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ABOUT ONE METHOD OF STOCHASTIC INTEGRAL REPRESENTATION OF BROWNIAN FUNCTIONAL

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Abstract. In the present work the question of representation of Brownian integral functionals in the form of Ito's stochastic integral with the explicit construction of integrand is studied (the existence of representation was studied by Clark in 1970). The considered class of functionals includes a case stochastically nonsmooth functionals and therefore it is impossible to use the well-known Clark-Ocone formula (1984). Besides, the considered class includes functionals for which even the conditional mathematical expectation isn't stochastically smooth and, consequently, neither our generalization of the Clark-Ocone formula (2017) is applicable to them.

Keywords and phrases: Brownian functional, stochastic integral representation, Clark-Ocone formula.

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1 Introduction and preliminaries. In the theory of stochastic processes, the representation of functionals of Brownian motion by stochastic integrals, states that a functional that is measurable with respect to the filtration generated by a Brownian motion can be written in terms of Ito's stochastic integral with respect to this Brownian motion. The theorem only asserts the existence of the representation and does not help to find it explicitly.

It is possible in many cases to determine the form of the representation using Malliavin calculus, if a functional is Malliavin differentiable. We consider nonsmooth (in Malliavin sense) functionals and have developed some methods of obtaining of constructive martingale representation theorems. The obtained results can be used to establish the existence of a hedging strategy in various European Options with corresponding payoff functions.

The first proof of the martingale representation theorem was implicitly provided by Ito (1951) himself. This theorem states that any square-integrable Brownian functional is equal to a stochastic integral with respect to Brownian motion. Many years later, Dellacherie (1974) gave a simple new proof of Ito's theorem using Hilbert space techniques.

Many other articles were written afterward on this problem and its applications but one of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark. One important property of the Ito stochastic integral: if adapted process $f \in L_2([0,T] \times \Omega)$ then the process $\xi_t = \int_0^t f(s,\omega)dB_s(\omega)$ is a martingale with respect to the filtration $\{\Im_t^B\}$. On the other hand, according to the well-known Clark formula (see [1]), if F is a square integrable $\Im_T^B$-measurable random variable, then there exist a square integrable $\Im_t^B$-adapted random process $\varphi(t,\omega)$ such that

$$F = EF + \int_0^T \varphi(t,\omega)dB_t(\omega).$$

It should be noted that finding an explicit expression for $\varphi(t,\omega)$ is a very difficult problem. In this direction, one general result, called Clark-Ocone formula (see [2]), is known according to which $\varphi(t,\omega) = E[D_t^B F|\Im_t^B]$, where D_t^B is the so called Malliavin stochastic derivative.

Those of Haussmann (1979), Ocone (1984), Ocone and Karatzas (1991) and Karatzas, Ocone and Li (1991) were also particularly significant. A different method for finding the process $\varphi(t,\omega)$ was proposed by Shiryaev, Yor and Graversen (2003, 2006), which was based on the Ito (generalized) formula and the Levy theorem for the Levy martingale $M_t = E[F|\Im_t^B]$ associated with F : let $M_T = \sup_{0\le t\le T} B_t$, then the following stochastic integral representation holds

$$M_T = EM_T + 2\int_0^T \left[1 - \Phi\left(\frac{M_t - B_t}{\sqrt{T-t}}\right)\right] dB_t.$$

later on, using the Clark-Ocone formula, Renaud and Remillard (2006) have established explicit martingale representations for path-dependent Brownian functionals. Let us define $B_t^\theta = B_t + \theta t$; $m_t^\theta = \inf_{0\le s\le t} B_s^\theta$; $M_t^\theta = \sup_{0\le s\le t} B_s^\theta$; $m_t = m_t^0$; $M_t = M_t^0$; $Div(G) = \partial_x G + \partial_y G + \partial_z G$; $Div_{x,y}(G) = \partial_x G + \partial_y G$; $Div_{x,z}(G) = \partial_x G + \partial_z G$; for $b < a < c$, $b < 0$, $c > 0$, and $\tau = T - t$:

$$f(a,b,c;t) = e^{-\frac{1}{2}\theta^2\tau} E[DivG(B_\tau + a, m_\tau + a, M_\tau + a)e^{\theta B_\tau} I_{\{m_\tau \le b-a, c-a\le M_\tau\}}$$

$$+Div_{x,y}G(B_\tau + a, m_\tau + a, c)e^{\theta B_\tau} I_{\{m_\tau\le b-a, M_\tau\le c-a\}}$$
$$+Div_{x,z}G(B_\tau + a, b, M_\tau + a)e^{\theta B_\tau} I_{\{b-a\le m_\tau, c-a\le M_\tau\}}$$
$$+\partial_x G(B_\tau + a, b, c)e^{\theta B_\tau} I_{\{b-a\le m_\tau, M_\tau\le c-a\}}].$$

If $G : R^3 \longrightarrow R$ is a continuously differentiable function with bounded partial derivatives or a Lipschitz function, then the Brownian functional $X = G(B_T^\theta, m_T^\theta, M_T^\theta)$ admits the following martingale representation:

$$X = EX + \int_0^T f(B_t^\theta, m_t^\theta, M_t^\theta; t)dB_t.$$

2 Main result. In all cases described above investigated functionals, were stochastically (in Malliavin sense) smooth. We study the task of stochastic integral representation of stochastically nonsmooth functional. In particular, we developed the method of obtaining integral representation for some type functionals using the Trotter-Meyer Theorem which establishes the relation between predictable square variation of semimartingale and its local time. Moreover, in [3] we consider the path-dependent Wiener functional

$F = (W_T - K)^- I_{\{\min\limits_{0\le t\le T} W_t \le c\}}$, which is also not stochastically smooth. For this functional the stochastic integral representation formula with an explicit form of integrand is obtained. Note that this functional is a typical example of payoff function for so called European barrier[1] and lookback[2] Options. Hence, obtained there stochastic integral representation formula could be used for calculating the explicit hedging portfolio of such barrier and lookback option.

It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (with prof. O. Glonti, 2014) considered Brownian functionals which are not stochastically differentiable. The considered class of functionals includes a case the stochastically nonsmooth functionals and therefore it is impossible to use the well-known Clark-Ocone formula. Besides, the considered class includes functionals for which even the conditional mathematical expectation isn't stochastically smooth and, consequently, neither our generalization of the Clark-Ocone formula (see [4]) is applicable to them. In particular, we generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding the integrand.

It is clear that there are also such functionals which don't satisfy even the weakened conditions, i.e. the nonsmooth functionals whose conditional mathematical expectations is not stochastically differentiable too (see, for example, [5]). In particular, to such functional belongs the integral type functional $\int_0^T u_s(\omega)ds$ with nonsmooth integrand $u_s(\omega)$.

Remark. It is well-known that if $u_s(\omega) \in D_{2,1}^B$ for all s, then (see [6]) $\int_0^T u_s(\omega)ds \in D_{2,1}^B$ and $D_t\{\int_0^T u_s(\omega)ds\} = \int_0^T D_t u_s(\omega)ds$. But if $u_s(\omega)$ is not differentiable in the Malliavin sense, then the Lebesgue average (with respect to ds) is not differentiable in the Malliavin sense either.

Indeed, in this case the conditional mathematical expectation is not stochastically smooth, because we have:

$$E\left[\int_0^T u_s(\omega)ds|\Im_t\right] = \int_0^t u_s(\omega)ds + \int_t^T E[u_s(\omega)|\Im_t]ds,$$

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable, but the second summand is differentiable in the Malliavin sense when u_s satisfied our weakened condition (if $E[u_s(\omega)|\Im_t] \in D_{2,1}^B$ for almost all s and $E[u_s(\omega)|\Im_t]$ is Lebesgue integrable for a.a. ω, then

$$\int_t^T E[u_s(\omega)|\Im_t]ds \in D_{2,1}^B\big).$$

[1]The barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option.

[2]The payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

We will consider the Brownian function of integral type $F = \int_0^T h(B_t)dt$. We introduce the notation

$$V(t,x) := E\left[\int_t^T h(B_s)ds|B_t = x\right].$$

Theorem. *If the deterministic function $V(t,x)$ satisfies the requirements of the generalized Ito theorem, then the following stochastic integral representation is fulfilled*

$$\int_0^T h(B_t)dt = E\left[\int_0^T h(B_t)dt\right] + \int_0^T V_x^{'}(t,B_t)dB_t.$$

Corollary. *The following stochastic integral representation is fulfilled*

$$\int_0^T I_{\{c_t^1 \le B_t \le c_t^2\}}dt = \int_0^T \left[\Phi\Big(x/\sqrt{t}\Big)\right]\Big|_{x=c_t^1}^{x=c_t^2}dt - \int_0^T \Big(\int_t^T \frac{1}{s-t}\varphi\Big(\frac{x-B_t}{\sqrt{u-t}}\Big)\Big|_{x=c_t^1}^{x=c_t^2}ds\Big)dB_t.$$

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