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Derived functors and the homology of n-types

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1. Introduction

Let X be a connected CW-space whose homotopy groups $\pi_i X$ are trivial in dimensions $i > n+1$. Such a space is termed a *homotopy* $(n+1)$*-type*. In the case $n = 0$, classical homological algebra provides a purely algebraic description of the integral homology $H_*(X)$ in terms of derived functors. For $n = 1$ it has recently been shown [1] (cf. [2]) that the homology can be realised as the non-abelian left derived functors of a certain abelianisation functor $\mathcal{A}$: (crossed modules) $\to$ (abelian groups). Crossed modules are convenient algebraic models of homotopy 2-types. More generally, homotopy $(n+1)$-types are modelled by cat^n-groups [3] or equivalently by crossed n-cubes [4]. Our aim in this paper is to explain how the methods of [1] extend to arbitrary $n \geqslant 0$ and lead to a natural isomorphism

$$H_{n+i+1}(X) \cong L_i^{\mathcal{A}}(G), \quad i \geqslant 1, \tag{1}$$

where $L_i^{\mathcal{A}}(-)$ is the ith non-abelian left derived functor of a certain abelianisation functor $\mathcal{A}$: (crossed n-cubes) $\to$ (abelian groups) and G is a suitable crossed n-cube. We also explain how the relationship between $H_{n+1}(X)$ and $L_0^{\mathcal{A}}(G)$ can

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be expressed as an algebraic formula for the homology of X analogous to the Hopf type formula for the higher homology of a group obtained in [5].

The paper handles only the cases $n = 1, 2$ in full detail. The routine modifications needed for $n \geqslant 3$ are largely left to the reader. In Section 2 we recall some terminology and results of D. Quillen [6,7] on homology in algebraic categories. In Section 3 we derive the following lemma from general results on simplicial objects: if G is a projective n-fold simplicial group then $\pi_0(G)$ is a free group and $\pi_i(G) = 0$ for $i \geqslant 1$. Isomorphism (1) is proved in Section 4, together with various Hopf-type formulae for $H_*(X)$.

We adopt the following notation. The category of sets is denoted by $\mathcal{S}$. The category of groups is denoted by $\mathcal{G}$. The category of simplicial objects of a category $\mathcal{C}$ is denoted by $\mathcal{SC}$. Accordingly, $\mathcal{SS}$ denotes the category of simplicial sets, $\mathcal{SG}$ the category of simplicial groups. The category of n-fold simplicial groups is denoted by $\mathcal{S}^n\mathcal{G}$. We always identify an object X of $\mathcal{C}$ with the constant simplicial object of $\mathcal{SC}$ whose simplicial operators are all equal to the identity morphism of X. The free group on a set X is denoted by $\langle X\rangle_{\mathrm{gr}}$. The standard n-simplex is denoted by Δ^n and characterised by the property $\mathrm{Hom}_{\mathcal{SS}}(\Delta^n, X) \cong X_n$ for all simplicial sets X. The integral homology of a simplicial set or CW-space X is denoted by $H_*(X)$. For a simplicial group G, we let $M_*(G)$ be the Moore complex of G, which is the non-abelian chain complex defined by

$$M_n(G) = \bigcap_{0<i\leqslant n} \operatorname{Ker} \partial_i^n,$$

with $d : M_n(G) \to M_{n-1}(G)$ induced by ∂_0^n.

2. Quillen homology

In this section we recall some ideas and results of Quillen (see [6,7]). Let $\mathcal{C}$ be a category with finite limits. We let $X \times_Y X$ denote the pull-back

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

of a morphism $f : X \to Y$ in $\mathcal{C}$. The morphism $f : X \to Y$ is said to be an *effective epimorphism* if, for any object T, the diagram of sets

$$\mathrm{Hom}_{\mathcal{C}}(Y, T) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, T) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(X \times_Y X, T)$$

is exact. This means that f^* is an injective map and, if $g : X \to T$ is a morphism such that $gp_1 = gp_2$, then there exists a (necessarily unique) morphism $h : Y \to T$ such that $g = hf$.

An object P of $\mathcal{C}$ is *projective* if for any diagram

$$\begin{array}{ccc} & & P \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f an effective epimorphism there exists a morphism $h : P \to X$ such that $g = fh$. We say that $\mathcal{C}$ has sufficiently many projective objects if for any object X there is a projective object P and an effective epimorphism $P \to X$.

Let us assume additionally that $\mathcal{C}$ possesses colimits. An object X is called *small* if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with filtered colimits.

A class $\mathcal{U}$ of objects of $\mathcal{C}$ is said to *generate* $\mathcal{C}$ if for every object X there is an effective epimorphism $Q \to X$ where Q is a coproduct of copies of members of $\mathcal{U}$.

The category $\mathcal{C}$ is said to be an *algebraic category* if it possesses finite limits and arbitrary colimits and has a set of small projective generators. We leave the proof of the following easy fact to the reader.

Lemma 1. *Let $\mathcal{C}$ be an algebraic category and $\mathcal{B} \subset \mathcal{C}$ be a full subcategory. Suppose that $\mathcal{B} \subset \mathcal{C}$ has a left adjoint $\mathcal{L} : \mathcal{C} \to \mathcal{B}$ and that the following condition holds*: *a morphism $f : X \to Y$ in $\mathcal{B}$ is an effective epimorphism in $\mathcal{C}$ if and only if it is an effective epimorphism in $\mathcal{B}$. Then $\mathcal{L}P$ is a projective object in $\mathcal{B}$ for each projective object P in $\mathcal{C}$. Thus $\mathcal{B}$ is an algebraic category.*

The following fundamental result is due to Quillen (see [6, Theorem 4, Chapter II, p. 4]).

Theorem 2. *Let $\mathcal{C}$ be an algebraic category. Then there exists a unique closed simplicial model category structure on the category $\mathcal{SC}$ of simplicial objects over $\mathcal{C}$ such that a morphism f in $\mathcal{SC}$ is a fibration* (*respectively weak equivalence*) *if and only if* $\mathrm{Hom}_{\mathcal{SC}}(P, f)$ *is a fibration* (*respectively weak equivalence*) *of simplicial sets for each projective object P of $\mathcal{C}$. Moreover, if X is a cofibrant object in $\mathcal{SC}$ then X_n is a projective object in $\mathcal{C}$ for all $n \geqslant 0$.*

A *simplicial resolution* of an object X of $\mathcal{C}$ can be defined as a fibration $Q \to X$ in $\mathcal{SC}$ which is also a weak equivalence. If additionally Q is a cofibrant object in $\mathcal{SC}$ then $Q \to X$ is called a simplicial *cofibrant* resolution. It is a formal consequence of Theorem 1 that simplicial cofibrant resolutions exist and are unique up to homotopy equivalence, and up to homotopy depend functorially on X.

Let $\mathcal{C}_{\mathrm{ab}}$ denote the category of abelian group objects in the algebraic category $\mathcal{C}$. It can be shown that $\mathcal{C}_{\mathrm{ab}}$ is an abelian category and that moreover the abelianisation functor

$$(-)_{\mathrm{ab}} : \mathcal{C} \to \mathcal{C}_{\mathrm{ab}}, \quad X \mapsto X_{\mathrm{ab}},$$

left adjoint to the forgetful functor $\mathcal{C}_{\mathrm{ab}} \subset \mathcal{C}$, exists. Following [7] one defines the Quillen homology of an object X in $\mathcal{C}$ as the homology of the chain complex associated to the simplicial object Q_{ab} obtained by applying $(-)_{\mathrm{ab}}$ dimensionwise to a simplicial cofibrant resolution Q of X. We let $D_*(X)$ denote the Quillen homology of X.

3. Projective objects in n-fold simplicial groups

The material in this section is well-known. Our goal is the following result: a projective object in $\mathcal{S}^n\mathcal{G}$ has no homotopy in dimensions $\geqslant 1$.

Let $\mathcal{I}$ be a small category and let $\mathcal{G}^{\mathcal{I}}$ denote the category of functors $\mathcal{I} \to \mathcal{G}$ from $\mathcal{I}$ to the category of groups.

Lemma 3. *A morphism $f : X \to Y$ in $\mathcal{G}^{\mathcal{I}}$ is an effective epimorphism if and only if $f(i)$ is surjective for all objects $i \in \mathcal{I}$.*

Proof. Assume $f(i)$ is surjective for all objects $i \in \mathcal{I}$. Then $\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T)$ is injective and, for any functor $T : \mathcal{I} \to \mathcal{G}$ and any natural transformation

$$g : X \to T$$

such that the diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{g} & T \end{array}$$

commutes, there exists a unique transformation

$$h : Y \to T$$

such that $g = hf$. The transformation h is given by

$$h(i)(y) = g(i)\big(f(i)^{-1}(y)\big), \quad y \in Y(i).$$

The condition on g implies that h is well-defined. Conversely, assume f is an effective epimorphism. Set $Y'(i) = \mathrm{Im}(f)(i) \subset Y(i)$. Then each $X(i) \to Y'(i)$ is surjective and hence $X \to Y'$ is an effective epimorphism. Therefore $\mathrm{Hom}(Y, Z)$ and $\mathrm{Hom}(Y', Z)$ are both equalisers of the same diagram and hence coincide. It follows that $Y' = Y$. $\square$

Lemma 4. *For each object $i \in \mathcal{I}$ let*

$$h_i : \mathcal{I} \to \mathcal{G}$$

be the functor given by

$$j \mapsto \langle \mathrm{Hom}_{\mathcal{I}}(i, j) \rangle_{\mathrm{gr}}.$$

Then the collection $(h_i)_{i \in \mathcal{I}}$ is a set of small projective generators in the category $\mathcal{G}^{\mathcal{I}}$.

Proof. The Yoneda lemma implies that

$$\mathrm{Hom}_{\mathcal{G}^{\mathcal{I}}}(h_i, T) \cong T(i)$$

for $i \in \mathcal{I}$, $T \in \mathcal{G}^{\mathcal{I}}$ and the result follows. □

Corollary 5. *The category $\mathcal{G}^{\mathcal{I}}$ is an algebraic category.*

In fact, one can prove that $\mathcal{C}^{\mathcal{I}}$ is an algebraic category for any algebraic category $\mathcal{C}$. As a consequence of Corollary 5, we see that the category of (n-fold) simplicial groups is algebraic. We need to identify the homotopy type of projective objects in $\mathcal{S}^n\mathcal{G}$.

Let Δ be the category of finite ordinals. We will assume that the objects are the sets

$$[n] = \{0, 1, \ldots, n\}, \quad n \geqslant 0$$

and morphisms are nondecreasing maps. Then $\mathcal{G}^{\Delta^{\mathrm{op}}}$ is the category of simplicial groups. Since

$$\Delta^n = \mathrm{Hom}_{\Delta^{\mathrm{op}}}([n], -)$$

is the standard n-simplex, Lemma 4 shows that any projective object in the category of simplicial groups is a retract of a simplicial group of the form

$$\left\langle \coprod_{n \geqslant 0} (S_n \times \Delta^n) \right\rangle_{\mathrm{gr}},$$

where $(S_n)_{n \geqslant 0}$ is a sequence of sets.

It is well known that the standard n-simplex is simplicially contractible (see, for example, ρ in [8, p. 151]) and therefore the projection of $\coprod_{n \geqslant 0} S_n \times \Delta^n$ to the constant simplicial set $\coprod_{n \geqslant 0} S_n$ is a simplicial homotopy equivalence. Since any degreewise extension of a functor $\mathcal{S} \to \mathcal{G}$ preserves the homotopy relation, we see that the simplicial group $\langle \coprod_{n \geqslant 0} S_n \times \Delta^n \rangle_{\mathrm{gr}}$ is homotopy equivalent to the constant simplicial group $\langle \coprod_{n \geqslant 0} S_n \rangle_{\mathrm{gr}}$. As a consequence we obtain the following corollary.

Corollary 6. *If P is a projective object in the category of simplicial groups, then P is degree-wise free. Moreover, $\pi_0 P$ is a free group and P is homotopy equivalent to a constant simplicial group, hence $\pi_i P = 0$ for $i \geqslant 1$.*

We now consider bisimplicial groups. Given a bisimplicial group G, we let G_n^{v} and G_m^{h} denote the nth vertical and mth horizontal parts; both are simplicial groups. We let $\pi_i^{\mathrm{v}} G$ and $\pi_i^{\mathrm{h}} G$ denote the simplicial groups obtained by taking the ith homotopy group of G_n^{v} and G_m^{h}. We let $\pi_i G$ denote the ith homotopy group of the diagonal simplicial group $(G_{n,n})_{n \geqslant 0}$. For simplicial sets X and Y we let $X \boxtimes Y$ denote the bisimplicial set whose (m,n)th component is $X_m \times Y_n$.

By Corollary 5 the category of bisimplicial groups is an algebraic category and any projective object is a retract of a bisimplicial group of the form

$$\left\langle \coprod_{n,m} S_{n,m} \times \left(\Delta^n \boxtimes \Delta^m\right) \right\rangle_{\mathrm{gr}},$$

where $(S_{n,m})_{n,m \geqslant 0}$ is a family of sets. Thus each vertical or horizontal part is a projective object in the category of simplicial groups. Moreover, if P is a projective object in the category of bisimplicial groups, then

$$\pi_n^{\mathrm{v}} P = 0 = \pi_n^{\mathrm{h}} P, \quad n \geqslant 1,$$

and both $\pi_0^{\mathrm{v}} P$ and $\pi_0^{\mathrm{h}} P$ are projective objects in the category of simplicial groups. Thus

$$\pi_i^{\mathrm{h}} \pi_j^{\mathrm{v}} P = 0$$

as soon as $i > 0$ or $j > 0$. So the spectral sequence [9]

$$E_{pq}^2 = \pi_p^{\mathrm{v}} \pi_q^{\mathrm{h}} P \Rightarrow \pi_{p+q} P$$

implies that $\pi_i P = 0$ for $i \geqslant 1$. Since $\pi_0 P = \pi_0^{\mathrm{v}} \pi_0^{\mathrm{h}} P$, we see that $\pi_0 P$ is a free group.

The situation for multisimplicial groups is analogous. We leave as an exercise to the reader the modifications required to obtain the following result.

Lemma 7. *Let G be a projective object in the category $\mathcal{S}^n \mathcal{G}$ of n-fold simplicial groups. In each direction G is homotopy equivalent to a constant simplicial object in the category $\mathcal{S}^{n-1} \mathcal{G}$ of $(n-1)$-fold simplicial groups, which is also a projective in $\mathcal{S}^{n-1} \mathcal{G}$. In particular, $\pi_i G = 0$ for $i \geqslant 1$, and $\pi_0 P$ is a free group, where $\pi_i G$ denotes the ith homotopy group of the diagonal of G.*

4. Homology of cat^n-groups and crossed n-cubes

Recall [3] that a cat^n-*group* consists of a group G together with $2n$ endomorphisms $s_i, t_i : G \to G$ satisfying

$$s_i s_i = s_i, \qquad s_i t_i = t_i, \qquad t_i t_i = t_i, \qquad t_i s_i = s_i,$$
$$s_i t_j = t_j s_i \quad (i \neq j),$$
$$\big[\mathrm{Ker}(s_i), \mathrm{Ker}(t_i)\big] = 0,$$

for $1 \leqslant i \leqslant n$. A *morphism* of cat^n-groups $(G, s_i, t_i) \to (G', s'_i, t'_i)$ is a group homomorphism $G \to G'$ that preserves the s_i and t_i. We let $\mathcal{CG}$ denote the category of cat^1-groups, and $\mathcal{C}^n\mathcal{G}$ the category of cat^n-groups. Note that a cat^0-group is just a group.

A cat^1-group (G, s, t) is equivalent to a category object in $\mathcal{G}$. The arrows are the elements of G, the identity arrows are the elements of $N = \mathrm{Im}(s) = \mathrm{Im}(t)$, the source and target maps are s and t, and composition of arrows $g, h \in G$ is given by $g \circ h = g(sg)^{-1}h$. Thus the nerve of a category provides a functor $\mathcal{N} : \mathcal{CG} \to \mathcal{SG}$.

Lemma 8. (i) *$\mathcal{N}$ is a full and faithful*;

(ii) *$\mathcal{N}$ possesses a left adjoint $\mathcal{T} : \mathcal{SG} \to \mathcal{CG}$ and the functor $\mathcal{N} \circ \mathcal{T} : \mathcal{SG} \to \mathcal{SG}$ preserves the homotopy relation. Moreover, for any simplicial group G one has*

$$\pi_i\big(\mathcal{N} \circ \mathcal{T}(G)\big) = \pi_i(G) \quad \text{if } i = 0, 1, \quad \text{and}$$
$$\pi_i\big(\mathcal{N} \circ \mathcal{T}(G)\big) = 0 \qquad \text{if } i > 1;$$

(iii) *A morphism f in $\mathcal{CG}$ is an effective epimorphism if and only if $\mathcal{N}f$ is an effective epimorphism in $\mathcal{SG}$*;

(iv) *$\mathcal{CG}$ is an algebraic category. Moreover if (G, s, t) is a projective object in the category $\mathcal{CG}$ of* cat^1*-groups, then $\pi_i(\mathcal{N}(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}(G))$ is free group.*

Proof. (i) is obvious. The statement (ii) is well-known and it follows for example from [10, Proposition 3]. By [10] the cat^1-group $\mathcal{T}G$ has underlying group $G_1/\partial_0^2(M_2(G))$; the maps s and t are induced by d_0^1 and d_1^1. One easily checks that for any simplicial group G the Moore complex of $\mathcal{N} \circ \mathcal{T}(G)$ is isomorphic to

$$\cdots \to 0 \to M_1(G)/\partial_0^2\big(M_2(G)\big) \to M_0(G)$$

and therefore $\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = \pi_i(G)$ if $i = 0, 1$ and $\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = 0$ if $i > 1$. In order to verify (iii), note that the argument given in the proof of Lemma 3 shows that f is an effective epimorphism if and only if f is surjective (as a homomorphism of groups). But f is surjective if and only if $\mathcal{N}f$ is degreewise surjective which, by Lemma 3, is equivalent to $\mathcal{N}f$ being an effective epimorphism. By (iii) the assumptions of Lemma 1 hold and therefore $\mathcal{CG}$ is an algebraic category. The last statement of (iv) follows easily from Lemma 6 and from (ii). □

A cat^2-group G is equivalent to a category object in $\mathcal{CG}$. It is thus equivalent to a group endowed with two compatible category structures, a horizontal one and a vertical one. The nerve $\mathcal{N}^{\mathrm{v}}G$ of the vertical category structure is a category object in $\mathcal{SG}$. By then taking the nerve of the horizontal category structure, we obtain a full and faithful functor

$$\mathcal{N}^2 = \mathcal{N}^{\mathrm{h}}\mathcal{N}^{\mathrm{v}} : \mathcal{C}^2\mathcal{G} \to \mathcal{S}^2\mathcal{G}$$

into bisimplicial groups. Moreover a morphism f is an effective epimorphism (i.e. surjective as a group homomorphism) in $\mathcal{C}^2\mathcal{G}$ if and only if $\mathcal{N}^2 f$ is an effective epimorphism (i.e. dimensionwise surjective) in $\mathcal{S}^2\mathcal{G}$. The functor $\mathcal{N}^2$ admits a left adjoint

$$\mathcal{T}^2 : \mathcal{S}^2\mathcal{G} \to \mathcal{C}^2\mathcal{G}$$

which is defined by first applying $\mathcal{T}$ dimensionwise to a bisimplicial group G to obtain a simplicial cat^1-group $\mathcal{T}G$, and then applying $\mathcal{T}$ again to obtain a cat^2-group $\mathcal{T}^2G$.

By Corollary 5 and Lemma 1, the category $\mathcal{C}^2\mathcal{G}$ of cat^2-groups is an algebraic category. Moreover, if (G, s_1, s_2, t_1, t_2) is a projective object in the category $\mathcal{C}^2\mathcal{G}$ of cat^2-groups, then the horizontal and vertical cat^1-groups are projective in the category of cat^1-groups. Moreover, $\pi_i(\mathcal{N}^2(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}^2(G))$ is free group. These facts follows easily from Lemma 7 because $\mathcal{T}$ respects the homotopy relations.

The situation for cat^n-groups is similar. We leave as an exercise for the reader the routine modifications needed to establish the following lemma.

Lemma 9. *The category $\mathcal{C}^n\mathcal{G}$ of* cat^n*-groups is an algebraic category. Moreover, if (G, s_i, t_i), $i = 1, \dots, n$ is a projective object in the category $\mathcal{C}^n\mathcal{G}$, then each 'face' of G is a projective object in the category $\mathcal{C}^{n-1}\mathcal{G}$. Furthermore $\pi_i(\mathcal{N}^n(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}^n(G))$ is a free group.*

An abelian group object in $\mathcal{C}^n\mathcal{G}$ is just a cat^n-group whose underlying group is abelian. The abelianisation functor

$$(-)_{\mathrm{ab}} : \mathcal{C}^n\mathcal{G} \to \big(\mathcal{C}^n\mathcal{G}\big)_{\mathrm{ab}}$$

sends a cat^n-group $G = (G, s_i, t_i)$ to the cat^n-group with underlying group $G_{\mathrm{ab}} = G/[G, G]$ and induced homomorphisms $s_i, t_i : G_{\mathrm{ab}} \to G_{\mathrm{ab}}$. The Quillen homology of a cat^n-group G is obtained from a cofibrant simplicial resolution $Q \to G$ by abelianising the simplicial cat^n-group Q dimensionwise and taking the homology of the associated chain complex or associated Moore complex:

$$D_i(G) = \pi_i(Q_{\mathrm{ab}}).$$

Note that $D_i(G)$ is an abelian cat^n-group for each $i \geqslant 0$. Below we define the group $H_i(G)_{\mathrm{Quillen}}$ as a subgroup of the underlying group of $D_{i-1}(G)$.

There is an alternative way to define the homology of a cat^n-group G, based on the composite functor

$$\mathcal{B} : \mathcal{C}^n\mathcal{G} \xrightarrow{\mathcal{N}^n} \mathcal{S}^n\mathcal{G} \xrightarrow{\mathcal{N}} \mathcal{S}^{n+1}\mathcal{S} \xrightarrow{\text{diagonal}} \mathcal{SS}$$

from cat^n-groups to simplicial sets ($n \geqslant 0$). The functor $\mathcal{N}:\mathcal{S}^n\mathcal{G} \to \mathcal{S}^{n+1}\mathcal{S}$ is defined by considering groups as categories and taking the nerve degreewise. The geometric realization $|\mathcal{B}G|$ is by definition the *classifying space* of the cat^n-group G and induces an equivalence between the (suitably defined) homotopy categories of cat^n-groups and connected CW-spaces X with $\pi_i X = 0$ for $i \geqslant n+2$ (see [3]). The integral homology of $|\mathcal{B}G|$ is a natural homology to associate to G, and so we set

$$H_i(G)_{\mathrm{Top}} = H_i\big(|\mathcal{B}G|\big), \quad i \geqslant 0.$$

We refer the reader to [11,12] for more information on $H_i(G)_{\mathrm{Top}}$ in the case $n = 1$. Our principal aim in this paper is a comparison of the algebraically defined homology $D_*(G)$ with the topologically defined homology $H_*(G)_{\mathrm{Top}}$.

We remark that the classifying functor $\mathcal{B}$ behaves nicely with respect to the inclusion functor $\mathcal{C}^n\mathcal{G} \to \mathcal{S}\mathcal{C}^n\mathcal{G}$ and also with respect to the inclusion functors $\mathrm{incl}_j:\mathcal{C}^n\mathcal{G} \longrightarrow \mathcal{C}^{n+1}\mathcal{G}$ ($1 \leqslant j \leqslant n+1$) which insert identity morphisms s_j, t_j. By taking nerves and diagonals appropriately one obtains a functor $\mathcal{B}:\mathcal{S}\mathcal{C}^n\mathcal{G} \to \mathcal{S}\mathcal{S}$ from simplicial cat^n-groups to simplicial sets such that the triangle of functors

$$\begin{array}{ccc} \mathcal{C}^n\mathcal{G} & \xrightarrow{\text{inclusion}} & \mathcal{S}\mathcal{C}^n\mathcal{G} \\ & \searrow^{\mathcal{B}} & \downarrow{\scriptstyle \mathcal{B}} \\ & & \mathcal{S}\mathcal{S} \end{array}$$

commutes. The triangle of functors

$$\begin{array}{ccc} \mathcal{C}^n\mathcal{G} & \xrightarrow{\mathrm{incl}_j} & \mathcal{C}^{n+1}\mathcal{G} \\ & \searrow^{\mathcal{B}} & \downarrow{\scriptstyle \mathcal{B}} \\ & & \mathcal{S}\mathcal{S} \end{array}$$

also commutes for each j.

To facilitate the comparison of $H_*(G)_{\mathrm{Top}}$ and $H_*(G)_{\mathrm{Quillen}}$ we recall from [4] some details on the categorical equivalence between cat^n-groups and crossed n-cubes. A *crossed n-cube* consists of a collection of groups M_α indexed by the 2^n subsets $\alpha \subset \{1,\dots,n\}$, together with homomorphisms $\lambda_i : M_\alpha \to M_{\alpha\setminus\{i\}}$ for $i \in \alpha$ and commutator type functions $h : M_\alpha \times M_\beta \to M_{\alpha\cup\beta}$. For present purposes it is unnecessary to recall precise details of the commutator functions or the axioms satisfied by the structure. A crossed 1-cube

$$M_{\{1\}} \xrightarrow{\lambda_1} M_\emptyset$$

is just a crossed module, the action being given by $M_\emptyset \times M_{\{1\}} \to M_{\{1\}}$, $(x, y) \mapsto h(x, y)y$. A crossed 2-cube

$$\begin{array}{ccc} M_{\{1,2\}} & \xrightarrow{\lambda_1} & M_{\{2\}} \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_2} \\ M_{\{1\}} & \xrightarrow{\lambda_1} & M_\emptyset \end{array}$$

coincides with the notion of a crossed square introduced by Loday [3]. A morphism $(M_\alpha) \to (M'_\alpha)$ of crossed n-cubes is a family of structure preserving group homomorphisms $M_\alpha \to M'_\alpha$. We let $\mathcal{XG}$ denote the category of crossed modules, and $\mathcal{X}^n\mathcal{G}$ the category of crossed n-cubes.

It has long been known that a crossed module is equivalent to a category object in $\mathcal{G}$, that is, to a cat^1-group (see [13]). Loday [3] proved that crossed squares are equivalent to cat^2-groups, and this equivalence was extended [4] to one between crossed n-cubes and cat^n-groups. The functorial equivalence

$$\mathcal{E}: \mathcal{C}^n\mathcal{G} \to \mathcal{X}^n\mathcal{G}$$

sends a cat^n-group $G = (G, s_i, t_i)$ to the crossed n-cube $\mathcal{E}G$ with

$$\mathcal{E}G_\alpha = \bigcap_{i\in\alpha} \mathrm{Ker}(s_i) \cap \bigcap_{j\in\bar{\alpha}} \mathrm{Im}(s_j),$$

where $\bar{\alpha}$ denotes the complement of α in $\{1, \ldots, n\}$. The morphisms $\lambda_i : \mathcal{E}G_\alpha \to \mathcal{E}G_{a\setminus\{i\}}$ are the restriction of t_i, and the functions h are all given by commutation in the group G. It is convenient to let σG denote the group

$$\sigma G = \mathcal{E}G_{\{1,\ldots,n\}} = \bigcap_{1\leqslant i\leqslant n} \mathrm{Ker}(s_i).$$

The inverse equivalence $\mathcal{E}^{-1}: \mathcal{X}^n\mathcal{G} \to \mathcal{C}^n\mathcal{G}$ is described in [4]. For a crossed n-cube M we set $\mathcal{B}M = \mathcal{B}(\mathcal{E}^{-1}M)$.

The equivalence $\mathcal{E}: \mathcal{CG} \to \mathcal{XG}$ induces an equivalence $\mathcal{E}: \mathcal{SCG} \to \mathcal{SXG}$ such that the diagram

$$\begin{array}{ccc} \mathcal{CG} & \xrightarrow{\text{inclusion}} & \mathcal{SCG} \\ \downarrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{E}} \\ \mathcal{XG} & \xrightarrow{\text{inclusion}} & \mathcal{SXG} \end{array}$$

commutes.

We need the following easily verified description of the crossed n-cube $\mathcal{E}(G_{\mathrm{ab}})$ associated to the abelianisation of a cat^n-group G.

Lemma 10. *Let G be a* cat^n*-group with associated crossed n-cube $\mathcal{E}G = (M_\alpha)$. Then the crossed n-cube associated to G_{ab} has the form $\mathcal{E}(G_{\mathrm{ab}}) = (\overline{M}_\alpha)$ where*

$$\overline{M}_\alpha = M_\alpha \Big/ \prod_{\beta\cup\gamma=\alpha,\ \beta\cap\gamma=\emptyset} [M_\beta, M_\gamma],$$

are commutator subgroups being defined via commutation in the underlying group of G.

The comparison of $H_*(G)_{\mathrm{Top}}$ with $D_*(G)$ is facilitated by setting

$$H_i(G)_{\mathrm{Quillen}} = \sigma D_{i-1}(G), \quad i \geqslant 1, \qquad H_0(G)_{\mathrm{Quillen}} = \mathbf{Z}.$$

We also denote by $\overline{H}_i(G)_{\mathrm{Quillen}}$ the corresponding reduced groups. Thus

$$\overline{H}_0(G)_{\mathrm{Quillen}} = 0 \quad \text{and} \quad \overline{H}_i(G)_{\mathrm{Quillen}} = H_i(G)_{\mathrm{Quillen}} \quad \text{for } i > 0.$$

Then both $H_*(G)_{\mathrm{Top}}$ and $H_*(G)_{\mathrm{Quillen}}$ are functors $\mathcal{C}^n\mathcal{G} \to \mathsf{Ab}$ to the category of abelian groups. When $n = 0$ we have functors $H_*(-)_{\mathrm{Top}}, H_*(-)_{\mathrm{Quillen}} : \mathcal{G} \to \mathsf{Ab}$ and it is well known that

$$H_*(G)_{\mathrm{Top}} \cong H_*(G)_{\mathrm{Quillen}}$$

in this case. We denote both of these homology functors by $H_*(G)$.

Let us now consider $n = 1$. A cat^1-group G is equivalent to a crossed module $\lambda_1 : \mathcal{E}G_{\{1\}} \to \mathcal{E}G_\emptyset$ which for simplicity we denote by $\lambda : M \to P$. To the group P we can associate the crossed module $0 \to P$. The inclusion morphism of crossed modules $(0 \xrightarrow{0} P) \longrightarrow (M \xrightarrow{\lambda} P)$ induces a map of simplicial sets

$$f_G : \mathcal{B}\begin{pmatrix} 0 \\ \downarrow \\ P \end{pmatrix} \to \mathcal{B}\begin{pmatrix} M \\ \downarrow\lambda \\ P \end{pmatrix}.$$

We denote by $\mathrm{Cof}(G)$ the homotopy cofibre of f_G. The following theorem, modulo some notation, was proved in [1]. (A more general version of the result for homology and cohomology with arbitrary coefficient module is contained in [14].)

Theorem 11. *For any* cat^1*-group G there is an isomorphism*

$$\overline{H}_i(G)_{\mathrm{Quillen}} \cong H_{i+1}\big(|\mathrm{Cof}(G)|\big) \quad (i \geqslant 0)$$

and consequently an exact sequence

$$\cdots \to H_{i+1}(P) \to H_{i+1}(G)_{\mathrm{Top}} \to H_i(G)_{\mathrm{Quillen}} \to H_i(P) \to \cdots \quad (i \geqslant 1).$$

We wish to explain how this result generalises to cat^n-groups, $n \geqslant 1$. To pave the way we recall the proof for the case $n = 1$.

Proof. Let $Q \to G$ be a cofibrant simplicial resolution of G, that is a fibration in $\mathcal{S}(\mathcal{CG})$ which is also a weak equivalence and where Q is cofibrant. Then

$\mathcal{B}Q \to \mathcal{B}G$ is a weak equivalence in $\mathcal{SS}$. Moreover, it is readily checked that $\mathcal{B}(\mathcal{E}Q_\emptyset) \to \mathcal{B}(\mathcal{E}G_\emptyset)$ is also a weak equivalence. The map f_Q and cofibre $\mathrm{Cof}(Q)$ are defined analogously to f_G and $\mathrm{Cof}(G)$. The homology exact sequences associated to the cofibrations f_G, f_Q show that $\mathrm{Cof}(Q) \to \mathrm{Cof}(G)$ induces an isomorphism in homology. Since $\mathrm{Cof}(Q)$ and $\mathrm{Cof}(G)$ are both 1-connected it follows that $\mathrm{Cof}(Q) \to \mathrm{Cof}(G)$ is a weak equivalence.

The simplicial set $\mathrm{Cof}(Q)$ is obtained as the diagonal of a bisimplicial set X with $X_{*p} = \mathrm{Cof}(Q_p)$, where Q_p is the pth component of Q. The homology spectral sequence for the bisimplicial set X has the form

$$E^1_{pq} = H_q\big(\mathrm{Cof}(Q_p)\big) \Rightarrow H_{p+q}\big(\mathrm{Cof}(G)\big).$$

Now $\mathrm{Cof}(Q_p)$ is the cofibre of the map $\mathcal{B}(P_p) \longrightarrow \mathcal{B}(M_p \to P_p)$ where $M_p \to P_p$ is the crossed module equivalent to Q_p. Since Q_p is a projective cat^1-group it follows that $M_p \to P_p$ is a projective crossed module. It is readily seen that P_p must be a free group. Part (iv) of Lemma 8 implies that both classifying spaces here have free fundamental group and trivial higher homotopy groups. So $\mathrm{Cof}(Q_p)$ is simply connected and the homology exact sequence of a cofibration implies that $H_i(\mathrm{Cof}(Q_p)) = 0$ for $i > 2$ and

$$H_2\big(\mathrm{Cof}(Q_p)\big) \cong \mathrm{Ker}\big((P_p)_{\mathrm{ab}} \to (P_p/M_p)_{\mathrm{ab}}\big).$$

Lemma 8 implies that P_p/M_p is free. Hence $H_2(P_p/M_p) = 0$ and

$$0 \to M_p \to P_p \to P_p/M_p \to 0$$

is a split short exact sequence. It follows that

$$\mathrm{Ker}\big((P_p)_{\mathrm{ab}} \to (P_p/M_p)_{\mathrm{ab}}\big) \cong M_p/[M_p, P_p].$$

Thus

$$H_2\big(\mathrm{Cof}(Q_p)\big) \cong M_p/[M_p, P_p].$$

Hence

$$E^1_{pq} = 0 \quad \text{if } q \neq 0 \text{ or } 2, \qquad E^1_{p0} = \mathbf{Z}, \quad \text{and} \quad E^1_{p2} = M_p/[M_p, P_p].$$

Thus E^1_{p0} is a constant simplicial abelian group. Hence $E^2_{p0} = 0$ for $p > 0$. Therefore the spectral sequence degenerates and gives the isomorphism

$$H_{i+2}\big(\mathrm{Cof}(G)\big) \cong \pi_i\big(M_*/[M_*, P_*]\big), \quad i \geqslant 0.$$

Lemma 10 implies

$$\frac{M_*}{[M_*, P_*]} \cong \sigma\left(\frac{Q_*}{[Q_*, Q_*]}\right)$$

and so

$$\pi_i\big(M_*/[M_*, P_*]\big) \cong H_{i+1}(G)_{\mathrm{Quillen}}. \qquad \square$$

Corollary 12. *Let $M \to P$ denote the crossed module associated to the* cat^1-*group G. If P is a free group then there are natural isomorphisms*

$$H_{i+1}(G)_{\mathrm{Top}} \cong H_i(G)_{\mathrm{Quillen}} \quad (i \geqslant 2),$$
$$H_2(G)_{\mathrm{Top}} \cong \mathrm{Ker}\big(M/[M,P] \to P/[P,P]\big).$$

The description of $H_2(G)_{\mathrm{Top}}$ given in the corollary can be viewed as a generalization of Hopf's formula for the second integral homology of a group K. To see this, note that if $\pi_1 G = K$, $\pi_2 G = 0$ in the corollary, then M is a normal subgroup of the free group P with $K \cong P/M$, and $H_2(G)_{\mathrm{Top}} \cong H_2(K, \mathbf{Z})$. We thus recover Hopf's formula $H_2(K, \mathbf{Z}) \cong M \cap [P,P]/[M,P]$.

Consider now $n = 2$. An arbitrary cat^2-group G is equivalent to a crossed square $\mathcal{E}G$ which, for simplicity, we denote by

$$\begin{array}{ccc} L & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{array}.$$

By applying the classifying functor $\mathcal{B}: \mathcal{X}^2\mathcal{G} \to \mathcal{SS}$ to a diagram of crossed squares we obtain the following diagram of simplicial sets:

$$\begin{array}{ccc} \mathcal{B}\begin{pmatrix} 0 \to 0 \\ \downarrow \quad \downarrow \\ 0 \to P \end{pmatrix} & \xrightarrow{f_G^1} & \mathcal{B}\begin{pmatrix} 0 \to N \\ \downarrow \quad \downarrow \\ 0 \to P \end{pmatrix} \\ \Big\downarrow g_G^1 & & \Big\downarrow g_G^2 \\ \mathcal{B}\begin{pmatrix} 0 \to 0 \\ \downarrow \quad \downarrow \\ M \to P \end{pmatrix} & \xrightarrow{f_G^2} & \mathcal{B}\begin{pmatrix} L \to N \\ \downarrow \quad \downarrow \\ M \to P \end{pmatrix} \end{array}.$$

There is a natural map

$$g_G : \mathrm{cofibre}\big(f_G^1\big) \to \mathrm{cofibre}\big(f_G^2\big)$$

from the homotopy cofibre of f_G^1 to the homotopy cofibre of f_G^2. We denote by $\mathrm{Cof}(G)$ the cofibre of this map g_G.

Lemma 13. *Let G be a* cat^2-*group equivalent to the crossed square*

$$\begin{array}{ccc} L & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{array}.$$

If G is a projective object in the category $\mathcal{C}^2\mathcal{G}$, then $|\mathrm{Cof}(G)|$ is homotopy equivalent to a wedge of 3-*spheres and*

$$H_3\big(|\mathrm{Cof}(G)|\big) \cong \frac{L}{[M,N][L,P]}.$$

Proof. By Lemma 9 both $N \to P$ and $M \to P$ are projective objects in the category of crossed modules and hence are injections. By [1, Proposition 1] $\mathrm{Cof}(f_G^1)$ is a wedge of 2-spheres and

$$H_2\bigl(\bigl|\mathrm{Cof}\bigl(f_G^1\bigr)\bigr|\bigr) \cong \frac{N}{[P,N]}.$$

The map f_G^2 yields the following epimorphism of free groups after applying the functor π_1

$$P/M \to P/MN.$$

Since $|\mathrm{Cof}(f_G^2)|$ is connected it follows that $|\mathrm{Cof}(f_G^2)|$ is 1-connected. On the other hand, both spaces $B(M \to P)$ and $B(G)$ are homotopy equivalent to wedges of 1-spheres thanks to Lemmas 8 and 9. Thus it follows from the homology exact sequence that $|\mathrm{Cof}(f_G^2)|$ is homotopy equivalent to the wedge of 2-spheres and the sequence

$$0 \to H_2\bigl(\bigl|\mathrm{Cof}\bigl(f_G^2\bigr)\bigr|\bigr) \to (P/M)_{\mathrm{ab}} \to (P/MN)_{\mathrm{ab}} \to 0$$

is exact. Since G is projective, we have $L = M \cap N$ because $\pi_2(\mathcal{N}(G)) = 0$. Thus

$$H_2\bigl(\bigl|\mathrm{Cof}\bigl(f_G^2\bigr)\bigr|\bigr) \cong \frac{N}{[P,N] \cap M}.$$

The map $\mathrm{Cof}(f_G^1) \to \mathrm{Cof}(f_G^2)$ yields the following epimorphism of groups by applying the functor H_2:

$$\frac{N}{[P,N]} \to \frac{N}{[P,N] \cap M}.$$

Hence the homology exact sequence shows that $|\mathrm{Cof}(G)|$ is a wedge of 3-spheres and that $H_3(\mathrm{Cof}(Q)) = L/[N,P] \cap M$. Since P/N is a free group the Hopf formula for $H_2(P/N)$ implies that $[N,P] = N \cap [P,P]$ and hence that $H_3(\mathrm{Cof}(Q)) = L/L \cap [P,P]$. The Hopf type formula for the third integral homology of a group [5] states that

$$H_3(P/MN) \cong \frac{L \cap [P,P]}{[M,N][L,P]}.$$

Since P/MN is a free group it follows that

$$H_3\bigl(\mathrm{Cof}(Q)\bigr) \cong \frac{L}{[M,N][L,P]}. \qquad \square$$

The following theorem is the main result.

Theorem 14. *For any* cat^2*-group G there is an isomorphism*

$$\overline{H}_i(G)_{\mathrm{Quillen}} \cong H_{i+2}\bigl(|\mathrm{Cof}(G)|\bigr), \quad (i \geqslant 0).$$

Proof. Let $Q \to G$ be a cofibrant simplicial resolution of G. The cofibre $\mathrm{Cof}(Q)$ is defined analogously to $\mathrm{Cof}(G)$. It is readily checked that there are weak equivalences

$$\begin{aligned} \mathcal{B}(Q) &\to \mathcal{B}(G), \\ \mathcal{B}(\mathcal{E}Q_{\{1\}} \to \mathcal{E}Q_{\emptyset}) &\to \mathcal{B}(\mathcal{E}G_{\{1\}} \to \mathcal{E}G_{\emptyset}), \\ \mathcal{B}(\mathcal{E}Q_{\{2\}} \to \mathcal{E}Q_{\emptyset}) &\to \mathcal{B}(\mathcal{E}G_{\{2\}} \to \mathcal{E}G_{\emptyset}), \\ \mathcal{B}(\mathcal{B}Q_{\emptyset}) &\to \mathcal{B}(\mathcal{G}_{\emptyset}). \end{aligned}$$

The homology exact sequences associated to the cofibrations

$$\begin{aligned} \mathcal{B}(\mathcal{E}G_{\{1\}} \to \mathcal{E}G_{\emptyset}) &\to \mathcal{B}(G) \to \mathrm{cofibre}\big(f_G^2\big), \\ \mathcal{B}(\mathcal{E}Q_{\{1\}} \to \mathcal{E}Q_{\emptyset}) &\to \mathcal{B}(Q) \to \mathrm{cofibre}\big(f_Q^2\big) \end{aligned}$$

show that the map $\mathrm{cofibre}(f_Q^2) \to \mathrm{cofibre}(f_G^2)$ is a homology equivalence and hence a weak equivalence. Similarly the map $\mathrm{cofibre}(f_Q^1) \to \mathrm{cofibre}(f_G^1)$ is a weak equivalence. Hence there is a weak equivalence

$$\mathrm{Cof}(Q) \xrightarrow{\simeq} \mathrm{Cof}(G).$$

The simplicial set $\mathrm{Cof}(Q)$ is obtained as the diagonal of a bisimplicial set X with $X_{*p} = \mathrm{Cof}(Q_p)$, where Q_p is a projective cat^2-group. The homology spectral sequence for the bisimplicial set X has the form $E^1_{pq} = H_q(\mathrm{Cof}(Q_p)) \Rightarrow H_{p+q}(\mathrm{Cof}(G))$.

Now Q_p is equivalent to a projective crossed square

$$\begin{array}{ccc} L_p & \longrightarrow & N_p \\ \downarrow & & \downarrow \\ M_p & \longrightarrow & P_p \end{array}.$$

According to Lemma 13 we have

$$E^1_{pq} = 0 \quad \text{if } q \neq 0 \text{ or } 3, \qquad E^1_{p0} = \mathbf{Z}, \quad \text{and} \quad E^1_{p3} = \frac{L_p}{[M_p, N_p][L_p, P_p]}.$$

So $E^2_{p0} = 0$ for $p > 0$ and the spectral sequence yields the isomorphism

$$H_{i+3}\big(\mathrm{Cof}(G)\big) \cong \pi_i\left(\frac{L_p}{[M_p, N_p][L_p, P_p]}\right), \quad i \geqslant 0.$$

Lemma 10 implies

$$\frac{L_p}{[M_p, N_p][L_p, P_p]} \cong \sigma\left(\frac{Q_*}{[Q_*, Q_*]}\right);$$

and so

$$\pi_i\left(\frac{L_p}{[M_p, N_p][L_p, P_p]}\right) \cong H_{i+1}(G)_{\mathrm{Quillen}}. \qquad \square$$

Corollary 15. *In the crossed square associated to a* cat^2*-group G suppose that the group P is free and the crossed modules $M \to P$, $N \to P$ are projective in $\mathcal{X}G$. Then*

$$H_{i+2}(G)_{\mathrm{Top}} \cong H_i(G)_{\mathrm{Quillen}} \quad (i \geqslant 2),$$
$$H_3(G)_{\mathrm{Top}} \cong \mathrm{Ker}\left(\frac{L}{[M,N][L,P]} \to \frac{P}{[P,P]}\right).$$

Proof. The isomorphism follows from the homology exact sequences arising from the various cofibration sequences involved in the construction of $\mathrm{Cof}(G)$. □

The description of $H_3(G)_{\mathrm{Top}}$ in the corollary can be viewed as a generalization of the Hopf-type formula for the third integral homology of a group given in [5]. Interestingly, the formula in [5] plays a key role in the proof of this generalisation.

We leave as an exercise for the reader the formulation and proof of Theorem 14 and Corollary 15 for the case $n \geqslant 3$.

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