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ON THE SMOOTHNESS OF CONDITIONAL MEAN OF SOME STOCHASTICALLY NONSMOOTH FUNCTIONALS

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Abstract. We generalize the Clark-Ocone's stochastic integral representation formula in case, when the Wiener functional isn't stochastically (in the Malliavin sense) smooth, but its conditional mathematical expectation with respect to natural filtration (generated by Wiener process) is stochastically differentiable and established the method of finding the corresponding integrand.

Keywords and phrases: Wiener functional, stochastic derivative, Clark's integral representation formula, Clark-Ocone's formula.

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In the stochastic process theory, the representation of functionals of Wiener process by stochastic integrals, also known as martingale representation theorem, it is stated that a functional that is measurable with respect to the filtration generated by a Wiener process can be written in terms of Ito's stochastic integral with respect to this Wiener process. The theorem only asserts the existence of the representation and does not help to find it explicitly.

It is possible in many cases to determine the form of the representation using Malliavin calculus, if a functional is Malliavin differentiable. We consider nonsmooth (in the Malliavin sense) functionals and have developed some methods of obtaining constructive martingale representation theorems. The obtained results can be used to establish the existence of a hedging strategy in various European Options with corresponding pay off functions.

The first proof of the martingale representation theorem was implicitly provided by Ito (1951) himself. This theorem states that any square-integrable Wiener functional is equal to a stochastic integral with respect to the Wiener process. Many years later, Dellacherie (1974) gave a simple new proof of Its theorem using Hilbert space techniques.

Many other articles were written afterwards on this problem and its applications but one of the pioneer works on explicit descriptions of the integrand is certainly the one by Clark (1970). Those of Haussmann (1979), Ocone (1984), Ocone and Karatzas (1991) and Karatzas, Ocone and Li (1991) were also particularly significant. A nice survey article on the problem of martingale representation was written by Davis (2005).

In spite of the fact that this problem is closely related to important issues in applications, for example finding hedging portfolios in finance, much of the work on the subject did not seem to consider explicitness of the representation as the ultimate goal. In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the explicitness point of view: the integrands in the stochastic integral representations always involve predictable projections or conditional expectations and some kind of gradients.

Shiryaev and Yor (2003) proposed a method based on Ito's formula to find explicit martingale representations for Wiener functionals which yields in particular the explicit martingale representation of the running maximum of the Wiener process. Even though they consider Clark-Ocone formula as a general way to find stochastic integral representations, they raise the question if it is possible to handle it efficiently even in simple cases.

On the probability space $(\Omega, \Im, P)$ is given the standard Wiener process $w = (w_t)$, $t \in [0, T]$ and $(\Im_t^w)$, $t \in [0, T]$ is the natural filtration generated by the Wiener process w. The stochastic integral as a process from the adapted square integrable process represents the square integrable martingale. The well-known Clark's theorem ([1]) gives an answer to the inverse question. If F is the square integrable $\Im_T^w$-measurable random variable, then there exists square integrable random process ψ_t, adapted to the filtration $\Im_t^w$, such that

$$F = EF + \int_0^T \psi_t dw_t.$$

If the functional is stochastically smooth then Clark-Ocone's ([2]) formula proves, that the integrand from Clark representation of this functional, represents the conditional expectation of Malliavin derivative, i.e. $\psi_t = E(D_t F|\Im_t^w)$, where D_t is the so called Malliavin's stochastic derivative.

Despite the fact that Clark-Ocone's formula gives integrand construction, there are problems with practical realizations (from the viewpoint of stochastic derivative calculation, as well as conditional mathematical expectation). We generalized this result ([4]) in case, when the functional isn't stochastically smooth, but its filter (conditional mathematical expectation) is stochastically differentiable and established the method of finding this integrand. This method demands smoothness only for conditional mathematical expectation of the considered functional, instead of the usual requirement of smoothness of the functional (as it was in the Clark-Ocone's formula). For example, the offered method allows us obtain the integral representations for the indicator $I_{\{w_t \le x\}}$, which is not differentiable in the Malliavin sense (indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one). Our aim to characterize a class of such functionals. Below we try to make the first step in this direction.

It is well known, that if a random variable is stochastically differentiable (in the Malliavin sense), then its conditional mathematical expectation is differentiable too ([3]): in particular, if $F \in D_{2,1}$, then $E(F|\Im_s^w) \in D_{2,1}$ and

$$D_t[E(F|\Im_s^w)] = E(D_t F|\Im_s^w) I_{[0,s]}(t),$$

where $D_{2,1}$ denotes the Hilbert space which is the closure of the class of smooth Wiener functionals S[1] with respect to the norm

$$||F||_{1,2} = \{E[F^2] + E[||D.F||^2_{L_2([0,T])}]\}^{1/2}.$$

On the other hand, it is possible that conditional mathematical expectation can be smooth even if the random variable isn't stochastically smooth ([4]): for example, $I_{\{w_T>x\}} \notin D_{2,1}$ (as it was mentioned above), but

$$[E[I_{\{w_T>x\}}|\Im_t^w] = 1 - \phi(\frac{x - w_t}{\sqrt{T-t}}) \in D_{2,1},$$

where ϕ is the standard normal distribution function.

Remark 1. However, it should be noted that in many practical cases (which are interesting from the point of view of financial mathematics) it is impossible to count on smoothnesses even of conditional mathematical expectation of the functional. For example, if the functional is represented as the Lebesgue integral (with respect to the time variable) the from square integrable process which isn't stochastically smooth, but its filter is a stochastically smooth process, then the conditional mathematical expectation of the functional won't be stochastically smooth. Indeed, on the one hand (see, Theorem 2 in [5]), if the square integrable random process u_s for almost all $s \in [0,T]$ does not belong to $D_{2,1}$, then the average process $\int_0^T u_s(\omega)ds$ does not belong to the space $D_{2,1}$ either. On the other hand, it is well-known (see, for example, [3]) that if $u_s(\omega) \in D_{2,1}$ for all s, then $\int_0^T u_s(\omega)ds \in D_{2,1}$ and

$$D_t\{\int_0^T u_s(\omega)ds\} = \int_0^T D_t u_s(\omega)ds.$$

Therefore, in this case the conditional mathematical expectation of the functional $\int_0^T u_s(\omega)ds$ is not stochastically smooth, because we have:

$$E[\int_0^T u_s(\omega)ds|\Im_t^w] = \int_0^t u_s(\omega)ds + \int_t^T E[u_s(\omega)|\Im_t^w]ds,$$

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable, but the second summand is differentiable in the Malliavin sense when u_s satisfied our weakened condition (if $E[u_s(\omega)|\Im_t] \in D_{2,1}$ for almost all s and $E[u_s(\omega)|\Im_t^w]$ is Lebesgue integrable for a.a. ω, then (see, for example, [3])

$$\int_0^T E[u_s(\omega)|\Im_t]ds \in D_{2,1}).$$

[1] Here S denotes the class of random variables which has the form $F = f(w_{t_1}, ..., w_{t_n})$, $f \in C_p^\infty(R^n)$, $t_i \in [0,T]$, $n \geq 1$, where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

Let $p(s,t,w_s,A)$ be the transition probability of the Wiener process w, i.e. $p(w_t \in A|\Im_s^w) = p(s,t,w_s,A)$, where $0 \le s \le t$, A is a Borel subset of R and

$$p(s,t,x,A) = \frac{1}{\sqrt{t-s}} \int_A \varphi\left(\frac{y-x}{\sqrt{t-s}}\right) dy$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_A \exp\{-\frac{(y-x)^2}{2(t-s)}\} dy.$$

For the computation of conditional mathematical expectation below we use the well-known statement:

Proposition 1. For all bounded or positive measurable functions f we have the relation

$$E[f(w_t)|\Im_s^w] = \int_R f(y) p(s,t,w_s,dy) \quad (P-a.s.).$$

Let $L_2([0,T]\times\Omega) = L_2([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\Im, \lambda\times P)$ (where $\mathcal{B}([0,T])$ is the Borel $\sigma-$ algebra on $[0,T]$ and λ is the Lebesgue measure) the set of square integrable processes, and $L_a^2([0,T]\times\Omega)$ represents the subspace of adapted (to the filtration $(\Im_t^w)$, $t\in[0,T]$) processes; $L_2([0,T]) = L_2([0,T], \mathcal{B}([0,T]), \lambda)$. Let $L_{2,T}$ denote the set of measurable functions $h: R \to R$, such that $h(\cdot)\rho(\cdot,T) \in L_2 := L_2(R, \mathcal{B}(R), \lambda)$, where $\rho(x,T) = \exp\{-\frac{x^2}{2T}\}$.

Proposition 2. $L_{2,T}$ be a Banach space with basis $\{x^n \rho(x,T)\}$, $n = 0,1,2,....$

Proof. The proposition directly follows from the statement VIII.4.3 [6], according to which if the measurable function f on (a,b) $(-\infty \le a < b \le +\infty)$ differs from zero and satisfies the condition $|f(x)| \le Ce^{-\delta|x|}$, where $\delta > 0$, then the system of functions $x^n f(x)$ $(n = 0,1,...)$ is full in $L_2(a,b)$.

As it was in the Malliavin calculus, we introduce the space $L^{2,1}$ (see Definition 3. 3 [7]).

Definition 1. Let $L^{2,1}$ denote the class of scalar processes $u \in L_2([0,T]\times\Omega)$ such that $u_t \in D_{2,1}$ for a.a. t and there exists a measurable version of $D_s u_t$ verifying

$$E\int_0^T \int_0^T |D_s u_t|^2 ds dt < \infty.$$

$L^{2,1}$ is the Hilbert space with the norm

$$||u||_{L^{2,1}} := (E\int_0^T |u_t|^2 dt)^{1/2} + (E\int_0^T \int_0^T |D_s u_t|^2 ds dt)^{1/2}.$$

Denote

$$L_C^{2,1} := \{F \in L_2(\Omega) : E(F|\Im_t^w) \in L^{2,1} \ \ for\ a.a.\ t\},$$

$$||F||_{L_C^{2,1}} := (EF^2)^{1/2} + (E\int_0^T |E(F|\Im_t^w)|^2 dt)^{1/2}$$

$$+(E\int_0^T \int_0^T |D_s E(F|\Im_t^w)|^2 ds dt)^{1/2}.$$

Definition 2. Let $\overline{L}_C^{2,1}$ denote the space which is the closure of the space $L_C^{2,1}$ with respect to the norm $||\cdot||_{L_C^{2,1}}$, i.e. if $G \in \overline{L}_C^{2,1}$, then there exist a sequence $F_n \in L_C^{2,1}$ such that

$$\lim_{n\longrightarrow\infty} ||F_n - G||_{L_C^{2,1}} = 0.$$

Theorem 1. The conditional mathematical expectation of the functional $w_T^n I_{\{w_T>K\}}$ is stochastically smooth and the following relation:

$$D_s\{E[w_T^n I_{\{w_T>K\}}|\Im_t^w]\} = \frac{K^n}{\sqrt{T-t}}\varphi(\frac{K-w_t}{\sqrt{T-t}})I_{[0,t]}(s)$$

$$+\frac{n}{\sqrt{T-t}}\int_K^\infty y^{n-1}\varphi(\frac{y-w_t}{\sqrt{T-t}})dy I_{[0,t]}(s) \qquad (1)$$

takes place (where φ is the standard normal distribution density function).

Moreover, for all $s < t$ we have

$$E[D_s\{E[w_T^n I_{\{w_T>K\}}|\Im_t^w]\}|\Im_s^w] = \frac{K^n}{\sqrt{T-s}}\varphi(\frac{K-w_s}{\sqrt{T-s}})$$

$$+\frac{n}{\sqrt{T-s}}\int_K^\infty y^{n-1}\varphi(\frac{y-w_s}{\sqrt{T-s}})dy. \qquad (2)$$

Proof. Due to Proposition 1, it is not difficult to see that

$$D_s\{E[w_T^n I_{\{w_T>K\}}|\Im_t^w]\} = D_s\{\frac{1}{\sqrt{T-t}}\int_{-\infty}^\infty y^n I_{\{y>K\}}\varphi(\frac{y-w_t}{\sqrt{T-t}})dy\}$$

$$= \frac{1}{\sqrt{2\pi(T-t)}}\int_K^\infty y^n D_s[\exp\{-\frac{(y-w_t)^2}{2(T-t)}\}]dy$$

$$= \frac{1}{\sqrt{2\pi}(T-t)^{3/2}}\int_K^\infty y^n(y-w_t)\exp\{-\frac{(y-w_t)^2}{2(T-t)}\}dy\cdot I_{[0,t]}(s).$$

Therefore, using the standard technique of integration and the well-known property of the normal distribution density function, we easily ascertain that (1) is fulfilled. Hence, according to Proposition I.2.3 [3], the conditional mathematical expectation of the considered functional is stochastically smooth.

On the other hand, using again Proposition 1, for all $s < t$ we have

$$E[D_s\{E[w_T^n I_{\{w_T>K\}}|\Im_t^w]\}|\Im_s^w]$$

$$= \frac{-w_s}{(t-s)\sqrt{2\pi(T-s)}}\int_K^\infty y^n \exp\{-\frac{(y-w_s)^2}{2(T-s)}\}dy$$

$$+\frac{1}{(T-s)^{3/2}\sqrt{2\pi}}\int_K^\infty y^{n+1}\exp\{-\frac{(y-w_s)^2}{2(T-s)}\}dy$$

$$+\frac{w_s(T-t)}{(t-s)(T-s)^{3/2}\sqrt{2\pi}}\int_K^\infty y^n \exp\{-\frac{(y-w_s)^2}{2(T-s)}\}dy.$$

From here, by analogy of the transformations made at calculation of the conditional mathematical expectation (1), it is not difficult to obtain the relation (2).

Corollary 1. In the case $n = 1$ we have

$$D_s\{E[w_T I_{\{w_T>K\}}|\Im_t^w]\} = \frac{K}{\sqrt{T-t}}\varphi(\frac{K-w_t}{\sqrt{T-t}})I_{[0,t]}(s)$$

$$+\frac{1}{\sqrt{T-t}}[1-\phi(\frac{y-w_t}{\sqrt{T-t}})]I_{[0,t]}(s).$$

Theorem 2. Let the function $f \in C^1$ such that $|f(x)| \leq c\exp\{\alpha x^2\}$ for some constant $0 < \alpha < 1/[2(T-t)]$. Then the conditional mathematical expectation of the functional $f(w_T)I_{\{w_T>K\}}$ is stochastically smooth and we have the following relation:

$$D_s\{E[f(w_T)I_{\{w_T>K\}}|\Im_t^w]\} = \frac{f(K)}{\sqrt{T-t}}\varphi(\frac{K-w_t}{\sqrt{T-t}})I_{[0,t]}(s)$$

$$+\frac{1}{\sqrt{T-t}}\int_K^\infty f'(y)\varphi(\frac{y-w_t}{\sqrt{T-t}})dyI_{[0,t]}(s). \tag{3}$$

Remark 2. It should be noted that in the future we are going to investigate functionals from the space $\overline{L}_C^{2,1}$.

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R E F E R E N C E S

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