Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

HEDGING OF BARRIER TYPE ONE EUROPEAN OPTION

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Abstract. We consider barrier type one European Option in the case of Black-Scholes financial market model, which payoff function is not differentiable in Malliavin sense and investigate the hedging problem. The Clark-Haussmann-Ocone's type stochastic integral representation formula for corresponding wiener functional with the explicit form of integrand is established.

Keywords and phrases: Black-Scholes model, European option, barrier option, Clark-Haussmann-Ocone's formula.

AMS subject classification (2010): 60H07, 60H30, 62P05.

1 Introduction. European options are contracts that give the owner the right, but not the obligation, to buy or sell the underlying security at a specific price, known as the strike price, on the option's expiration date. The payoffs for a standard European call option and European put options are as follows: Call option payoff is $(S-K)^+$ and Put option payoff is $(S-K)^-$, where S is equal to the spot price of the underlying security and K equals the strike price of the option. The barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option. It turns out that in modern financial mathematics (see Harrison and Pliska, 1981) extremely important role plays the so-called Martingale representation theorem (including Girsanovs measure transformation theorem). Karatzas and Ocone (1991) have shown how to use Clark-Haussmann-Ocone formula in financial mathematics, in particular for constructing of hedging strategies in the complete financial markets driven by Wiener process.

The martingale representation theorem states that a square integrable Wiener functional can be written in terms of Ito's stochastic integral. It is possible in many cases to determine the form of the representation using Malliavin calculus if a functional is Malliavin differentiable. We consider nonsmooth (in Malliavin sense) functional and have developed some methods of obtaining of constructive martingale representation theorem. It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We, with prof. O. Glonti [1] (see also [2]) considered Wiener functionals which are not stochastically differentiable. In particular, we generalized the Clark-Haussmann-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding of integrand.

Let on the probability space $(\Omega, \Im, P)$ be given the Wiener process $W=(W_t)$, $t \in [0,T]$ and $(\Im_t^W)$, $t \in [0,T]$ is the natural filtration generated by the Wiener process W. Consider the Black-Scholes model with risk-free asset price and risky asset price evolutions

$dB_t = rB_t dt$; $B_0 = 1$ and $dS_t = \mu S_t dt + \sigma S_t dW_t$; $S_0 = 1$, where $r \geq 0$ is interest rate, $\mu \in R$ is appreciation rate and $\sigma > 0$ is volatility coefficient.

Let $Z_T = exp\{-(\mu - r)W_T/\sigma - (\mu - r)^2 T/2\sigma^2\}$ and let $\widetilde{P}_T$ be the measure on $(\Omega, \Im_T^W)$ such that $d\widetilde{P}_T = Z_T dP$. From Girsanov's Theorem it follows that under this measure (martingale risk neitral measure) $\widetilde{W}_t = W_t + (\mu - r)t/\sigma$ is the standard Wiener process and $dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$, $S_0 = 1$ or $S_t = exp\{\sigma \widetilde{W}_t + (r - \sigma^2/2)t\}$.

We consider the problem of "replication" the European Option with the payoff function $F = (W_T - C_1)^- I_{\{S_T \leq C_2\}}$, i.e. one needs to find a trading strategy (β_t, γ_t), $t \in [0, T]$ such that the capital process $X_t = \beta_t B_t + \gamma_t S_t$, $X_T = F$ under the self-financing condition $dX_t = \beta_t dB_t + \gamma_t dS_t$. Hence, we have $dX_t = r(\beta_t B_t + \gamma_t S_t)dt + \sigma \gamma_t S_t d\widetilde{W}_t$, with $X_T = F$.

2 Main results

Theorem 1. *For any real numbers C_1 and $C_2 > 0$ the functional $F = (W_T - C_1)^- I_{\{S_T \leq C_2\}}$ admits the following stochastic integral representation*

$$F = \widetilde{E}F + \int_0^T \left[\frac{C_1}{\sqrt{T-u}} \varphi\left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}} \right) - \Phi\left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}} \right) \right] d\widetilde{W}_u, \tag{1}$$

where $\Phi(\cdot)$ is a standard normal distribution function and

$$C_3 = \min\{C_1 + (\mu - r)T\sigma^{-1}, (\ln(C_2) - (r - \sigma^2/2)T)\sigma^{-1}\} := \min\{\widetilde{C}_1, \widetilde{C}_2\}.$$

Proof. According to the Markov property of the Wiener process and the well-known properties of conditional mathematical expectation, using the transition probabilities of the Wiener process, we have

$$g_t = \widetilde{E}[F|\Im_t^{\widetilde{W}}] = \{\widetilde{E}[(W_T - C_1)^- I_{\{S_T \leq C_2\}} | \widetilde{W}_t = y]\}|_{y=\widetilde{W}_t}$$

$$= \left[-\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\min\{\widetilde{C}_1, \widetilde{C}_2\}} (x-y) \exp\left\{ -\frac{(x-y)^2}{2(T-t)} \right\} dx \right] \Big|_{y=\widetilde{W}_t}$$

$$+ \left[(C_1 - y) \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{C_3} \exp\left\{ -\frac{(x-y)^2}{2(T-t)} \right\} dx \right] \Big|_{y=\widetilde{W}_t} := [I_1 + I_2]|_{y=\widetilde{W}_t}.$$

Using the standard technique of integration it is not difficult to see that

$$I_1 = \frac{T-t}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{C_3} d\left(\exp\left\{ -\frac{(x-y)^2}{2(T-t)} \right\} \right) = \sqrt{T-t}\,\varphi\left(\frac{C_3 - y}{\sqrt{T-t}} \right)$$

and

$$I_2 = (C_1 - y)\Phi\left(\frac{C_3 - y}{\sqrt{T-t}} \right).$$

Hence, due to Proposition 1.2.3 [3], we easily conclude that g_t is stochastically differentiable and one can write that

$$D_u g_t = \frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}} \varphi\left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}}\right) I_{[0,t]}(u) - \Phi\left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}}\right) I_{[0,t]}(u)$$

$$+ \frac{C_1 - \widetilde{W}_t}{\sqrt{T-t}} \varphi\left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}}\right) I_{[0,t]}(u) := [J_1 - J_2] I_{[0,t](u)}.$$

Due to the transition probabilities of the Wiener process, using again the standard technique of integration and the well-known property of the normal distribution density function, it is easy to see that

$$\widetilde{E}[J_1|\Im_u^{\widetilde{W}}] := \widetilde{E}\left[\frac{C_3 + C_1 - \widetilde{W}_t}{\sqrt{T-t}} \varphi\left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}}\right) \Big| \Im_u^{\widetilde{W}}\right]$$

$$= \frac{C_3 + C_1}{2\pi\sqrt{(T-t)(t-u)}} \exp\left\{-\frac{(C_3 - \widetilde{W}_u)^2}{2(T-u)}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{\left[x - \frac{C_3(t-u) + \widetilde{W}_u(T-t)}{T-u}\right]^2}{2\frac{(T-t)(t-u)}{T-u}}\right\} dx$$

$$+ \frac{\frac{(T-t)(t-u)}{T-u}}{2\pi\sqrt{(T-t)(t-u)}} \exp\left\{-\frac{(C_3 - \widetilde{W}_u)^2}{2(T-u)}\right\} \int_{-\infty}^{\infty} d\left(\exp\left\{-\frac{\left[x - \frac{C_3(t-u) + \widetilde{W}_u(T-t)}{T-u}\right]^2}{2\frac{(T-t)(t-u)}{T-u}}\right\}\right)$$

$$- \frac{\frac{C_3(t-u) + \widetilde{W}_u(T-t)}{T-u}}{2\pi\sqrt{(T-t)(t-u)}} \exp\left\{-\frac{(C_3 - \widetilde{W}_u)^2}{2(T-u)}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{\left[x - \frac{C_3(t-u) + \widetilde{W}_u(T-t)}{T-u}\right]^2}{2\frac{(T-t)(t-u)}{T-u}}\right\} dx$$

$$= \left[\frac{C_3 + C_1}{\sqrt{T-u}} - \frac{C_3(t-u) + \widetilde{W}_u(T-t)}{\sqrt{(T-u)^3}}\right] \varphi\left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}}\right).$$

Further, we have

$$\widetilde{E}[J_2|\Im_u^{\widetilde{W}}] := \widetilde{E}\left[\Phi\left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}}\right) \Big| \Im_u^{\widetilde{W}}\right]$$

$$= \frac{1}{\sqrt{2\pi(t-u)}} \int_{-\infty}^{\infty} \Phi\left(\frac{C_3 - x}{\sqrt{T-t}}\right) \exp\left\{-\frac{(x - W_u)^2}{2(t-u)}\right\} dx.$$

Hence, according to the relation $\lim\limits_{t\uparrow T} \Phi(\frac{x}{\sqrt{T-t}}) = 0.5 I_{\{x=0\}} + I_{\{x>0\}}$, using the dominated convergence theorem, we ascertain that

$$\lim_{t\uparrow T} \widetilde{E}\left\{J_2 I_{[0,t]}(u) | \Im_u^{\widetilde{w}}\right\} = \Phi\left(\frac{C_3 - \widetilde{w}_u}{\sqrt{T-u}}\right) I_{[0,T]}(u).$$

Combining now all the relations obtained above, we conclude that

$$\nu_u := \lim_{t \longrightarrow T} \widetilde{E}[D_u g_t | \Im_u^{\widetilde{W}}] = \left[\frac{C_1}{\sqrt{T-u}} \varphi \left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}} \right) - \Phi \left(\frac{C_3 - \widetilde{w}_u}{\sqrt{T-u}} \right) \right] I_{[0,T]}(u).$$

Hence, due to Theorem 1 [1] (see also Theorem 2.3 [2]), we easily obrtain (1).

On the other hand, it is clear that

$$\widetilde{E}F = \widetilde{E}(W_T - C_1)^- I_{\{S_T \le C_2\}} = \widetilde{E}(\widetilde{W}_T - \widetilde{C}_1)^- I_{\{\widetilde{W}_T \le \widetilde{C}_2\}}$$

$$= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{C_3} (x - \widetilde{K}) \exp\{-\frac{x^2}{2T}\} dx = \sqrt{T} \varphi \left(\frac{C_3}{\sqrt{T}} \right) + \widetilde{K} \Phi \left(\frac{C_3}{\sqrt{T}} \right).$$

□

Theorem 2. *The hedging strategy $\pi = (\beta_t, \gamma_t)$, $t \in [0,T]$, the capital process X_t and the price C in problem of "replication" of barrier type European Option (with payoff F given in Theorem 1) in case of Black-Scholes financial market model are defined by relations*

$$\gamma_t = \frac{1}{\gamma_t S_t} \left[\frac{C_1}{\sqrt{T-t}} \varphi \left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}} \right) - \Phi \left(\frac{C_3 - \widetilde{W}_t}{\sqrt{T-t}} \right) \right], \quad \beta_t = \frac{1}{B_t}(X_t - \gamma_t S_t),$$

$$X_t = \widetilde{E}F + \int_0^t \left[\frac{C_1}{\sqrt{T-u}} \varphi \left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}} \right) - \Phi \left(\frac{C_3 - \widetilde{W}_u}{\sqrt{T-u}} \right) \right] d\widetilde{W}_u,$$

$$C = \widetilde{E}F = \sqrt{T} \varphi \left(\frac{C_3}{\sqrt{T}} \right) + \widetilde{K} \Phi \left(\frac{C_3}{\sqrt{T}} \right).$$

R E F E R E N C E S

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Received 13.05.2017; revised 25.09.2017; accepted 20.10.2017.

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