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ON SOME NONLOCAL BOUNDARY VALUE PROBLEMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH DELAY

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In the paper, nonlocal boundary value problems are studied for higher order nonlinear ordinary differential equations with delay. More precisely, on a finite interval $[a,b]$, the differential equation $u^{(n)}(t) = f(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t)))$ is considered with the boundary conditions $u^{(i-1)}(a) = c_i$ $(i = 1,\dots,n-1)$, $\ell(u) = c_n$, where $n \geqslant 2$, $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function having continuous partial derivatives in the last n arguments, $\tau_i : [a,b] \to [a,b]$ $(i = 1,\dots,n)$ are continuous functions satisfying the inequalities $\tau_i(t) \leqslant t$ for $a \leqslant t \leqslant b$ $(i = 1,\dots,n)$, $\ell : C^{n-1}([a,b]) \to \mathbb{R}$ is a linear bounded functional, and c_i $(i = 1,\dots,n)$ are real constants. Sufficient conditions are established for the unique solvability of that problem. An analogue of Fredholm's first theorem is obtained. The conditions of the main theorems guarantee also the well-posedness of that problem. An example is constructed showing the optimality of the obtained conditions.

MSC: 34B10, 34B15.
Key words: nonlocal boundary value problem, ordinary differential equation, nonlinear, delay, unique solvability.

Introduction. On a finite interval $[a,b]$, we consider the differential equation

$$u^{(n)}(t) = f\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(a) = c_i \ \ (i = 1,\dots,n-1), \quad \ell(u) = c_n. \tag{2}$$

Here, $n \geqslant 2$, $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function having continuous partial derivatives in the last n arguments, $\tau_i : [a,b] \to [a,b]$ $(i = 1,\dots,n)$ are continuous functions satisfying the inequalities

$$\tau_i(t) \leqslant t \ \text{ for } \ a \leqslant t \leqslant b \ \ (i = 1,\dots,n),$$

$\ell : C^{n-1}([a,b]) \to \mathbb{R}$ is a linear bounded functional, and c_i $(i = 1,\dots,n)$ are real constants.

Important particular cases of (2) are the multi-point boundary conditions

$$u^{(i-1)}(a) = c_i \ \ (i = 1,\dots,n-1), \quad u^{(m)}(b) = \sum_{k=0}^{m} \alpha_k u^{(k)}(a_k) + c_n \tag{3}$$

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and

$$u^{(i-1)}(a) = c_i \ \ (i = 1, \dots, n-1), \quad \sum_{k=0}^{n} \beta_k u^{(k-1)}(b_k) = c_n, \tag{4}$$

where

$$m \in \{0, \dots, n-2\}, \ \ a \leqslant a_k < b \ \ (k = 0, \dots, m), \quad \sum_{k=0}^{m} \frac{(b-a)^{m-k}}{(m-k)!}[\alpha_k]_+ \leqslant 1, \tag{5}$$

and

$$a < b_k \leqslant b, \ \ \beta_k \geqslant 0 \ \ (k = 1, \dots, n), \quad \sum_{k=1}^{n} \beta_k > 0. \tag{6}$$

For the differential equation without delay

$$u^{(n)}(t) = f\big(t, u(t), \dots, u^{(n-1)}(t)\big), \tag{7}$$

boundary value problems of the above mentioned type are subjects of numerous investigations (see, i.e., [1]– [10], [12]– [16], and the references therein).

Problems of the type (1), (2) are investigated by I. Kiguradze and Z. Sokhadze [11] in the case, where

$$f(t, x_1, \dots, x_n)x_1 \geqslant 0 \ \text{ for } \ a \leqslant t \leqslant b, \ \ x_k \operatorname{sgn} x_1 \geqslant r \ \ (k = 1, \dots, n), \tag{8}$$

where r is a sufficiently large positive constant.

I. T. Kiguradze and T. I. Kiguradze [8] have proved a Fredholm type theorem for problem (7), (2), and based on that theorem they have established efficient conditions guaranteeing the unique solvability of that problem. In the present paper, analogous results are obtained for problem (1), (2). These results contain also the case where condition (8) is violated.

Main Results. Before formulating the main results we introduce notations and definitions used in the paper.

$$[x]_+ = \frac{|x| + x}{2}, \quad [x]_- = \frac{|x| - x}{2}.$$

$C([a,b])$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm

$$\|u\|_{C([a,b])} = \max\big\{|u(t)| : \ a \leqslant t \leqslant b\big\}.$$

$C^{n-1}([a,b])$ is the Banach space of $(n-1)$-times continuously differentiable functions $u : [a,b] \to \mathbb{R}$ with the norm

$$\|u\|_{C^{n-1}([a,b])} = \sum_{i=1}^{n} \|u^{(i-1)}\|_{C([a,b])}.$$

Definition 1. *We say that a vector function* $\big(h_{11}, \dots, h_{1n}; h_{21}, \dots, h_{2n}\big) : [a,b] \to \mathbb{R}^{2n}$ *belongs to the set* $\mathcal{U}_\ell(\tau_1, \dots, \tau_n)$ *if for any measurable functions* $h_i : [a,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ *satisfying almost everywhere on* $[a,b]$ *the inequalities*

$$h_{1i}(t) \leqslant h_i(t) \leqslant h_{2i}(t) \ \ (i = 1, \dots, n),$$

the linear boundary value problem

$$v^{(n)}(t) = \sum_{i=1}^{n} h_i(t) v^{(i-1)}(\tau_i(t)),$$
$$v^{(i-1)}(a) = 0 \ \ (i = 1, \dots, n-1), \quad \ell(v) = 0$$

has only a trivial solution.

Definition 2. *A linear bounded functional* $\ell : C^{n-1}([a,b]) \to \mathbb{R}$ *is said to be* ***positive*** *if for any function* $u \in C^{n-1}([a,b])$, *satisfying the conditions*

$$u^{(i-1)}(t) > 0 \ \ \text{for} \ \ a < t \leqslant b \ \ (i = 1, \dots, n),$$

the inequality

$$\ell(u) > 0$$

holds.

Theorem 1. *Let on the set* $[a,b] \times \mathbb{R}^n$ *the inequalities*

$$h_{1i}(t) \leqslant \frac{\partial f(t, x_1, \dots, x_n)}{\partial x_i} \leqslant h_{2i}(t) \ \ (i = 1, \dots, n)$$

hold, where

$$\big(h_{11}, \dots, h_{1n}; h_{21}, \dots, h_{2n}\big) \in \mathcal{U}_\ell(\tau_1, \dots, \tau_n).$$

Then problem (1), (2) *has one and only one solution.*

This theorem is an analogue of Fredholm's first theorem for problem (1), (2).

Along with (1), (2) we consider the perturbed problem

$$v^{(n)}(t) = f\big(t, v(\tau_1(t)), \dots, v^{(n-1)}(\tau_n(t))\big) + q(t), \tag{9}$$
$$v^{(i-1)}(a) = \widetilde{c}_i \ \ (i = 1, \dots, n-1), \quad \ell(v) = \widetilde{c}_n. \tag{10}$$

The following theorem is valid.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then there exists a positive constant* r *such that for any* $q \in C([a,b])$ *and* $\widetilde{c}_i \in \mathbb{R}$ $(i = 1, \dots, n)$ *problem* (9), (10) *has one and only one solution* v *admitting the estimate*

$$\|v - u\|_{C^{n-1}([a,b])} \leqslant r \left(\sum_{i=1}^{n} |\widetilde{c}_i - c_i| + \|q\|_{C([a,b])} \right),$$

where u *is a solution of problem* (1), (2).

Consequently, the conditions of Theorem 1 guarantee not only the unique solvability but also the well-posedness of problem (1), (2).

Theorem 3. *Let on the set* $[a,b] \times \mathbb{R}^n$ *the inequalities*

$$-h_i(t) \leqslant \frac{\partial f(t,x_1,\dots,x_n)}{\partial x_i} \leqslant h_0 \quad (i=1,\dots,n) \tag{11}$$

hold, where h_0 *is a positive constant, and* $h_i : [a,b] \to [0,+\infty)$ $(i=1,\dots,n)$ *are continuous functions such that*

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_a^b (\tau_i(t)-a)^{n-i} h_i(t)dt < 1. \tag{12}$$

If, moreover, ℓ *is a positive functional, then problem* (1), (2) *has one and only one solution.*

For the linear equation

$$u^{(n)}(t) = \sum_{i=1}^{n} p_i(t) u^{(i-1)}(\tau_i(t)) + q(t), \tag{13}$$

where $p_i \in C([a,b])$ $(i=1,\dots,n)$ and $q \in C([a,b])$, Theorem 3 has the following form.

Corollary 1. *If*

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_a^b (\tau_i(t)-a)^{n-i} [p_i(t)]_- dt < 1 \tag{14}$$

and the functional ℓ *is positive, then problem* (13), (2) *has one and only one solution.*

It is easy to verify that the following lemma is true.

Lemma. *Let either*

$$\ell(u) = u^{(m)}(b) - \sum_{k=0}^{m} \alpha_k u^{(k)}(a_k)$$

and conditions (5) *hold, or*

$$\ell(u) = \sum_{k=0}^{n} \beta_k u^{(k-1)}(b_k)$$

and conditions (6) *hold. Then the functional* ℓ *is positive.*

By virtue of the above formulated lemma, Theorem 3 and Corollary 1 result in the following propositions.

Corollary 2. *Let on the set* $[a,b] \times \mathbb{R}^n$ *inequalities* (11) *be satisfied, where* h_0 *is a positive constant, and* $h_i : [a,b] \to [0,+\infty)$ $(i=1,\dots,n)$ *are continuous functions satisfying inequality* (12). *If, moreover, conditions* (5) *(conditions* (6)*) are fulfilled, then problem* (1), (3) *(problem* (1), (4)*) has one and only one solution.*

Corollary 3. *If along with* (14) *conditions* (5) *(conditions* (6)*) are fulfilled, then problem* (13), (3) *(problem* (13), (4)*) has one and only one solution.*

Example. *Consider the problem*

$$u^{(n)}(t) = -p(t)u^{(n-1)}(a) + q(t), \tag{15}$$

$$u^{(i-1)}(a) = 0 \;\; (i = 1,\dots,n-1), \quad u^{(n-1)}(b) = 0, \tag{16}$$

where p, $q \in C([a,b])$, *and*

$$p(t) \geqslant 0 \;\; \text{for} \;\; a \leqslant t \leqslant b.$$

This problem can be obtained from problem (13), (2) *in the case, where*

$$p_i(t) \equiv 0 \;\; (i = 1,\dots,n-1), \;\; p_n(t) \equiv -p(t), \;\; \tau_n(t) \equiv a, \;\; \ell(u) \equiv u^{(n-1)}(b).$$

If

$$\int_a^b p(t)dt < 1,$$

then according to Corollary 1, problem (15), (16) *has a unique solution. Assume now that*

$$\int_a^b p(t)dt = 1, \tag{17}$$

$$\int_a^b q(t)dt \neq 0, \tag{18}$$

and problem (15), (16) *has a solution* u. *If we integrate both sides of equality* (15) *from* a *to* b, *then in view of* (16) *and* (17) *we find*

$$-u^{(n-1)}(a) = -u^{(n-1)}(a) + \int_a^b q(t)dt,$$

which contradicts inequality (18). *Consequently, if conditions* (17) *and* (18) *hold, then problem* (15), (16) *has no solution. On the other hand, in this case for problem* (15), (16) *all the conditions of Corollary 1 are satisfied except the strict inequality* (14), *instead of which we have*

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_a^b (\tau_i(t) - a)^{n-i} [p_i(t)]_- dt = 1.$$

The above constructed example shows that the strict inequality (12) in Theorem 3 (the strict inequality (14) in Corollary 1) cannot be replaced by non-strict one.

Партцванія Н.

Про деяку нелокальну крайову задачу для нелінійного звичайного диференціального рівняння із запізненням

Резюме

В статті вивчається нелокальна крайова задача для нелінійних диференціальних рівнянь вищого порядку із запізненням. А саме, на скінченому інтервалі $[a,b]$ розглядається диференціальне рівняння $u^{(n)}(t) = f(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t)))$ з крайовими умовами

$u^{(i-1)}(a) = c_i$ $(i = 1,\dots,n-1)$, $\ell(u) = c_n$, де $n \geqslant 2$, $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ — неперервна функція, яка має неперервні частинні похідні за останніми n аргументами, $\tau_i : [a,b] \to [a,b]$ $(i = 1,\dots,n)$ є неперервними функціями, що задовольняють нерівностям $\tau_i(t) \leqslant t$ для $a \leqslant t \leqslant b$ $(i = 1,\dots,n)$, $\ell : C^{n-1}([a,b]) \to \mathbb{R}$ є обмеженим лінійним функціоналом, а c_i $(i = 1,\dots,n)$ — дійсні константи. Отримані достатні умови єдиності розв'язку такої задачі та аналог першої теореми Фредгольма. Умови основної теореми гарантують також коректність задачі. Сконструйований приклад, який демонструє оптимальність отриманих умов.
Ключові слова: нелокальна крайова задача, звичайне диференціальне рівняння, нелінійний, запізнення, единий розв'язок .

Партцвания Н.

О некоторой нелокальной краевой задаче для нелинейного обыкновенного дифференциального уравнения с запаздыванием

Резюме

В статье изучаэться нелокальная краевая задача для нелинейных дифференциальных уравнений высшего порядка с запаздыванием. А именно, на конечном интервале $[a,b]$ рассматривается дифференциальное уравнение $u^{(n)}(t) = f(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t)))$ с краевыми условиями $u^{(i-1)}(a) = c_i$ $(i = 1,\dots,n-1)$, $\ell(u) = c_n$, где $n \geqslant 2$, $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ — непрерывная функция, которая имеет непрерывные частные производные по последним n аргументам, $\tau_i : [a,b] \to [a,b]$ $(i = 1,\dots,n)$ являются непрерывными функциями, удовлетворяющими неравенствам $\tau_i(t) \leqslant t$ для $a \leqslant t \leqslant b$ $(i = 1,\dots,n)$, $\ell : C^{n-1}([a,b]) \to \mathbb{R}$ — ограниченный линейный функционал, а c_i $(i = 1,\dots,n)$ — вещественные константы. Получены достаточные условия единственности решения такой задачи и аналог первой теоремы Фредгольма. Условия основной теоремы гарантируют также корректность задачи. Сконструирован пример, демонстрирующий оптимальность полученных условий.
Ключевые слова: нелокальная краевая задача, обыкновенное дифференциальное уравнение, нелинейный, запаздывание, единственное решение .

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