Stable K-theory is bifunctor homology (after A. Scorichenko)

Vincent FRANJOU*and Teimuraz PIRASHVILI†

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V.F.: Laboratoire Jean Leray, Université de Nantes
2, rue de la Houssinière
BP 92208,
44322 Nantes cedex 3, France
franjou@math.univ-nantes.fr

T.P.: A. M. Razmadze Mathematical Institute
Aleksidze str. 1,
Tbilisi 380093, Republic of Georgia
pira@rmi.acnet.ge

*membre de l'UMR 6629 Université de Nantes/CNRS

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Contents

1 Introduction

The purpose of this chapter is to present A. Scorichenko's work for his dissertation at Northwestern.

Theorem 1.1 [19] *For a ring R, let $\mathbb{P}(R)$ be the category of finitely generated projective left R-modules, and let $D : \mathbb{P}(R)^{op} \times \mathbb{P}(R) \to \mathrm{Ab}$ be a bifunctor. If D has finite degree with respect to both variables, then there is an isomorphism between Waldhausen's stable K-theory and the homology of $\mathbb{P}$:*

$$K_*^{st}(R, D) \to \mathrm{H}_*(\mathbb{P}(R), D) \ .$$

This proves a conjecture stated in [4]. The conjecture first appeared in [14] for biadditive bifunctors, a case proved in [6] (see also [20] for the outline of another approach). In the case of finite fields, the conjecture was proved for general bifunctor coefficients in [3] and in [9, Appendix].

Stable K-theory is precisely related to homology of invertible matrices: Waldhausen explained [21, Section 6] that stable K-theory gives access to homology of the general linear group, with twisted coefficients, through the spectral sequence discussed in sections 2 and 6. One point of the theorem is that although stable K-theory is defined in terms of invertible matrices, it is equal to a more manageable theory, expressed in terms of all matrices. The isomorphism of Theorem 1.1 is induced by the inclusion of invertible matrices in all matrices. There are variations on this, as will be seen with Scorichenko's use of the category of epimorphisms.

The conjecture has been a motivation for developing computation tools in categories of functors. Indeed, the homology $\mathrm{H}_*(\mathbb{P}(R), D)$ can be expressed purely in terms of homological algebra in categories of functors.

2 Homology of general linear groups and stable K-theory

Let R be a ring and $\mathrm{GL}_n(R)$ be the group of invertible matrices over R. For a bimodule P over R, the R-bimodule of $n \times n$-matrices $\mathrm{gl}_n(P)$ is a $\mathrm{GL}_n(R)$-module for the conjugation action: $X * M := X^{-1}MX$. We embed $\mathrm{GL}_n(R)$ as a subgroup in $\mathrm{GL}_{n+1}(R)$ by: $X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$, and define the direct limit $\mathrm{GL}(R) = \bigcup_n \mathrm{GL}_n(R)$. We embed $\mathrm{gl}_n(P)$ in $\mathrm{gl}_{n+1}(P)$ by: $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, and define the direct limit $\mathrm{gl}(P) = \bigcup_n \mathrm{gl}_n(P)$. This yields the conjugation action of $\mathrm{GL}(R)$ on $\mathrm{gl}(P)$.

The homology groups with twisted coefficients $\mathrm{H}_*(\mathrm{GL}(R); \mathrm{gl}(P))$ appear as the E^2_{n1}-terms of the following change of rings spectral sequence. Let:

$$0 \to P \to S \to R \to 0$$

be a singular extension of rings. Thus S is a ring and P is a two-sided ideal of S such that $P^2 = 0$ and $R = S/P$. Then there is a short exact sequence of groups

$$0 \to \mathrm{gl}(P) \to \mathrm{GL}(S) \to \mathrm{GL}(R) \to 1 \ ,$$

where the inclusion is by the exponential map $x \mapsto 1 + x$. It yields a Hochschild-Serre spectral sequence

$$E^2_{pq} = \mathrm{H}_p(\mathrm{GL}(R), \mathrm{H}_q(\mathrm{gl}(P))) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{GL}(S)) \ .$$

Since $\mathrm{gl}(P)$ is an abelian group, its homology $\mathrm{H}_*(\mathrm{gl}(P))$ is known [18, Section 8]. Here is a way to put these groups in a more general framework.

Let $\mathbb{P}(R)$, or simply $\mathbb{P}$, be the category of finitely generated projective left R-modules. The category $\mathbb{P}$ is equivalent to a small category and therefore we can do homological algebra in $\mathrm{Func}(\mathbb{P}, \mathrm{Ab})$. For a bifunctor $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ the abelian group $D(R^n, R^n)$ has a natural $\mathrm{GL}_n(R)$-module structure, with action on both variable. Define $p_n : R^{n+1} \to R^n$ and $i_n : R^n \to R^{n+1}$ by: $p_n(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_n)$ and $i_n(x_1 \cdots, x_n) = (x_1, \cdots, x_n, 0)$. They yield an homomorphism

$$D(p_n, i_n) : D(R^n, R^n) \to D(R^{n+1}, R^{n+1})$$

which is compatible with the inclusions $\mathrm{GL}_n(R) \subset \mathrm{GL}_{n+1}(R)$. At the limit, one gets a $\mathrm{GL}(R)$-module $D_\infty := \mathrm{colim}_n D(R^n, R^n)$. For example, when $D(X, Y) = \mathrm{Hom}_R(X, P \otimes_R Y)$ for a given bimodule P, then $D_\infty = \mathrm{gl}(P)$. Considering the bifunctor defined by $D(X, Y) = \mathrm{H}_q(\mathrm{Hom}_R(X, P \otimes_R Y))$ recovers $D_\infty = \mathrm{H}_q(\mathrm{gl}(P))$.

Therefore we are left with the general problem of understanding the groups $\mathrm{H}_*(\mathrm{GL}(R), D_\infty)$. This is achieved by comparing it with an appropriate notion of homology of a small category for the category $\mathbb{P}$ (see Section 3.5). The group $\mathrm{GL}_n(R)$ appears as the subcategory of $\mathbb{P}$ consisting of the automorphisms of R^n, and this inclusion induces an homomorphism

$$\psi_* : \mathrm{H}_*(\mathrm{GL}(R), D_\infty) \to \mathrm{H}_*(\mathbb{P}(R), D) \ .$$

The homology of the right hand side is well understood in many cases (see [7, 8, 9] or the article *Introduction to functor homology* in this volume). For example, when $D(X, Y) = \mathrm{Hom}_R(X, P \otimes_R Y)$, $\mathrm{H}_*(\mathbb{P}(R), D)$ is canonically isomorphic to the topological Hochschild homology [16] and to the MacLane homology [11] of R with coefficients in P.

Unfortunately the homomorphism ψ_* is very far from being an isomorphism in general. Indeed, if D is a constant bifunctor, then $\mathrm{H}_*(\mathbb{P}, D)$ vanishes in positive dimensions, because $\mathbb{P}$ has a zero object, while the homology of the general linear group is highly nontrivial in general. There is a trick due to Waldhausen [21, pp 387–388], which simplifies the situation. Define the stable K-theory $K^{st}_*(R, D)$ of R with coefficients in D as the homology of the homotopy fiber of

$B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$, with twisted coefficients in D_∞. In the resulting Serre spectral sequence

$$E^2_{pq} = \mathrm{H}_*(\mathrm{GL}(R), K^{st}_*(R, D)) \Longrightarrow \mathrm{H}_*(\mathrm{GL}(R), D_\infty) \qquad (*)$$

the action of $\mathrm{GL}(R)$ on $K^{st}_*(R, D)$ is **trivial** (see [12]). The spectral sequence $(*)$ degenerates at E^2 in many cases (see [4], or Section 6 in this paper). Moreover there is a natural transformation

$$\nu_* : K^{st}_*(R, D) \to \mathrm{H}_*(\mathbb{P}(R), D)$$

because $\mathrm{H}_*(\mathbb{P}, -)$ is a universal sequence of functors defined on $\mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab})$ (see Lemma 3.1).

Scorichenko's theorem 1.1 states that ν_* is an isomorphism, if D has finite degree with respect to both variables. For the definition of functors of finite degree we refer the reader to Section 4. Symmetric, exterior or divided powers all have finite degree, as does indeed the bifunctor defined by $D(X, Y) = \mathrm{H}_q(\mathrm{Hom}_R(X, P \otimes_R Y))$, which is relevant to the above change of rings spectral sequence.

3 Preliminaries from homological algebra

3.1 Universal sequences of functors

We assume the reader to be familiar with the basics of homological algebra and category theory, as in [5]. We recall the following axiomatic characterization of derived functors, to be used several times in this paper. Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. A *connected sequence of functors* is a sequence of additive functors $(T_n : \mathbf{A} \to \mathbf{B})_{n \geq 0}$ together with homomorphisms

$$\partial_n : T_{n+1}(C) \to T_n(A)$$

for each exact sequence in $\mathbf{A}$

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{s} C \longrightarrow 0$$

which are natural in respect of maps of short exact sequences. A connected sequence is *exact* if for each exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{s} C \longrightarrow 0$ in $\mathbf{A}$, the long sequence in $\mathbf{B}$

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\partial} T_n(A) \xrightarrow{i_*} T_n(B) \xrightarrow{s_*} T_n(C) \longrightarrow \cdots \longrightarrow T_0(C) \longrightarrow 0$$

is exact. Assume $\mathbf{A}$ has enough projective objects. A *universal sequence of functors* is an exact connected sequence of functors such that $T_n(P) = 0$ for all positive n and all projective P. The following is a particular case of [5, Proposition III.5.2].

Proposition 3.1 *Let $T : \mathbf{A} \to \mathbf{B}$ be an additive covariant functor. Its left derived functors $(L_nT : \mathbf{A} \to \mathbf{B})_{n\geq 0}$ form a universal sequence of functors. Conversely, if $(T_n : \mathbf{A} \to \mathbf{B})_{n\geq 0}$ is an exact connected sequence of functors, then there is a unique morphism of connected sequence of functors $(\xi_n : T_n \to L_n(T_0))_{n\geq 0}$ such that $\xi_0 : T_0 \to L_0T_0$ is the canonical isomorphism. Furthermore ξ_n is an isomorphism for all $n \geq 0$ provided $(T_n : \mathbf{A} \to \mathbf{B})_{n\geq 0}$ is a universal sequence of functors.*

3.2 A lemma on collapsing spectral sequences

We now extend these notions to spectral sequences of functors. A *∂-spectral sequence* is for each $A \in \mathbf{A}$ a upper-half-plane spectral sequence $(E^r_{pq}(A), d^r)_{r\geq 2}$ in $\mathbf{B}$, which is natural in $A \in \mathbf{A}$, together with homomorphisms

$$\partial_r : E^r_{pq}(C) \to E^r_{p,q-1}(A)$$

for each short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{s} C \longrightarrow 0$ in $\mathbf{A}$, which are natural in respect of maps of short exact sequences, and such that:

1. for each $r \geq 2$, ∂_{r+1} is the map induced in homology by ∂_r
2. the diagrams

$$\begin{array}{ccc} E^r_{pq}(C) & \xrightarrow{d^r} & E^r_{p-r,q+r-1}(C) \\ \partial\downarrow & & \downarrow\partial \\ E^r_{p,q-1}(A) & \xrightarrow{d^r} & E^r_{p-r,q+r-2}(A) \end{array}$$

   commute for all integers p, q, and $r \geq 2$.

Lemma 3.2 *Let $\mathbf{A}$ be an abelian category and let $(E^r_{pq})_{r\geq 2}$ be a ∂-spectral sequence. Assume that the following condition holds: For any C in $\mathbf{A}$, there is a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{A}$ such that the maps $\partial^2 : E^2_{pq}(C) \to E^2_{p,q-1}(A)$ are monomorphisms. Then the spectral sequence $(E^r_{pq}(C), d^r)_{r\geq 2}$ stops at E^2 for any C in $\mathbf{A}$.*

Proof: We need to show that $d^r = 0$ for each r. Let C be in $\mathbf{A}$, and let $0 \to A \to B \to C \to 0$ be a short exact sequence as in the statement. Starting at the E^2-level, let us consider the commutative diagram:

$$\begin{array}{ccc} E^2_{pq}(C) & \xrightarrow{d^2} & E^2_{p-2,q+1}(C) \\ \partial\downarrow & & \downarrow\partial \\ E^2_{p,q-1}(A) & \xrightarrow{d^2} & E^2_{p-2,q}(A) \end{array}$$

By hypothesis, the right vertical map is mono. When $q = 0$, the left bottom term is 0: hence $d^2_{p0} = 0$. We then proceed by induction on q, applying the induction hypothesis to A to show that the bottom map is 0.

At the next stage, we have $E^3 = E^2$ and $\partial_3 = \partial_2$, by the first condition of a ∂-spectral sequence. Hence the conditions on the E^2-term carry over to the E^3-term, and we repeat the argument *ad lib.*

3.3 Categories of functors

For a small category $\mathcal{C}$ and a category $\mathbf{A}$ we let $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ be the category of all functors from $\mathcal{C}$ to $\mathbf{A}$ and natural transformations between them. The category $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ carries lots of the properties of $\mathbf{A}$. It has limits (resp. colimits) provided $\mathbf{A}$ has limits (resp. colimits). The limits and colimits in $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ are computed pointwise. In particular, if $\mathbf{A}$ is an abelian category, then $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ is also an abelian category: A sequence

$$0 \to F \to G \to H \to 0$$

is an exact sequence in $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ if

$$0 \to F(X) \to G(X) \to H(X) \to 0$$

is exact for all $X \in \mathcal{C}$.

We are especially interested in the case when $\mathbf{A}$ is the category R-Mod of left modules over a ring R. We restrict to this case for the rest of the section. To describe projective generators in the category $\mathrm{Func}(\mathcal{C}, \mathbf{A})$, we recall the Yoneda lemma.

Lemma 3.3 [13] *Let X be an object in $\mathcal{C}$. For any functor*

$$T : \mathcal{C} \to \mathrm{Sets}$$

to the category of sets, there is a natural (in X) bijection

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C},\mathrm{Sets})}(\mathrm{Hom}_{\mathcal{C}}(X, -), T) \;\cong\; T(X) \;,$$

which assigns $\xi_X(1_X) \in T(X)$ to a natural transformation $\xi : \mathrm{Hom}_{\mathcal{C}}(X, -) \to T$. Its inverse associates to each a in $T(X)$ the natural transformation $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to T(Y)$ given by evaluation $f \mapsto T(f)(a)$.

For any $X \in \mathcal{C}$, let us define $P_X \in \mathrm{Func}(\mathcal{C}, \mathbf{A})$ by

$$P_X(Y) := R[\mathrm{Hom}(X, Y)] = \bigoplus_{f : X \to Y} R.$$

Here and elsewhere $R[S]$ denotes the free left R-module generated by a set S (it is a covariant functor of S). Sometimes, to emphasize the category $\mathcal{C}$ we write $P^{\mathcal{C}}_X$ instead of P_X.

Corollary 3.4 *Let* $\mathbf{A}$ *be the category of left* R*-modules.*

i) *For any* $X \in \mathcal{C}$ *and any functor* $F : \mathcal{C} \to \mathbf{A}$ *there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C},\mathbf{A})}(P_X, F) \;\cong\; F(X).$$

ii) *For any* $X \in \mathcal{C}$*, the functor* P_X *is a projective object in* $\mathrm{Func}(\mathcal{C}, \mathbf{A})$*.*

iii) *Any projective object in* $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ *is a direct summand in a coproduct of objects* P_X*.*

iv) *For any object* $F \in \mathrm{Func}(\mathcal{C}, \mathbf{A})$ *there is an epimorphism* $P \to F$ *with projective* P*.*

Proof. The first statement is an immediate consequence of the Yoneda lemma. The functor $\mathrm{Hom}_{\mathrm{Func}(\mathcal{C},\mathbf{A})}(P_X, -)$ is an exact functor, thanks to i) and we obtain ii). Take any functor $F : \mathcal{C} \to \mathbf{A}$ and an element $x \in F(X)$. Thanks to i) we have a morphism $\xi_x : P_X \to F$ such that $(\xi_x)_X(1_X) = x$. The collection of all (ξ_x), $x \in F(X)$, where X runs over the isomorphism classes of objects of the category $\mathcal{C}$, yields a homomorphism

$$\xi = (\xi_x) : \bigoplus_X \bigoplus_{x \in F(X)} P_X \to F$$

which is clearly an epimorphism. This implies iii) and iv).

□

3.4 Tor **in functor categories**

We now discuss Tor groups in categories of functors. Assume $M : \mathcal{C} \to R$-Mod and $N : \mathcal{C}^{op} \to$ Mod-R are functors to the category of left and right R-modules respectively. We let $N \otimes_{\mathcal{C}} M$ be the abelian group generated by all symbols $x \otimes y$, where $x \in N(A)$, $y \in M(A)$ and $A \in \mathcal{C}$, subjected to the following relations:

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

$$(xr) \otimes y = x \otimes (ry), \quad \alpha^*(x') \otimes y = x' \otimes \alpha_*(y)\ ,$$

for $\alpha\colon A \to B$ a morphism in $\mathcal{C}$, x_1, x_2, x in $N(A)$, y_1, y_2, y in $M(B)$, x' in $N(B)$ and r in R. In other words: $N \otimes_{\mathcal{C}} M$ is the quotient of $\bigoplus_{A \in \mathcal{C}} N(A) \otimes_R M(A)$ by the relations $\alpha^*(x') \otimes y = x' \otimes \alpha_*(y)$. The bifunctor $- \otimes_{\mathcal{C}} -$ is right exact with respect to each variable and preserves direct sums.

Example 3.5 *If* R *denotes the constant functor with value* R*, then* $R \otimes_{\mathcal{C}} M$ *is isomorphic to the colimit of* $M : \mathcal{C} \to R$-Mod*.*

Lemma 3.6 *For any functors* $M : \mathcal{C} \to R$-Mod*,* $N : \mathcal{C} \to$ Mod-R *and any* $A \in \mathcal{C}$*, there exist natural isomorphisms*

$$N \otimes_{\mathcal{C}} P_A^{\mathcal{C}} \cong N(A)$$

$$P_A^{\mathcal{C}^{op}} \otimes_{\mathcal{C}} M \cong M(A).$$

Where as usual $P_A^{\mathcal{C}} = R[\mathrm{Hom}_{\mathcal{C}}(A, -)]$ *and* $P_A^{\mathcal{C}^{op}} = R^{op}[\mathrm{Hom}_{\mathcal{C}}(-, A)]$.

Proof. We construct mutually inverse homomorphisms $f : N \otimes_{\mathcal{C}} P_A^{\mathcal{C}} \to N(A)$ and $g : N(A) \to N \otimes_{\mathcal{C}} P_A^{\mathcal{C}}$ by $f(x \otimes \alpha) = N(\alpha)(x)$ and $g(a) = a \otimes 1_A$ respectively. Here $a \in N(A)$, $x \in N(X)$ and $\alpha : A \to X$ is a morphism in $\mathcal{C}$. Similarly for the second isomorphism. □

As usual, the left derived functors of $- \otimes_{\mathcal{C}} -$ are denoted by $\mathrm{Tor}_*^{\mathcal{C}}(-, -)$.

The following lemma is similar to a change of rings in Tor-groups. Note that any functor $f : \mathcal{C} \to \mathcal{D}$ between small categories yields a functor $f^* : \mathrm{Func}(\mathcal{D}, \mathcal{E}) \to \mathrm{Func}(\mathcal{C}, \mathcal{E})$ defined by pre-composition: $f^*R = R \circ f$.

Lemma 3.7 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be small categories. Let* $l : \mathcal{C} \to \mathcal{D}$ *and* $r : \mathcal{D} \to \mathcal{C}$ *form a couple of adjoint functors. For any* $F : \mathcal{C} \to R$-Mod *and* $G : \mathcal{D}^{op} \to$ Mod-R, *there is an isomorphism*

$$\mathrm{Tor}_*^{\mathcal{D}}(G, r^*F) \cong \mathrm{Tor}_*^{\mathcal{C}}(l^*G, F) \ .$$

Proof. For $A \in \mathcal{C}$ and $B \in \mathcal{D}$ one has

$$r^* P_A^{\mathcal{C}} \cong P_{lA}^{\mathcal{D}}, \quad \text{and } l^* P_B^{\mathcal{D}^{op}} \cong P_{rB}^{\mathcal{C}^{op}}$$

Therefore l^* and r^* respect projective objects. Furthermore one has

$$l^* P_B^{\mathcal{D}^{op}} \otimes_{\mathcal{C}} P_A^{\mathcal{C}} \cong P_{rB}^{\mathcal{C}^{op}} \otimes_{\mathcal{C}} P_A^{\mathcal{C}} \cong R[\mathrm{Hom}_{\mathcal{C}}(A, rB)]$$

and

$$P_B^{\mathcal{D}^{op}} \otimes_{\mathcal{C}} r^* P_A^{\mathcal{C}} \cong P_B^{\mathcal{C}^{op}} \otimes_{\mathcal{D}} P_{lA}^{\mathcal{D}} \cong R[\mathrm{Hom}_{\mathcal{D}}(lA, B)]$$

Thus $l^*G \otimes_{\mathcal{C}} F \cong G \otimes_{\mathcal{D}} r^*F$ as abelian groups, provided both G and F are projective objects. Since $\otimes$ is right exact it follows that the isomorphism exists for any F and G. This proves the lemma in dimension 0. Since r^* and l^* are exact functors and send projective objects to projectives, the result is also true in all dimensions. □

3.5 Homology of small categories

Let $\mathcal{C}$ be a small category and let

$$D : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$$

be a bifunctor, which is contravariant with respect to the first argument and covariant with respect to the second argument. Thus, for $x \in D(X, Y)$, $f : Y \to Z$ and $g : W \to X$, one has: $f_*x \in D(X, Z)$ and $g^*x \in D(W, Y)$.

Consider diagrams in $\mathcal{C}$:

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} X_n \ .$$

As usual, let $\mathrm{N}_n(\mathcal{C})$ be the set of all such diagrams, which we denote, for short, by f or, for $n > 0$, $(f_1, \cdots, f_n)$. Define:

$$\mathrm{F}_n(\mathcal{C}, D) := \bigoplus_{f \in \mathrm{N}_n(\mathcal{C})} D(X_0, X_n) \ .$$

For $n = 0$,

$$\mathrm{F}_0(\mathcal{C}, D) = \bigoplus_{X_0 \in \mathcal{C}} D(X_0, X_0) \ .$$

A typical generator of $\mathrm{F}_n(\mathcal{C}, D)$ is denoted by $(a; f_1, \cdots, f_n)$, $a \in D(X_0, X_n)$. The boundary map

$$d : \mathrm{F}_n(\mathcal{C}, D) \to \mathrm{F}_{n-1}(\mathcal{C}, D) \ , \quad n > 0,$$

is defined by

$$d(a; f_1, \cdots, f_n) =$$

$$(f_1^* a, f_2 \cdots, f_n) + \sum_{i=1}^{n-1} (-1)^i (a; f_1, \cdots, f_i f_{i+1}, \cdots, f_n) + (-1)^n (f_{n*} a, f_1, \cdots, f_{n-1}).$$

The homology $\mathrm{H}_*(\mathcal{C}, D)$ of the category $\mathcal{C}$ with coefficients in the bifunctor D is defined as the homology of the complex $\mathrm{F}_*(\mathcal{C}, D)$. Sometimes we write $\mathrm{H}^*(\mathcal{C}, (X, Y) \mapsto D(X, Y))$ to make explicit the values of the bifunctor D. The category of bifunctors $\mathrm{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathrm{Ab})$ is a category of functors, hence it is an abelian category with enough projective and injective objects. Lemma 3.4 applied to the category $\mathcal{C}^{op} \times \mathcal{C}$ says that projective generators are given by

$$P_{A,B} = \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(-, A) \times \mathrm{Hom}_{\mathcal{C}}(B, -)], \quad A, B \in \mathcal{C}.$$

Lemma 3.8 *For any A, B in $\mathcal{C}$ one has:*

$$\mathrm{H}_n(\mathcal{C}, P_{A,B}) = 0 \ \text{ if } \ n > 0, \quad \text{and} \quad \mathrm{H}_0(\mathcal{C}, P_{A,B}) = \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(B, A)] \ .$$

Proof. Since $\mathrm{F}_0(\mathcal{C}, P_{A,B})$ is the free abelian group on the set of diagrams $A \leftarrow X \leftarrow B$, composition yields an homomorphism

$$\mathrm{F}_0(\mathcal{C}, P_{A,B}) \to \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(B, A)] \ .$$

Let $\mathrm{F}_{-1}(\mathcal{C}, P_{A,B})$ be $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(B, A)]$. For $n \geq -1$, $\mathrm{F}_n(\mathcal{C}, P_{A,B})$ is the free abelian group spanned by the diagrams

$$A \xleftarrow{f} X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} X_n \xleftarrow{g} B \ .$$

A contracting homotopy $h_n : \mathrm{F}_n(\mathcal{C}, P_{A,B}) \to \mathrm{F}_{n+1}(\mathcal{C}, P_{A,B})$, $n \geq -1$ is defined by

$$h_n(f, f_1, \cdots, f_n, g) = (Id_A, f, f_1, \cdots, f_n, g) \ .$$

□

Corollary 3.9 *The sequence of functors*

$$\big(\mathrm{H}_n(\mathcal{C},-) : \mathrm{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathrm{Ab}) \to \mathrm{Ab}\big)_{n\geq 0}$$

is universal.

Proof. Since the functor $\mathrm{F}_n(\mathcal{C},-) : \mathrm{Func}(\mathcal{C}^{op}\times\mathcal{C},\mathrm{Ab})$ is exact it follows that $\mathrm{H}_n(\mathcal{C},-)$ is an exact connected sequence of functors. It is universal thanks to Lemma 3.8. □

We now express the homology of small categories as Tor-groups. Let R be a ring and let $F : \mathcal{C}^{op} \to \text{Mod-}R$ be a contravariant functor to the category of right R-modules. Let $T : \mathcal{C} \to R\text{-Mod}$ be a covariant functor to the category of left R-modules. Then

$$(X,Y) \mapsto F(X)\otimes_R T(Y)$$

defines a bifunctor $T \boxtimes_R F : \mathcal{C}^{op}\times\mathcal{C} \to \mathrm{Ab}$.

Proposition 3.10 *Assume that the values of F or T are projective R-modules. Then, for each $i \geq 0$, there is an isomorphism*

$$\mathrm{Tor}_i^{\mathcal{C}}(F,T) \cong \mathrm{H}_i(\mathcal{C}, F\boxtimes_R T)\ .$$

Proof. Since

$$\mathrm{H}_0(\mathcal{C}, F\boxtimes_R T) = \mathrm{Coker}\Big(\bigoplus_{f:X\to Y} F(Y)\otimes_R T(X) \to \bigoplus_X F(X)\otimes_R T(X)\Big) \cong F\otimes_{\mathcal{C}} T$$

we have the expected isomorphism for all T and F when $i=0$. Assume now that the values of F are projective. Varying T we obtain an exact connected sequence of functors

$$\mathrm{H}_n(\mathcal{C}, F\boxtimes_R(-)) : \mathrm{Func}(\mathcal{C}, R-\mathrm{Mod}) \to \mathrm{Ab},\ \ n\geq 0\ .$$

Thus it suffices to show $\mathrm{H}_n(\mathcal{C}, F\boxtimes_R P_A) = 0$ for $n>0$, where as usual $P_A(Y) = R[\mathrm{Hom}_{\mathcal{C}}(A,Y)]$. Since

$$\mathrm{F}_0(\mathcal{C}, F\boxtimes_R P_A) \cong \bigoplus_{X\leftarrow A} F(X)$$

projection on the identity factor yields a map $\mathrm{F}_0(\mathcal{C}, F\boxtimes_R P_A) \to F(A)$. Let $\mathrm{F}_{-1}(\mathcal{C}, F\boxtimes_R P_A)$ be $F(A)$. For $n\geq -1$,

$$\mathrm{F}_n(\mathcal{C}, F\boxtimes_R P_A) = \bigoplus_{X_0\leftarrow\cdots\leftarrow X_n\leftarrow A} F(X_0)\ .$$

A contracting homotopy $h_n : \mathrm{F}_n(\mathcal{C}, F\boxtimes_R P_A) \to \mathrm{F}_{n+1}(\mathcal{C}, F\boxtimes_R P_A)$, $n\geq -1$ is defined by:

$$h_n(a; f_1,\cdots,f_n,g) \mapsto (a; f_1,\cdots,f_n,g,Id_A)\ .$$

□

Remark 3.11 Indeed the following more general result is true (compare with [11, Theorem B]): If F and T are arbitrary functors, then there is a spectral sequence

$$E^2_{pq} = \mathrm{H}_p(\mathcal{C}, (X,Y) \mapsto \mathrm{Tor}^R_q(FX, TY)) \Longrightarrow \mathrm{Tor}^{\mathcal{C}}_*(F,T) \ .$$

This is a consequence of Grothendieck's spectral sequence for a composite of functors.

Remark 3.12 Take $\mathcal{C}$ to be the category $\mathbb{P}(R)$ of finitely generated projective modules over a ring R. For any functor $T : \mathbb{P}(R) \to R$-Mod, the *MacLane homology* $\mathrm{HML}_*(R,T)$ *of* R *with coefficient in* T is $\mathrm{Tor}^{\mathbb{P}}_*(Id^*, T)$, where $Id^* = \mathrm{Hom}_R(-,R)$. Proposition 3.10 shows that $\mathrm{HML}_*(R,T) \cong \mathrm{H}_*(\mathbb{P}(R), D_T)$, where $D_T(X,Y) = \mathrm{Hom}_R(X, T(Y))$.

Another application of Proposition 3.10 is the following. Consider the constant functor with value R, still denoted by R. Note that

$$(R \boxtimes_R T)(X,Y) = T(Y)$$

is a bifunctor which is constant with respect to the contravariant argument. Let us denote this bifunctor again by T. Thus

$$\mathrm{H}_*(\mathcal{C}, T) \cong \mathrm{Tor}^{\mathcal{C}}_*(R, T).$$

Since $R \otimes_{\mathcal{C}} T = \mathrm{colim}\ T$, the sequence of functors $\mathrm{H}_*(\mathcal{C}, -) : \mathrm{Func}(\mathcal{C}, R\text{-Mod}) \to \mathrm{Ab}$ is isomorphic to the left derived functors of the functor

$$\mathrm{colim} : \mathrm{Func}(\mathcal{C}, \mathrm{R\text{-}Mod}) \to \mathrm{Ab}.$$

Corollary 3.13 *If the category* $\mathcal{C}$ *has a terminal object* C*, then for any functor* $T : \mathcal{C} \to R\text{-Mod}$*, the group* $\mathrm{H}_i(\mathcal{C}, T) = 0$ *for positive* i*, and* $\mathrm{H}_0(\mathcal{C}, T) = T(C)$.

Proof. It is clear that $\mathrm{colim}(T) = T(C)$. Thus colim is an exact functor and the result follows. □

Proposition 3.14 *Let* $\mathcal{C}$ *be a category, with finite coproducts and finite products. For any bifunctor* $D : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$*, let* $D_{\coprod}$ *and* $D_{\prod}$ *be the bifunctors defined on* $\mathcal{C}$ *by* $D_{\coprod}(X,Y) = D(X \coprod X, Y)$ *and* $D_{\prod}(X,Y) = D(X, Y \times Y)$*. There is an isomorphism*

$$\mathrm{H}_*(\mathcal{C}, D_{\coprod}) \cong \mathrm{H}_*(\mathcal{C}, D_{\prod}).$$

Proof. By varying $D \in \mathrm{Func}(\mathcal{C}^{op} \times \mathcal{C}, \mathrm{Ab})$ we obtain two exact connected sequence of functors $D \mapsto \mathrm{H}_n(\mathcal{C}, D_{\coprod})$, $n \geq 0$ and $D \mapsto \mathrm{H}_n(\mathcal{C}, D_{\prod})$, $n \geq 0$. It suffices to show that both of them are universal and take the same values on projective objects. Consider a projective $P_{A,B}$: $P_{A,B}(X,Y) = \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(X,A) \times \mathrm{Hom}_{\mathcal{C}}(B,Y)]$.

One finds: $(P_{A,B})_{\coprod} \cong P_{A\times A,B}$ and $(P_{A,B})_{\prod} \cong P_{A,B\coprod B}$. It follows from Lemma 3.8 that homology vanishes in positive dimensions for both functors, and it equals

$$\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(B, A\times A)] \cong \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(B\coprod B, A)]$$

in dimension 0. Lemma 3.1 finishes the proof. □

Remark. The adjonction at the end of the proof shows the similarity with Lemma 3.7.

4 Finite degree functors

4.1 Cross-effects

Let $F : \mathcal{C} \to \mathbf{A}$ be a functor from an additive category to an abelian category. For all X and Y in $\mathcal{C}$, the projections induce a natural map:

$$F(X \oplus Y) \to F(X) \oplus F(Y) \ .$$

Suppose $F(0) = 0$. This map is an epimorphism, which is naturally split by the map induced by the inclusions of X and Y in $X \oplus Y$. The *second cross-effect* of F is the bifunctor defined by

$$(\mathrm{Cr}_2 F)(X, Y) : = \mathrm{Ker}(F(X \oplus Y) \to F(X) \oplus F(Y)) \ .$$

Since the cross-effect fits in a natural splitting:

$$F(X \oplus Y) \cong F(X) \oplus F(Y) \oplus \mathrm{Cr}_2 F(X, Y) \ ,$$

the functor F is additive if and only if $\mathrm{Cr}_2 F = 0$.

In order to define the third cross-effect of F, we proceed as follows. We consider the second cross-effect $(\mathrm{Cr}_2 F)(X, Y)$. We fix Y and let X vary. In this way we obtain the functor $X \mapsto (\mathrm{Cr}_2 F)(X, Y)$ which we take the second cross-effect of. We can continue this process and define the n-th cross-effect $(\mathrm{Cr}_n F)(X_1, \cdots, X_n)$. Alternatively, $\mathrm{Cr}_n F(X_1, \cdots, X_n)$ is isomorphic to the kernel of the natural homomorphism

$$F(X_1 \oplus \cdots \oplus X_n) \to \bigoplus_{i=1}^{n} F(X_1 \oplus \cdots \oplus \hat{X}_i \cdots \oplus X_n)$$

This shows that the n-th cross-effect $(\mathrm{Cr}_n F)(X_1, \cdots, X_n)$ is symmetric on $X_1, \cdots, X_n$. Another consequence of the definition is the following natural splitting:

$$F(X_1 \oplus X_2 \oplus \cdots \oplus X_n) \cong \bigoplus_{k=1}^{n} \bigoplus_{1\le i_1 < \cdots < i_k \le n} (\mathrm{Cr}_k F)(X_{i_1}, \cdots, X_{i_k}) \qquad (\dagger)$$

One observes that an arbitrary functor $F : \mathcal{C} \to \mathbf{A}$ has a natural decomposition $F \cong F(0) \oplus F'$ with $F'(0) = 0$. This allows to define the n-th cross-effects of F to be the cross-effects of F' for any $n \ge 2$. It will be convenient to call the functor F' the *first* cross-effect of F.

4.2 Functors of finite degree

Definition 4.1 [10] *A functor* $F\colon \mathcal{C} \to \mathbf{A}$ *is of degree* n *if its* $(n+1)$*-st cross-effect functor vanishes, but its* n*-th cross-effect does not. We then write:* $\deg(F) = n$.

Example 4.2 *Assume* $\mathcal{C}$ *and* $\mathbf{A}$ *are the category of (left) modules over a commutative ring* R*. Then the functors* $X \mapsto \Lambda^n X,\ S^n X,\ X^{\otimes n}$ *are of degree* n*, while the functor* $X \mapsto R[X]$ *is not of finite degree.*

For an integer d, we let $\mathrm{Func}_d(\mathcal{C}, \mathbf{A})$ be the full subcategory of $\mathrm{Func}(\mathcal{C}, \mathbf{A})$ of functors of degree $\leq d$. By definition, $\mathrm{Func}_0(\mathcal{C}, \mathbf{A})$ consists of constant functors. The subcategory $\mathrm{Func}_d(\mathcal{C}, \mathbf{A})$ is closed in respect of coproducts and products. It is closed also in respect of subobjects, quotients and extensions. It follows that the category $\mathrm{Func}_d(\mathcal{C}, \mathbf{A})$ is an abelian category, and the inclusion $\mathrm{Func}_d(\mathcal{C}, \mathbf{A}) \subset \mathrm{Func}(\mathcal{C}, \mathbf{A})$ is an exact functor.

We now construct a left adjoint to this inclusion functor. Such a left adjoint can be seen as a Taylor expansion at order d. Consider the following "codiagonal" morphism:

$$(1_X, \cdots, 1_X) : X^{\oplus(d+1)} = X \oplus X \ldots \oplus X \to X\ .$$

For a functor $F : \mathcal{C} \to \mathbf{A}$, the codiagonal induces a natural map $F(X \oplus \cdots \oplus X) \to F(X)$, whose restriction on $(\mathrm{Cr}_{d+1}F)(X, \cdots, X)$ defines a morphism

$$\varrho_{d,X}(F) : (\mathrm{Cr}_{d+1}F)(X, \cdots, X) \to F(X).$$

Note (for use in Definition 5.2) that the above formula defines a natural transformation to F which we denote $\varrho_d(F)$ (of course also natural in F). A left adjoint functor $t_d : \mathrm{Func}(\mathcal{C}, \mathbf{A}) \to \mathrm{Func}_d(\mathcal{C}, \mathbf{A})$ is given by:

$$(t_d F)(X) = \mathrm{Coker}\,(\varrho_{D,X}(F) : (\mathrm{Cr}_{d+1}F)(X, \cdots, X) \to F(X)).$$

Similarly, a right adjoint functor $t^d : \mathrm{Func}(\mathcal{C}, \mathbf{A}) \to \mathrm{Func}_d(\mathcal{C}, \mathbf{A})$ is defined using the diagonal morphism $X \to X^{\oplus(d+1)}$. Since $(\mathrm{Cr}_{d+1})F(X, \cdots, X)$ is a direct summand of $F(X^{d+1})$, we obtain a natural transformation

$$\vartheta_{D,X}(F) : F(X) \to (\mathrm{Cr}_{d+1}F)(X, \cdots, X)$$

and a right adjoint is defined by: $(t^d F)(X) = \mathrm{Ker}(\vartheta_{D,X}(F))$. Observe that the natural transformation $F \to t_d F$ is an isomorphism if and only if $\deg(F) = d$. Similarly, $t^d F \to F$ is an isomorphism if and only if $\deg(F) = d$. Since $\mathrm{Func}(\mathcal{C}, R\text{-Mod})$ has enough projective and injective objects the same is also true for $\mathrm{Func}_d(\mathcal{C}, R\text{-Mod})$. We sum up this discussion with the following lemma.

Lemma 4.3 *The inclusion functor* $\mathrm{Func}_d(\mathcal{C}, \mathbf{A}) \subset \mathrm{Func}(\mathcal{C}, \mathbf{A})$ *has a left adjoint (and a right adjoint). The category* $\mathrm{Func}_d(\mathcal{C}, R\text{-Mod})$ *is an abelian category with enough projectives (and enough injectives).*

4.3 A cancellation lemma

For a bifunctor $D : \mathcal{C} \times \mathcal{C} \to \mathrm{Ab}$ we let $\deg_1 D$ (resp. $\deg_2 D$) be the degree with respect to the first (resp. second) variable. Similarly, $\mathrm{Cr}^1_n D$ and $\mathrm{Cr}^2_n D$ denote the n-th cross-effect functor with respect to the first and the second variable respectively. The following vanishing result is a variant of the vanishing result of the second author [15] (see for example the page ? in this volume). The lemma states that, after taking homology, cross-effects can be moved from one argument of the bifunctor coefficients to the other.

Lemma 4.4 *Let $\mathcal{C}$ be an additive category and let $D : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ be a bifunctor. There is a natural isomorphism:*

$$\mathrm{H}_*(\mathcal{C}, (X, Y) \mapsto (\mathrm{Cr}^1_n)D(X, \ldots, X; Y)) \cong \mathrm{H}_*(\mathcal{C}, (X, Y) \mapsto (\mathrm{Cr}^2_n D)(X; Y, \ldots, Y)) \;.$$

In particular, if $\deg_1 D < n$, then:

$$\mathrm{H}_*(\mathcal{C}, (X, Y) \mapsto (\mathrm{Cr}^2_n D)(X; Y, \ldots, Y)) = 0 \;.$$

Proof. By a slight generalization of Lemma 3.14 to $n + 1$ factors:

$$\mathrm{H}_*(\mathcal{C}, (X, Y) \mapsto D(X \oplus \ldots \oplus X; Y)) \cong \mathrm{H}_*(\mathcal{C}, (X, Y) \mapsto D(X; Y \oplus \ldots \oplus Y)) \;.$$

Consideration of the splitting (†) in 4.1 allows to deduce the first isomorphism. If $n > \deg_1 D$, the left hand side is zero, thus the same is true for the right hand side. □

5 Proof of Scorichenko's Theorem

Scorichenko's proof uses an intermediary category, the category $\mathbb{E}$, which still has finitely generated projective R-modules as objects, but where morphisms are epimorphisms of R-modules. Scorichenko observed that the definition of stable K-theory is still meaningful for bifunctors from $\mathbb{E}^{op} \times \mathbb{P}$. This leads to Theorem 5.1 computing stable K-theory as homology of $\mathbb{E}$, for those bifunctors having finite degree in the covariant argument. Theorem 5.4 then compares the homology of $\mathbb{E}$ and the homology of $\mathbb{P}$, this time for bifunctors having finite degree in the contravariant argument. The two comparisons together give Theorem 1.1.

5.1 Stable K-theory is homology of the category $\mathbb{E}$

The first main step is similar in spirit to the approach given in [4], where stable K-theory was obtained as the derived functor of the functor

$$K_0^{st} : \mathrm{Func}(\mathbb{P}^{op}(R) \times \mathbb{P}(R), \mathrm{Ab}) \to \mathrm{Ab}$$

under restricted conditions on the ring R (see Remark 6.3). Extending the domain of definition of stable K-theory to the category $\mathrm{Func}(\mathbb{E}^{op}\times\mathbb{P},\mathrm{Ab})$ overcomes this difficulty.

We let μ be the inclusion $\mathbb{E}\subset\mathbb{P}$. Thus μ is identity on objects.

Theorem 5.1 *Let $D:\mathbb{E}^{op}(R)\times\mathbb{P}(R)\to\mathrm{Ab}$ be a bifunctor, and let μ^*D be the composite:*

$$\mathbb{E}(R)^{op}\times\mathbb{E}(R)\xrightarrow{Id\times\mu}\mathbb{E}(R)^{op}\times\mathbb{P}(R)\xrightarrow{D}\mathrm{Ab}$$

If D has finite degree with respect to the covariant argument, then

$$K_*^{st}(R,D)\cong\mathrm{H}_*(\mathbb{E}(R),\mu^*(D))\ .$$

Proof. First we consider the case when D is constant with respect to the first variable. In this case, the right hand side is the homology of the category $\mathbb{E}$ with coefficients in a functor $T=D(0,-)$. Because 0 is the terminal object of $\mathbb{E}$, this homology vanishes in positive dimensions and it equals $T(0)$ in dimension 0 thanks to Corollary 3.13. The corresponding statement for stable K-theory is a result of Betley [2].

Next, we extend the comparison to bifunctors $Z_{U,B}$ defined by

$$Z_{U,B}=B(Y)[\mathrm{Hom}_{\mathbb{E}}(X,U)]=P_U(X)\otimes B(Y)\ ,$$

where $B:\mathbb{P}\to\mathrm{Ab}$ is a finite degree functor, and $U\in Ob(\mathbb{E})=Ob(\mathbb{P})$. Lemma 3.10 tells us that the right-hand side $\mathrm{H}_*(\mathbb{E},\mu^*D)$ is isomorphic to $\mathrm{Tor}_*^{\mathbb{E}}(P_U,\mu^*B)$. Since P_U is a projective in $\mathrm{Func}(\mathbb{E},\mathrm{Ab})$, these groups vanish in positive dimensions and they are isomorphic to $B(U)$ in dimension 0. Let us again show the same result for $K_*^{st}(R,Z_{U,B})$.

Having fixed a projection $\pi:R^\infty\to U$ and the corresponding stabilizer $Stab(\pi)$ we get an isomorphism of $\mathrm{GL}(R)$-modules

$$Z_{U,B}^\infty\cong Ind_{Stab(\pi)}^{\mathrm{GL}(R)}B^\infty.$$

It follows from the Shapiro lemma in group homology that

$$\mathrm{H}_*(\mathrm{GL}(R),Z_{U,B}^\infty)\cong\mathrm{H}_*(Stab(\pi),B^\infty)\ .$$

It is known that $\mathrm{H}_*(Stab(\pi),B^\infty)\cong\mathrm{H}_*(\mathrm{GL}(R),B(U\oplus\infty))$. Because B has finite degree, we can still use the result of Betley [2] to conclude that $\mathrm{H}_*(\mathrm{GL}(R),Z_{U,B}^\infty)\cong\mathrm{H}_*(\mathrm{GL}(R),B(U))$ where $\mathrm{GL}(R)$ acts trivially on $B(U)$. Now the spectral sequence $(*)$ yields that K^{st} vanishes on $Z_{U,B}$ in positive dimensions and it equals $B(U)$ in dimension 0.

To conclude, consider, for each integer d, the abelian subcategory of $\mathrm{Func}(\mathbb{E}\times\mathbb{P},\mathrm{Ab})$ consisting of bifunctors $D:\mathbb{E}(R)\times\mathbb{P}(R)\to\mathrm{Ab}$ which have degree d with respect to the first variable. Both terms in the statement form an exact connected sequence of functors defined on this subcategory. The result follows from Proposition 3.1. □

5.2 Another cancellation lemma

As pointed out in Example 4.2, while most of the usual functors have finite degree, the cross-effect on projective functors has the opposite property of enlarging their size. Scorichenko applies Lemma 4.4 to bifunctors which are of finite degree in one variable, but have this opposite property with respect to the other variable. Lemma 5.3's homology cancellation was known to the first author in the special case when R is a prime field and when D is the bifunctor defined by: $D(X,Y) = \mathrm{Hom}(F(X), P_A(Y))$ for a finite degree functor F and a projective P_A (for a nice proof due to L. Schwartz, see [17]).

Definition 5.2 *Let $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ be a bifunctor and let d be an integer. Consider the natural transformation $\varrho_d(D)$:*

$$\varrho_{d,X,Y}(D) : (\mathrm{Cr}^2_{d+1} D)(X; Y, \dots, Y) \to D(X,Y) \ .$$

*The bifunctor D is called S_d-*acyclic *if for each $X, Y \in \mathbb{P}$ the morphism $\varrho_{d,X,Y}$ has a section*

$$s_{X,Y} : D(X,Y) \to (\mathrm{Cr}^2_{d+1} D)(X; Y, \dots, Y)$$

which is natural on $X \in \mathbb{P}$ and on $Y \in \mathbb{E}$.

Lemma 5.3 *Let $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ be a bifunctor and assume that D is of degree $\leq d$ with respect to the first variable. If D is S_d-acyclic then:* $\mathrm{H}_*(\mathbb{P}, D) = 0$.

Proof. By assumption the natural transformation ϱ_d is surjective on D. We let C be the kernel of this transformation. Direct computation checks that if D is S_d-acyclic then C is S_d-acyclic as well. It follows from Lemma 4.4 that $\mathrm{H}_0(\mathbb{P}, D) = 0$ and $\mathrm{H}_{i+1}(\mathbb{P}, D) \cong \mathrm{H}_i(\mathbb{P}, C)$. Now one can use induction to finish the proof. □

5.3 The homology of the category $\mathbb{E}$

We now turn to Scorichenko's second theorem.

Theorem 5.4 *Let $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ be a bifunctor. If D has finite degree with respect to the contravariant argument, then the inclusion $\mu : \mathbb{E} \subset \mathbb{P}$ yields an isomorphism in homology*

$$\mathrm{H}_*(\mathbb{E}, \mu^* D) \cong \mathrm{H}_*(\mathbb{P}, D) \ ,$$

where $\mu^* D : \mathbb{E}^{op} \times \mathbb{E} \to \mathrm{Ab}$ is the composite

$$\mathbb{E}^{op} \times \mathbb{E} \xrightarrow{\mu \times \mu} \mathbb{P}^{op} \times \mathbb{P} \xrightarrow{D} \mathrm{Ab} \ .$$

We refer to the original paper [19] for the proof of this theorem in full generality. Here we give the proof in the case, when submodules of finitely generated projective left R-modules are still finitely generated and projective. This holds for instance when R is the ring $\mathbb{Z}$, and more generally when R is left noetherian and $gl.dim(R) \leq 1$. In the rest of Section 5 we shall assume that R satisfies this condition. Let $\mu : \mathbb{E} \to \mathbb{P}$ be the inclusion. As does any functor between small categories, it yields a functor

$$\mu^* : \mathrm{Func}(\mathbb{P}, \mathrm{Ab}) \to \mathrm{Func}(\mathbb{E}, \mathrm{Ab}), \quad T \mapsto T \circ \mu,$$

which has both left and right adjoint functors known as *left* and *right Kan extensions of the functor* μ [13]. We shall need only the left Kan extension $\mu_!$. The hypothesis on the ring allows canonical factorisation of linear maps by epimorphisms, and thus allows the following description of the left Kan extension of μ. For a functor $T : \mathbb{E} \to \mathrm{Ab}$ and a finitely generated projective left R-module P:

$$\mu_! T(P) := \bigoplus_{W \subset P} T(W)$$

where W runs through the submodules of P. The hypothesis on the ring R insures that any such W is also finitely generated and projective, and $T(W)$ is thus well-defined. A typical generator of $\mu_! T(P)$ is denoted by $(a; W)$, where $a \in T(W)$. If Q is an another finitely generated projective R-module and $f : P \to Q$ is R-linear, one defines $\mu_!(f) : \mu_! T(P) \to \mu_! T(Q)$ by

$$\mu_!(f)(a, W) = (T(f')(a), f(W)) \ ,$$

where $f' : W \to f(W)$ is the restriction of f. We obtain a functor

$$\mu_! : \mathrm{Func}(\mathbb{E}, \mathrm{Ab}) \to \mathrm{Func}(\mathbb{P}, \mathrm{Ab})$$

which is left adjoint to μ^*. The Kan extension $\mu_!$ is the functor defined by Suslin in the case when R is a finite field [9] (it is denoted $\tilde{a}$ there).

This construction bears some obvious variations. For example we also have the functor

$$\mu^* : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{E}, \mathrm{Ab}) \to \mathrm{Func}(\mathbb{E}^{op} \times \mathbb{E}, \mathrm{Ab}), \quad \mu^* B = B \circ (\mu \times Id_{\mathbb{E}})$$

and the functor

$$\mu_! : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{E}, \mathrm{Ab}) \to \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab})$$

which is given by: $(\mu_! B)(X, Y) = \bigoplus_{W \subset Y} B(X, W)$.

Lemma 5.5 *For each A and B, the equation*

$$S_{A,B}(X, Y) = \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(X, A) \times \mathrm{Hom}_{\mathbb{E}}(B, Y)]$$

defines an object $S_{A,B}$ *of the category* $\mathrm{Func}(\mathbb{P}^{op}\times\mathbb{E},\mathrm{Ab})$*. These objects are projective generators of the category* $\mathrm{Func}(\mathbb{P}^{op}\times\mathbb{E},\mathrm{Ab})$*. Furthermore, the following isomorphisms hold:*

$$\mathrm{H}_i(\mathbb{P},\mu_!S_{A,B}) = 0 = \mathrm{H}_i(\mathbb{E},\mu^*S_{A,B}),\quad \text{if } i>0,$$

and

$$\mathrm{H}_0(\mathbb{P},\mu_!S_{A,B}) \cong \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(B,A)] \cong \mathrm{H}_0(\mathbb{E},\mu^*S_{A,B}).$$

Proof. The statement on projective generators follows from Lemma 3.4. We have the following bijection

$$\mathrm{Hom}_{\mathbb{P}}(X,A) \cong \coprod_{W\subset A} \mathrm{Hom}_{\mathbb{E}}(X,W) \tag{$\ddagger$}$$

and it is natural on $(X,A)\in\mathbb{E}\times\mathbb{P}$. It follows that

$$\mu_!(S_{A,B})(X,Y) = \bigoplus_{U\subset Y} \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(X,A)\times\mathrm{Hom}_{\mathbb{E}}(B,U)] \cong \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(X,A)\times\mathrm{Hom}_{\mathbb{P}}(B,Y)]$$

Thus $\mu_!(S_{A,B})$ is a standard projective generator of $\mathrm{Func}(\mathbb{P}^{op}\times\mathbb{P},\mathrm{Ab})$ and therefore Lemma 3.8 shows that

$$\mathrm{H}_i(\mathbb{P},\mu_!S_{A,B}) = 0,\quad \text{if } i>0,\quad \text{and } \mathrm{H}_0(\mathbb{P},\mu_!S_{A,B}) = \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(B,A)]$$

Similarly, we have

$$\mu^*(S_{A,B})(X,Y) = \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(X,A)\times\mathrm{Hom}_{\mathbb{E}}(B,Y)] \cong \bigoplus_{W\subset A} \mathbb{Z}[\mathrm{Hom}_{\mathbb{E}}(X,W)\times\mathrm{Hom}_{\mathbb{E}}(B,Y)]$$

Thus: $\mu^*(S_{A,B}) \cong \bigoplus_{W\subset B} P_{W,B}$, and it is projective in $\mathrm{Func}(\mathbb{E}^{op}\times\mathbb{E},\mathrm{Ab})$. Lemma 3.8 still applies to get: $\mathrm{H}_i(\mathbb{E},\mu^*S_{A,B}) = 0$, if $i>0$ and

$$\mathrm{H}_0(\mathbb{E},\mu^*S_{A,B}) \cong \bigoplus_{W\subset B} \mathbb{Z}[\mathrm{Hom}_{\mathbb{E}}(W,A)] \cong \mathbb{Z}[\mathrm{Hom}_{\mathbb{P}}(B,A)].$$

□

Lemma 5.6 *Let* $B:\mathbb{P}^{op}\times\mathbb{E}\to\mathrm{Ab}$ *be a bifunctor. For bifunctors*

$$\mu_!B:\mathbb{P}^{op}\times\mathbb{P}\to\mathrm{Ab} \quad\text{and}\quad \mu^*B:\mathbb{E}^{op}\times\mathbb{E}\to\mathrm{Ab}$$

there is an isomorphism

$$\mathrm{H}_*(\mathbb{E},\mu^*B) \cong \mathrm{H}_*(\mathbb{P},\mu_!B).$$

Proof. The sequences of functors

$$\mathrm{H}_n(\mathbb{E}, \mu^*(-)) : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{E}, \mathrm{Ab}) \to \mathrm{Ab}$$

and

$$\mathrm{H}_n(\mathbb{P}, \mu^*(-)) : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{E}, \mathrm{Ab}) \to \mathrm{Ab}$$

are exact connected sequences of functors. Lemma 5.5 shows that both of them are universal. Moreover, in dimension 0, both of them are isomorphic on projective generators. Since H_0 is right exact and commutes with coproducts, it follows that both exact connected sequences of functors are isomorphic. □

Lemma 5.7 *For any bifunctor $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ the natural map $\epsilon : \mu_! \mu^* D \to D$ is an epimorphism and* $\mathrm{Ker}(\epsilon)$ *is S_d-acyclic for any $d \geq 0$.*

Proof. By definition: $(\mu_! \mu^* D)(X, Y) = \oplus_{W \subset Y} D(X, W)$. Let (a, W), $a \in D(X, W)$, be a typical generator. The co-unit ϵ sends (a, W) to $i_{W*}(a)$, where $i_W : W \to Y$ is the inclusion. The first statement is clear by considering the factor for $W = Y$. We now prove the S_d-acyclicity of $\mathrm{Ker}(\epsilon)$. Let us fix $d \geq 0$ and let us consider the map

$$s_{X,Y} : \mu_! \mu^* D(X, Y) = \bigoplus_{W \subset Y} D(X, W) \to \mu_! \mu^* D(X, Y^{d+1}) = \bigoplus_{V \subset Y^{d+1}} D(X, V)$$

defined by:

$$s_{X,Y}(a, W) = (-j_* a, W \oplus Y^d) + (a, W) \ .$$

Here $j : W \to W \oplus Y^d$ is given by $j(w) = (w, 0)$, while $W \oplus Y^d$ and W are considered as submodules of Y^{d+1} by embedding W in the first factor. Thus $s_{X,Y}$ is natural on $\mathbb{P}^{op} \times \mathbb{E}$, and the image of s lies in the subfunctor $\mathrm{Cr}^2_{d+1} D$. Furthermore $\varrho \circ s(a, W) = -(i_{W*}(a), Y) + (a, W)$. Therefore the restriction of $\varrho \circ s$ on $\mathrm{Ker}(\epsilon)$ is the identity. □

Proof of Theorem 5.4. Let $\deg_1 D = d$. Since $\mathrm{Ker}(\epsilon)$ is S_d-acyclic we have $\mathrm{H}_*(\mathbb{P}, \mathrm{Ker}(\epsilon)) = 0$ thanks to Lemma 5.3. It follows from the exact sequence $0 \to \mathrm{Ker}(\epsilon) \to \mu_! \mu^* D \to D \to 0$ that: $\mathrm{H}_*(\mathbb{P}, D) \cong \mathrm{H}_*(\mathbb{P}, \mu_! \mu^* D)$. We use Lemma 5.6 to finish the proof. □

6 General Linear homology with twisted coefficients

Let us recall from Section 2 the spectral sequence $(*)$ for a ring R:

$$E^2_{pq} = \mathrm{H}_p(\mathrm{GL}(R), K^{st}_q(R, D)) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{GL}(R), D_\infty) \ .$$

In this section we show that the spectral sequence $(*)$ degenerates at E^2, provided that the bifunctor D takes vector space values and has finite degree with respect to the second argument. It is clear that the spectral sequence $(*)$ is a functor of D.

We begin by showing that it is a ∂-spectral sequence. To define the maps ∂, we first extend the definition of stable K-theory to chain complexes of bifunctors. Since stable K-theory is defined by homology with twisted coefficients, it suffices to consider this case. Let Λ be a ring, let C_* be a complex of Λ-modules and let M be a Λ-module, and consider the twisted homology $\mathrm{H}_*(C_* \otimes_\Lambda M)$. One can replace M by a complex of modules and take the homology of the resulting total complex. When the complex C_* is free, the resulting homology sends weak equivalences to isomorphisms. This situation occurs for the twisted homology of spaces, hence for stable K-theory. With such a definition, the spectral sequence $(*)$ is now natural in respect of morphisms of chain complexes of bifunctors.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of bifunctors. It gives rise to a morphism from the complex

$$\cdots \to 0 \to 0 \to C$$

to the complex

$$\cdots \to 0 \to B \to C \ .$$

The latter is weakly equivalent to the complex $\cdots \to 0 \to A \to 0$. The induced map on spectral sequences is the expected one.

We now prove that the spectral sequence $(*)$ satisfies the conditions of 3.2. Let Func_d be the category of bifunctors $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$ such that $\deg_2 D \leq d$. Let

$$0 \to A \to P \to C \to 0$$

be a short exact sequence in Func_d such that P is a projective in Func_d. It follows from Theorem 5.1 that the connecting homomorphism $K_q^{st}(R, C) \to K_{q-1}^{st}(R, A)$ is an isomorphism for $q \geq 1$ and is a monomorphism for $q = 1$. This applies also when considering only those functors taking values in vector spaces over a field. When A takes vector space values, the monomorphism $K_1^{st}(R, C) \to K_0^{st}(R, A)$ splits. The same is true for the homomorphism $E^2_{pq}(C) \to E^2_{p,q-1}(A)$, because $\mathrm{GL}(R)$ acts trivially. We have proved:

Theorem 6.1 *Let D be a vector space values bifunctor which has finite degree with respect to the second argument. The spectral sequence $(*)$*

$$E^2_{pq} = \mathrm{H}_p(\mathrm{GL}(R), K_q^{st}(R, D)) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{GL}(R), D_\infty)$$

stops at the E_2-term.

Remark 6.2 If the ring R is such that

$$K_*^{st}(R,-) : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab}) \to \mathrm{Ab}$$

is a universal connected sequence of functors, then Theorem 6.1 is true for any bifunctor $D : \mathbb{P}^{op} \times \mathbb{P} \to \mathrm{Ab}$. This follows again from Lemma 3.2. Indeed, take $\mathbf{A} = \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab})$, and for a given $C \in \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab})$ take a standard projective B as in [4, Lemma 2.1]. Then the connecting homomorphism $K_q^{st}(R,C) \to K_{q-1}^{st}(R,A)$ is an isomorphism for $q \geq 1$ and it is a monomorphism for $q = 1$. Because $K_0^{st}(R,B)$ is a free abelian group, the monomorphism $K_1^{st}(R,C) \to K_0^{st}(R,A)$ splits. The above argument then applies. Indeed not only does the spectral sequence $(*)$ stop at E^2, but no extension problem appears: there is a non-natural isomorphism $\mathrm{H}_n(\mathrm{GL}(R), D_\infty) \cong \bigoplus_{p+q=n} \mathrm{H}_p(\mathrm{GL}(R), K_q(R,D))$ [4].

Remark 6.3 According to [4]

$$K_*^{st}(R,-) : \mathrm{Func}(\mathbb{P}^{op} \times \mathbb{P}, \mathrm{Ab}) \to \mathrm{Ab}$$

is a universal connected sequence of functors provided that the ring R is semisimple. It was claimed in [4] that it is still the case for any commutative integral domain of finite Krull dimension, but the proof given there is not correct. However, because Lemma 1.6 of [4] is true under the condition that R is a principal ideal domain, the statement holds for such rings.

Example 6.4 Take $R = \mathbb{Z}$, and $D(X,Y) = \mathrm{Hom}(X, Y \otimes \mathbb{Z}/2\mathbb{Z})$. By [7, paragraphe 9.2], $\mathrm{H}_i(\mathbb{P}(\mathbb{Z}), D)$ is $\mathbb{Z}/2\mathbb{Z}$ when $i \equiv 0,\ 3\ (mod\ 4)$ and 0 else. Scorichenko's theorem says it is the answer for $K_*^{st}(\mathbb{Z}, D)$ as well. Recently [1] the Hopf algebra $\mathrm{H}_*(\mathrm{GL}(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ was computed; its Poincaré series is given by: $\prod_{n\geq 1} \frac{1-t^{2n+1}}{1-t^n}$. Theorem 6.1 implies that the Poincaré series of $\mathrm{H}_*(\mathrm{GL}(\mathbb{Z}), \mathrm{gl}(\mathbb{Z}/2\mathbb{Z}))$ is:

$$\frac{1+t^3}{1-t^4} \prod_{n\geq 1} \frac{1-t^{2n+1}}{1-t^n} .$$

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