$Q_t = E(\xi^2) + \int_{(0.t]} {}^p[\overline{D}_s^M (\xi)^2] dM_s \quad (P-a.s.)$$

and, due to the proposition 1, using the well-known properties of the predictable projection, we have

$$Q_t = E(\xi^2) + \int_{(0.t]} E[(2\xi+1)\overline{D}_s^M \xi|\Im_{s-}] dM_s.$$

Further, according to the Ito's formula, we write

$$Q_t ln(Q_t) = E(\xi^2) ln[E(\xi^2)] + \int_{(0.t]} [1 + ln(Q_{s-})] dQ_s + [Q_t, ln(Q_t)]_t.$$

Taking the mathematical expectation from the both side of the above relation, using the well-known properties of the normal martingale, one can write that

$$E[\xi^2 lnE(\xi^2)] = E(\xi^2)ln[E(\xi^2)] + E\int_{(0.T]} \frac{\{E[(2\xi+1)\overline{D}_t^M \xi|\Im_{t-}]\}^2}{Q_t}dt. \quad (10)$$

On the other hand, according the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$, using the Proposition 2 and the Jensen inequality, it is not difficult to see that

$$E\int_{(0.T]} \frac{\{E[(2\xi+1)\overline{D}_t^M \xi|\Im_{t-}]\}^2}{Q_t}dt$$

$$= E\int_{(0.T]} \frac{\{2E(\xi\overline{D}_t^M \xi|\Im_{t-}) + E(\overline{D}_t^M \xi|\Im_{t-})\}^2}{Q_t}dt$$

$$\leq E\int_{(0.T]} \frac{2\{4E(\xi\overline{D}_t^M \xi|\Im_{t-})^2 + E(\overline{D}_t^M \xi|\Im_{t-})^2\}}{Q_t}dt$$

$$\leq E\int_{(0.T]} \frac{2\{4E(\xi^2|\Im_{t-})E[(\overline{D}_t^M \xi)^2|\Im_{t-}] + E[(\overline{D}_t^M \xi)^2|\Im_{t-}]\}}{Q_t}dt$$

$$= E\int_{(0.T]} \frac{2E[(\overline{D}_t^M \xi)^2|\Im_{t-}](4Q_t+1)}{Q_t}dt$$

$$= 8\int_{(0.T]} E[(\overline{D}_t^M \xi)^2]dt + 2E\int_{(0.T]} \frac{E[(\overline{D}_t^M \xi)^2|\Im_{t-}]}{E[\xi^2|\Im_{t-}]}\}dt$$

$$= 8E||(\overline{D}_t^M \xi)^2||^2_{L_2([0,T])} + 2E\int_{(0.T]} \frac{E[(\overline{D}_t^M \xi)^2|\Im_{t-}]}{\epsilon}\}$$

$$= 2E||(\overline{D}_t^M \xi)^2||^2_{L_2([0,T])}(4+\frac{1}{\epsilon}).$$

Combining now this last estimation with the relation (10), we complete the proof of theorem in the case when the random variable $\xi \in W^M_{1,2}$ is bounded.The general case we easily obtain by a standard localization argument based o monotone cut-offs of ξ.

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