ACADEMIA DE TRANSPORTURI, INFORMATICĂ ŞI COMUNICAŢII

INSTITUTUL DE CIBERNETICĂ ACADEMIA NAŢIONALĂ DE ŞTIINŢĂ A UCRAINEI

UNIVERSITATEA DE STAT DIN MOLDOVA

UNIVERSITATEA NAŢIONALĂ DIN KIEV

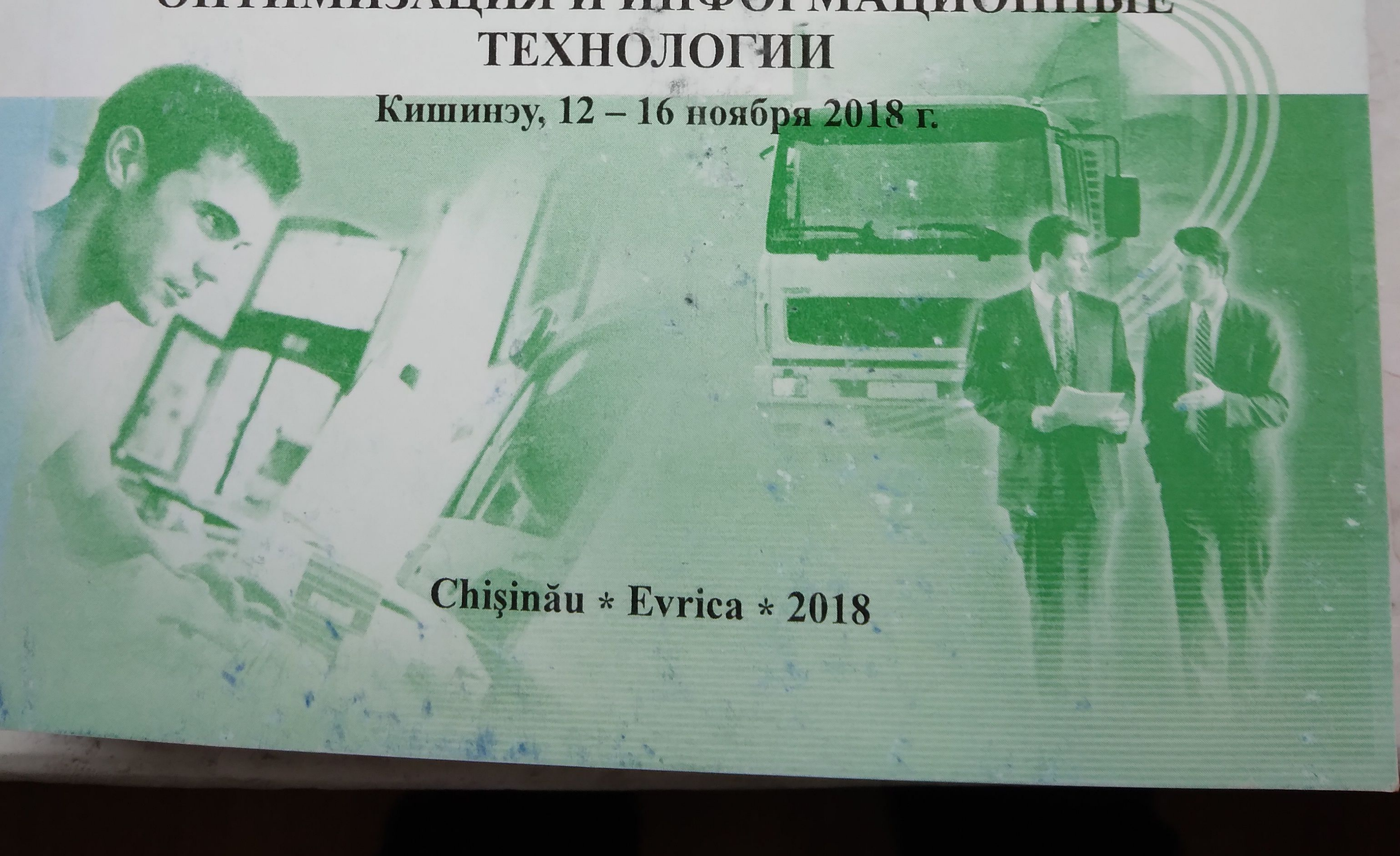
MODELARE MATEMATICĂ, OPTIMIZARE ŞI TEHNOLOGII INFORMAŢIONALE

Materialele Conferinţei Internaţionale
Chişinău, 12 – 16 noiembrie 2018
EDIŢIA A VI-A

Материалы 6-й международной конференции

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ, ОПТИМИЗАЦИЯ И ИНФОРМАЦИОННЫЕ ТЕХНОЛОГИИ

Кишинэу, 12 – 16 ноября 2018 г.



Chişinău * Evrica * 2018

ACADEMIA DE TRANSPORTURI,
INFORMATICĂ ŞI COMUNICAŢII

INSTITUTUL DE CIBERNETICĂ
ACADEMIA NAŢIONALĂ
DE ŞTIINŢĂ A UCRAINEI

UNIVERSITATEA DE STAT DIN MOLDOVA

UNIVERSITATEA NAŢIONALĂ DIN KIEV

MODELARE MATEMATICĂ, OPTIMIZARE ŞI TEHNOLOGII INFORMAŢIONALE

Materialele Conferinţei Internaţionale
Chişinău, 12 – 16 noiembrie 2018
EDIŢIA A VI-A

Материалы 6-й международной конференции

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ, ОПТИМИЗАЦИЯ И ИНФОРМАЦИОННЫЕ ТЕХНОЛОГИИ

Кишинэу, 12 – 16 ноября 2018

Chişinău • Evrica • 2018

STOCHASTIC INTEGRAL REPRESENTATION OF THE PAYOFF FUNCTIONS OF EUROPEAN EXOTIC TYPE OPTIONS

O.G. PURTUKHIA,
Ivane Javakhishvili Tbilisi State University,
Tbilisi, Georgia
o.purtukhia@gmail.com

The question of representation of wide class of Wiener functionals in the form of Ito's stochastic integral with the explicit construction of integrand is studied, which includes the stochastically non-smooth functionals and therefore it is impossible to use the well-known Clark-Ocone formula (1984), as well as our generalization of the Clark-Ocone formula (2017).
keywords: *European Option, payoff function, stochastic integral representation, Clark-Ocone formula.*

In contrast to the standard European Option payoff function (i.e. $(S_T - K)^+$), which is stochastically (in Malliavin sense) differentiable, we will discuss European type options with non-smooth payoff functions. The payoff functions of derivative securities with more complicated forms than standard European or American call and put options are known as exotic options.

One of such kind exotic option is so-called Binary Option. It is an option with discontinuous payoff function. The simplest examples of the Binary Options are call and put options "cash or nothing". The payoff function of the call option has the form $BC_T = QI_{\{S_T > K\}}$, and for the put option -- $BC_T = QI_{\{S_T < K\}}$, where K is the strike price at the time of execution T (it should be noted that indicator of event G is Malliavin differentiable if and only if probability $P(G)$ is equal to zero or one [3]). Moreover, so-called Asian Options also are type of Exotic Option.

Here we will explore the stochastically non-smooth Wiener functionals that can be considered in the future as a payoff function of a certain exotic European Option and study the issues of their stochastic integral representation, which it is known to play a significant role in the hedging problem of European Options. It is possible in many cases to determine the form of the representtation using Malliavin calculus, if a functional is Malliavin differentiable. We consider non-smooth (in Malliavin sense) functionals and have developed some methods of obtaining of constructive martingale representation theorems. The obtained results can be used to establish the existence of a hedging strategy in various European Options with corresponding payoff functions.

After Clark [1] obtained the formula for the stochastic integral representation for Wiener functionals, which asserted only the existence of this representation, many authors tried to find the integrand explicitly, and the corresponding results were obtained when the functionals were smooth in some sense. In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the explicitness point of view: the integrands in the stochastic integral representations always involve predictable projections or conditional expectations and some kind of gradients.

Theorem (Ocone, [2]). *If F is differentiable in Malliavin sense, $F \in D^W_{2,1}$, then the stochastic integral representation is fulfilled*

$$F = EF + \int_0^T E[D_t F \mid \Im_t^W] dW_t ,$$

Where $D_t F$ is the stoqhastic derivative of F.

Shiryaev, Yor and Graversen (2003, 2006) proposed a method based on Ito's formula to find explicit martingale representations for Wiener functionals which yields in particular the explicit martingale representation of the running maximum of Wiener process. Our approach with prof. Jaoshvili (2005-2009) in the framework of the classical Ito calculus, on the basis of the standard L_2 theory and the theory of weighted Sobolev spaces, made it possible to construct an explicit formula for the itegrand when the functional does not have the mentioned smoothness.

Theorem (Jaoshvili, Purtukhia, 2005). *Let the function* $f \in L_{2,T/\alpha}$, $0<\alpha<1$, *and it has the generalized derivative of the first order* $\partial f/\partial x$, *such that* $\partial f/\partial x \in L_{2,T/\beta}$, $0<\beta<1/2$, *then the following integral representation holds*

$$f(W_T) = Ef(W_T) + \int_0^T E[\frac{\partial f}{\partial x}(W_T) \mid \Im_t^W] dW_t \,,$$

where $L_{2,T}$ *denotes the set of measurable functions* u, *such that* $u(\cdot)\rho(\cdot,T) \in L_2 := L_2(R, B(R), \lambda)$ *(where* $B(R)$ *is the Borel* σ*-algebra on* R, λ *is the Lebesgue measure and* $\rho(x,T) = \exp\{-x^2/(2T)\}$.

As it has been already noted in all cases described above investigated functionals, were stochasticaly (in Malliavin sense) smooth. It has turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. We (with prof. O. Glonti, 2014) considered Wiener functionals which are not stochastically differentiable. In particular, we generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding of integrand.

Theorem (Glonti, Purtukhia, [5]). *Suppose that* $g_t := E[F \mid \Im_t^W]$ *is Malliavin differentiable (*$g_t \in D_{2,1}^W$*) for almost all* $t \in [0,T)$. *Then we have the stochastic integral representation*

$$g_T = F = EF + \int_0^T v_u dW_u \,, \quad (P\text{-a.s.})$$

where $v_u := \lim_{t\to T} E[D_u g_t \mid \Im_u^W]$ *in the* $L_2([0,T]\times\Omega)$.

Theorem 1. *For the Wiener functional* $F = (W_T - C_1)^- I_{\{\inf_{0\le t\le T} W_t \le C_2\}}$ *(*$C_2 \le 0$, $C_2 \le C_1$*) the following stochastic integral representation holds*

$$F = EF + \int_0^T \Phi(\frac{2C_2 - C_1 - W_t}{\sqrt{T-t}}) dW_t \,.$$

Next, we have considered functionals which didn't satisfy even these weakened conditions. To such functionals belong, for example, Lebesgue integral (with respect to time variable) from stochastically non smooth square integrable processes.

Theorem 2. *If the deterministic function* $V(t,x) := E[\int_t^T f(s,W_s)ds \mid W_t = x]$ *satisfies the requirements of the generalized Ito theorem, then the following stochastic integral representation is fulfilled*

$$\int_0^T f(t,W_t)dt = E\left[\int_0^T f(t,W_t)dt\right] + \int_0^T \frac{\partial}{\partial x} V(t,W_t)dW_t\,.$$

Corollary. *The following stochastic integral representation is fulfilled*

$$\int_0^T I_{\{c_i^1 \le W_t \le c_i^2\}} dt = \int_0^T \Phi(\frac{x}{\sqrt{t}})\Big|_{x=c_i^1}^{x=c_i^2} dt + \int_0^T\int_t^T \frac{1}{s-t}\varphi\left(\frac{x-W_t}{\sqrt{s-t}}\right)\Big|_{x=c_i^1}^{x=c_i^2} dsdW_t.$$

References.

1. Clark M. C. The representation of functionals of Brownian motion by stochastic integrals // J. The Annals of Mathematical Statistics. – 1970. – Vol. 41. – P. 1282–1295.
2. Ocone D. Malliavin calculus and stochastic integral representation formulas of diffusion processes // J. Stochastics. – 1984. – Vol. 12. – P. 161–185.
3. Nualart D. The Malliavin calculus and related topics. – Berlin, Springer-Verlag 2006. – 383 p.
4. Glonti O., Jaosshvili, V., Purtukhia, O. Hedging of European Option of Exotic Type // Proceedings of A. Razmadze Mathematical Institute. – 2015. – Vol. 168. – P. 25–40.
5. Glonti O., Purtukhia O. On One Integral Representation of Functionals of Brownian Motion // SIAM J. Theory of Probability and Its Applications. – 2017. – Vol. 61. – P. 133–139.