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OSCILLATORY PROPERTIES OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Necessary and sufficient conditions for oscillation and non-oscillation of the differential equation

$$u'' + f(t, u, u') = 0$$

are given, where $f : [a, +\infty) \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function such that $f(t, x, y)x \geq 0$ for $t \geq a$, x and $y \in \mathbf{R}$. The results obtained extend recent results of [**21**–**23**] stated for the equation $u'' + h(t)g(u) = 0$, where $g(x)x > 0$ for $x \neq 0$. Some examples illustrating the sharpness of oscillation criteria are also given.

1. Introduction. We consider the second order nonlinear differential equation

$$u'' + f(t, u, u') = 0 \tag{1.1}$$

on the infinite interval $[a, +\infty)$, where $a > 0$, and $f : [a, +\infty) \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function such that

$$f(t, x, y)x \geq 0 \quad \text{for } t \geq a, \ x \text{ and } y \in \mathbf{R}. \tag{1.2}$$

A solution u of this equation is said to be *proper* if it is defined on some interval $[a_0, +\infty) \subset [a, +\infty)$ and satisfies

$$\sup\{|u(s)| \ : \ s \geq t\} > 0 \quad \text{for } t \geq a_0.$$

A proper solution is said to be *oscillatory* if it has a sequence of zeros tending to $+\infty$, and it is said to be *non-oscillatory* in the opposite case.

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Equation (1.1) is said to be *oscillatory* if all its proper solutions are oscillatory, and it is said to be *non-oscillatory* otherwise. Equation (1.1) is said to be *strongly non-oscillatory* if all its proper solutions are non-oscillatory.

In the well-known works by Kneser [**16**], Hille [**8**], Nehari [**18, 19**], Atkinson [**2**], Kurzweil [**17**], Kiguradze [**9**–**13**], Belohorec [**4, 5**], Butler [**6**] and Wong [**25**–**27**], one can find conditions for oscillation, non-oscillation, and strong non-oscillation of various classes of equations of the type (1.1). These conditions are unimprovable, in a certain sense, and later have been generalized by a number of authors. Main results obtained in this direction are contained in the monographs [**1, 3, 14, 24**].

During the last decade, several papers have been devoted to the investigation of oscillatory properties of the differential equation

$$u'' + h(t)g(u) = 0, \tag{1.3}$$

where $h : [a, +\infty) \to [0, +\infty)$ and $g : \mathbf{R} \to \mathbf{R}$ are continuous functions such that

$$g(x)x > 0 \quad \text{for } x \neq 0,$$

see [**20**–**23, 28**] and the references therein. The oscillation (non-oscillation) theorems, given in these papers, concern the case where

$$t^2 h(t) \geq 1 \quad \text{for } t \geq a \quad (0 \leq t^2 h(t) \leq 1 \text{ for } t \geq a),$$

and there exists $r > 0$ (there exist $r > 0$ and $\sigma \in \{-1, 1\}$) such that

$$\frac{g(x)}{x} \geq \frac{1}{4} + \delta(|x|) \quad \text{for } |x| > r \quad \left(\frac{g(x)}{x} \leq \frac{1}{4} + \delta(|x|) \text{ for } \sigma x \geq r\right),$$

where $\delta : [r, +\infty) \to (0, +\infty)$ is a continuous function satisfying the condition

$$\lim_{x \to +\infty} \delta(x) = 0.$$

In particular, results in this direction are obtained by Sugie and Yamaoka [**22, 23**] which are formulated in terms of the iterated logarithms as follows.

Denote

$$e_0 = 0, \qquad e_k = \exp(e_{k-1}) \quad (k = 1, 2, \dots). \tag{1.4}$$

Following Bellman [**3**], for an arbitrary natural $n \geq 2$ and $t > e_{n-1}$, we set

$$\ln_1 t = \ln t, \qquad \ln_{k+1} t = \ln(\ln_k t) \quad (k = 1, \dots, n-1). \tag{1.5}$$

Moreover, throughout the paper, we assume that

$$\ln_0 t \equiv 1.$$

In [**22**] and [**23**], respectively, the following theorems are proved.

Theorem 1.1. *Let* $0 \leq t^2 h(t) \leq 1$ *for* $t \geq a$. *Suppose that there exist* $\sigma \in \{-1, 1\}$, $n \in \mathbf{N}$ *and a sufficiently large constant* $r > 0$ *such that*

$$\frac{g(x)}{x} \leq \frac{1}{4} \sum_{k=0}^{n} \left(\prod_{i=0}^{k} \ln_i x^2 \right)^{-2} \quad \text{for } \sigma x \geq r.$$

Then equation (1.3) *is strongly non-oscillatory.*

Theorem 1.2. *Let* $t^2 h(t) \geq 1$ *for* $t \geq a$. *Suppose that there exist* $n \in \mathbf{N}$, *a sufficiently large constant* $r > 0$ *and a constant* $\mu > 1$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} \left[\sum_{k=0}^{n-1} \left(\prod_{i=0}^{k} \ln_i x^2 \right)^{-2} + \mu \left(\prod_{i=1}^{n} \ln_i x^2 \right)^{-2} \right] \quad \text{for } |x| \geq r.$$

Then equation (1.3) *is oscillatory.*

Note that for $n = 1$ Theorems 1.1 and 1.2 are proved by Sugie and Kita [**21**]. It is also worth noting that in non-oscillation and oscillation theorems given in [**20, 28**], a stronger assumption on the function g is assumed than in Theorems 1.1 and 1.2.

In the present paper, necessary and sufficient conditions for the non-oscillation of equation (1.1) are given. We establish effective and optimal, in a certain sense, sufficient conditions for non-oscillation and oscillation of equation (1.1) when

$$\begin{gathered} \sigma f(t, x, y) \leq p(t)|x| + q(t, |x|)|x|^{\lambda} \\ \text{for } t \geq a_0, \quad \sigma x \geq r, \quad y \in \mathbf{R} \end{gathered} \tag{1.6}$$

and

$$f(t,x,y)\operatorname{sgn} x \geq p(t)|x| + q(t,|x|)|x|^{\lambda} \quad \text{for } t \geq a_0, \quad |x| \geq r, \quad y \in \mathbf{R}, \tag{1.7}$$

where $p : [a_0, +\infty) \to [0, +\infty)$ is a continuous function such that the equation

$$w'' + p(t)w = 0 \tag{1.8}$$

is non-oscillatory, $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ is continuous function, $\sigma \in \{-1, 1\}$, $\lambda \in \mathbf{R}$, $a_0 \geq a$, and $r > 0$ are constants.

The particular case of one of our oscillation theorems (see Theorem 3.2 and Corollary 3.2 below) is the above formulated Sugie-Kita-Yamaoka result (Theorem 1.2). Non-oscillation theorems given in our paper are not generalizations of Theorem 1.1, because under the conditions of these theorems, in contrast to equation (1.3), equation (1.1) admits the coexistence of nonoscillatory and oscillatory solutions, as Remark 2.2 below illustrates.

2. Non-oscillation theorems.

We start this section with the following two lemmas which play a crucial role in the proof of our general oscillation and nonoscillation theorems.

Lemma 2.1. *Let $u : [a_0, +\infty) \to \mathbf{R}$ be a twice continuously differentiable function satisfying the inequalities*

$$u(t) \geq 0, \qquad u''(t) \leq 0 \quad \textit{for } t \geq a_0.$$

Then

$$u'(t) \geq 0 \quad \textit{for } t \geq a_0.$$

Lemma 2.2. *Let there exist $a_0 \geq a$ and twice continuously differentiable functions $w_i : [a_0, +\infty) \to \mathbf{R}$ $(i = 1, 2)$ such that*

$$w_1(t) \leq w_2(t), \ (-1)^i \Big[w_i''(t) + f(t, w_i(t), w_i'(t)) \Big] \leq 0 \quad \textit{for } t \geq a_0 \ (i = 1, 2). \tag{2.1}$$

Let, moreover,

$$f(t,x,y)x \geq 0 \quad \textit{for } t \geq a_0, \qquad w_1(t) \leq x \leq w_2(t), \quad y \in \mathbf{R}. \tag{2.2}$$

Then equation (1.1) *has a solution* $u : [a_0, +\infty) \to \mathbf{R}$ *satisfying the inequalities*

$$w_1(t) \leq u(t) \leq w_2(t) \quad \textit{for } t \geq a_0. \tag{2.3}$$

Lemma 2.1 is a particular case of the well-known lemma by Kiguradze [**9**], and Lemma 2.2 is a particular case of Theorem 5.1 by Kiguradze and Shekhter [**15**].

Theorem 2.1. *Let condition* (1.2) *hold. Equation* (1.1) *is nonoscillatory if and only if there exist numbers* $a_0 > a$, $\sigma \in \{-1, 1\}$, *and a twice continuously differentiable function* $w : [a_0, +\infty) \to \mathbf{R}$ *such that*

$$\sigma w(t) > 0, \qquad \sigma\Big[w''(t) + f(t, w(t), w'(t))\Big] \leq 0 \quad \textit{for } t \geq a_0. \tag{2.4}$$

Proof. The necessity is obvious. Let us prove the sufficiency. For the sake of definiteness, we will assume that $\sigma = 1$, because the case $\sigma = -1$ can be treated analogously.

In view of (1.2) and (2.4) we have

$$w(t) > 0, \qquad w''(t) \leq 0 \quad \text{for } t \geq a_0.$$

Hence, by Lemma 2.1, it follows that $w'(t) \geq 0$ for $t \geq a_0$, and consequently,

$$w(t) \geq w(a_0) > 0 \quad \text{for } t \geq a_0. \tag{2.5}$$

Put

$$w_1(t) = w(a_0), \qquad w_2(t) = w(t).$$

Then, in view of (1.2), (2.4) and (2.5), inequalities (2.1) and (2.2) are satisfied. By Lemma 2.2, these conditions guarantee the existence of a solution $u : [a_0, +\infty) \to \mathbf{R}$ satisfying inequalities (2.3), i.e.,

$$0 < w(a_0) \le u(t) \le w(t) \quad \text{for } t \ge a_0.$$

Therefore, equation (1.1) is non-oscillatory. □

Corollary 2.1. *Let condition* (1.2) *hold. Suppose that there exist* $\sigma \in \{-1, 1\}$, $n \in \mathbf{N}$ *and a constant* $a_0 > a$ *such that the inequality*

$$\sigma f(t,x,y) \le \frac{|x|}{4t^2}\left[\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\ln_i t\right)^{-2} + \left(\prod_{k=1}^{n}\ln_k x^2\right)^{-2}\right] \tag{2.6}$$

is fulfilled on the set

$$\left\{(t,x,y) \ : \ t \ge a_0,\ \sigma x \ge \left(t\prod_{k=1}^{n}\ln_k t\right)^{1/2},\ y \in \mathbf{R}\right\}. \tag{2.7}$$

Then equation (1.1) *is non-oscillatory.*

Proof. We assume that $a_0 > e_n$. Let

$$w(t) = \sigma\left(t\prod_{k=1}^{n}\ln_k t\right)^{1/2}.$$

Then due to (1.4) and (1.5), the function $w : [a_0, +\infty) \to \mathbf{R}$ is twice continuously differentiable,

$$\ln_k t \ge 1 \ (k = 0, \dots, n)$$

and

$$w^2(t) = t\prod_{k=0}^{n}\ln_k t > t \quad \text{for } t \ge a_0.$$

We have (see e.g. [**3**])

$$w''(t) = -\frac{1}{4t^2}\sum_{m=0}^{n}\left(\prod_{k=0}^{m}\ln_k t\right)^{-2} w(t) \quad \text{for } t \ge a_0.$$

From here and (2.6) we get

$$\begin{aligned}\sigma\big[w''(t)+f(t,w(t),w'(t))\big]&\\ &\le \frac{|w(t)|}{4t^2}\bigg[-\bigg(\prod_{k=1}^{n}\ln_k t\bigg)^{-2}+\bigg(\prod_{k=1}^{n}\ln_k w^2(t)\bigg)^{-2}\bigg]\\ &\le 0 \quad \text{for } t\ge a_0.\end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied which guarantees the non-oscillation of equation (1.1). □

Corollary 2.2. *Let condition* (1.2) *hold. Suppose that there exist* $\sigma\in\{-1,1\}$, $n\in\mathbf{N}$ *and constants* $a_0>a$ *and* $r>0$ *such that*

$$\sigma f(t,x,y)\le \frac{|x|}{4t^2}\sum_{k=0}^{n}\bigg(\prod_{i=0}^{k}\ln_i x^2\bigg)^{-2} \tag{2.8}$$

for $t>a_0$, $\sigma x\ge r$, $y\in\mathbf{R}$. *Then equation* (1.1) *is non-oscillatory.*

Proof. Without loss of generality we can assume that $a_0>\max\{a,e_n,r\}$. Then

$$\ln_k t\ge 1,\quad \ln_k x^2\ge \ln_k t \text{ for } \sigma x\ge\bigg(t\prod_{k=1}^{n}\ln_k t\bigg)^{1/2},\quad t\ge a_0.$$

Thus (2.8) implies that inequality (2.6) is satisfied on the set (2.7). □

Corollary 2.3. *Let condition* (1.2) *hold. Suppose that there exist* $\sigma\in\{-1,1\}$, $n\in\mathbf{N}$, *and constants* $\alpha\in(0,(1/2))$, $\beta>0$, *and* $a_0>a+e_n$ *such that*

$$\sigma f(t,x,y)\le\frac{|x|}{t^2}\bigg[\alpha(1-\alpha)+\bigg(\frac{1}{2}-\alpha\bigg)^2\bigg(\frac{\ln_n t}{\ln_n x^2}\bigg)^{\beta}\bigg] \tag{2.9}$$

for $t\ge a_0$, $\sigma x\ge t^{1/2}$, $y\in\mathbf{R}$. *Then equation* (1.1) *is non-oscillatory.*

Proof. Let $w(t)=\sigma t^{1/2}$. Using (2.9) one can verify that the function w satisfies (2.4). □

Corollary 2.4. *Let condition* (1.2) *hold. Suppose that there exist numbers* $a_0 > a$, $r > 0$, $\sigma \in \{-1, 1\}$ *and a continuous function* $g : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *such that*

$$\sigma f(t, x, y) \leq g(t, |x|) \text{ for } t \geq a_0, \quad \sigma x \geq r, \; y \in \mathbf{R}, \tag{2.10}$$

and the equation

$$v'' + g(t, v) = 0$$

has at least one solution $v : [a_0, +\infty) \to [r, +\infty)$. *Then equation* (1.1) *is non-oscillatory.*

Proof. Let

$$w(t) = \sigma v(t).$$

Then obviously $\sigma w(t) > 0$ for $t \geq a_0$. Using (2.10) we have

$$\begin{aligned} \sigma\big[w''(t) + f(t, w(t), w'(t))\big] & \\ &\leq \sigma w''(t) + g(t, |w(t)|) = v''(t) + g(t, v(t)) \\ &= 0 \quad \text{for } t \geq a_0. \end{aligned}$$

Consequently, inequalities (2.4) are satisfied and, by Theorem 2.1, equation (1.1) is non-oscillatory. □

Let $p : [a, +\infty) \to \mathbf{R}$ be a continuous function. Following Hartman [7], a solution w of equation (1.8) is said to be *principal* if there exists an $a_0 > a$ such that

$$w(t) \neq 0 \text{ for } t \geq a_0, \qquad \int_{a_0}^{+\infty} w^{-2}(t)\, dt = +\infty.$$

It is well known that if equation (1.8) is non-oscillatory, then it has a principal solution which is determined uniquely up to a multiplicative constant (see [**7**, Chapter XI, Section 6, Theorem 6.4]).

Theorem 2.2. *Let conditions* (1.2) *and* (1.6) *hold, where* $a_0 > a$, $r > 0$, $\lambda \in \mathbf{R}$, $\sigma \in \{-1, 1\}$ *are constants and* $p : [a_0, +\infty) \to (0, +\infty)$ *and* $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *are continuous functions. Suppose* *that equation* (1.8) *is non-oscillatory and* $w : [a_0, +\infty) \to [1, +\infty)$ *is its principal solution. Let*

$$w_0(t) = \tau_0 + \int_{a_0}^{t} w^{-2}(s)\,ds, \tag{2.11}$$

where $\tau_0 \geq 1$, *the function* q *is non-increasing* (*non-decreasing*) *in the second argument, and either the condition*

$$\int_{a_0}^{+\infty} w^{1+\lambda}(t)\,w_0(t)\,q(t, cw(t))\,dt < +\infty \tag{2.12}$$

holds for some $c > r/w(a_0)$, *or the condition*

$$\int_{a_0}^{+\infty} w^{1+\lambda}(t)\,w_0^{\lambda}(t)\,q(t, cw_0(t)w(t))\,dt < +\infty \tag{2.13}$$

holds for some $c > r/(w_0(a_0)w(a_0))$. *Then equation* (1.1) *is non-oscillatory.*

Proof. For the sake of definiteness, we will assume that q is a non-increasing function in the second argument. The case when q is a non-decreasing function in the second argument can be considered analogously.

By Corollary 2.4, it is sufficient to prove that for some $t_0 > a_0$ the equation

$$v'' + p(t)v + q(t, v)v^{\lambda} = 0 \tag{2.14}$$

has a solution $v : [t_0, +\infty) \to [r, +\infty)$.

Using the Liouville transformation

$$\tau = w_0(t), \qquad v(t) = w(t)z(\tau), \tag{2.15}$$

equation (2.14) is reduced to the equation

$$z'' + \widetilde{q}(\tau, z)z^{\lambda} = 0, \tag{2.16}$$

where $'$ denotes the derivative with respect to τ and

$$\widetilde{q}(\tau,z) = w^{3+\lambda}(t)q(t,w(t)z).$$

Obviously, the function $\widetilde{q} : [\tau_0,+\infty)\times[r,+\infty)\to[0,+\infty)$ is continuous and non-increasing in the second argument.

If for some $\tau_1 \ge \tau_0$ equation (2.16) has a solution $z : [\tau_1,+\infty)\to[r,+\infty)$, then the function v, given by equalities (2.15), is a desired solution of equation (2.14). Therefore, it remains to prove that for some $\tau_1 \ge \tau_0$ equation (2.16) has a solution $z : [\tau_1,+\infty)\to[r,+\infty)$.

Let $c_0 > c$ and

$$c_1 = \begin{cases} c_0^\lambda & \text{for } \lambda \ge 0, \\ c^\lambda & \text{for } \lambda < 0. \end{cases}$$

Let $\phi : [\tau_0,+\infty)\to[a_0,+\infty)$ be a function inverse to w_0. Consider the functions

$$\begin{aligned} q_1(\tau) &= c_1 w^{3+\lambda}(t)q(t,cw(t)), \\ q_2(\tau) &= c_1 w^{3+\lambda}(t)w_0^\lambda(t)q(t,cw_0(t)w(t)), \end{aligned} \tag{2.17}$$

where $t=\phi(\tau)$. Then

$$0 \le \widetilde{q}(\tau,z) \le q_1(\tau) \text{ for } \tau\ge\tau_0, \quad c\le z\le c_0, \tag{2.18}$$

$$0 \le \widetilde{q}(\tau,z) \le q_2(\tau) \text{ for } \tau\ge\tau_0, \quad c\tau\le z\le c_0\tau. \tag{2.19}$$

If condition (2.12) is fulfilled, then due to (2.11) and (2.17) we have

$$\int_{\tau_0}^{+\infty} \tau q_1(\tau)d\tau = c_1 \int_{a_0}^{+\infty} w^{1+\lambda}(t)w_0(t)q(t,cw(t))\,dt < +\infty; \tag{2.20}$$

similarly, if condition (2.13) is fulfilled, then

$$\int_{\tau_0}^{+\infty} q_2(\tau)d\tau = c_1 \int_{a_0}^{+\infty} w^{1+\lambda}(t)w_0^\lambda(t)q(t,cw_0(t)w(t))\,dt < +\infty. \tag{2.21}$$

By Corollary 8.2 in [**14**], conditions (2.18) and (2.20) (conditions (2.19) and (2.21)) guarantee the existence of a number $\tau_1 > \tau_0$ and a solution $z : [\tau_1,+\infty)\to[r,+\infty)$ of equation (2.16) such that

$$\lim_{\tau\to+\infty} z(\tau) = c \quad \Big(\lim_{\tau\to+\infty}\big(z(\tau)/\tau\big) = c\Big). \qquad \square$$

Remark 2.1. If

$$p(t) = \frac{1}{4t^2} \left(p(t) = \frac{\alpha(1-\alpha)}{t^2}, \text{ where } 0 < \alpha < \frac{1}{2} \right),$$

then

$$w(t) = t^{1/2}, \qquad w_0(t) = \ln t \left(w(t) = t^{\alpha}, \; w_0(t) = \frac{1}{1-2\alpha} t^{1-2\alpha} \right),$$

and conditions (2.12) and (2.13), respectively, take the forms

$$\int_{a_0}^{+\infty} t^{(1+\lambda)/2} \ln t \, q(t, ct^{1/2}) dt < +\infty \left(\int_{a_0}^{+\infty} t^{\alpha(\lambda-1)+1} q(t, ct^{\alpha}) dt < +\infty \right),$$

$$\int_{a_0}^{+\infty} t^{(1+\lambda)/2} \ln^{\lambda} t \, q(t, ct^{1/2} \ln t) dt < +\infty \left(\int_{a_0}^{+\infty} t^{\alpha+\lambda(1-\alpha)} q(t, ct^{1-\alpha}) dt < +\infty \right).$$

Remark 2.2. The conditions of the above-given theorems and their corollaries do not guarantee the strong non-oscillation of equation (1.1). Indeed, if

$$f(t, x, y) = \begin{cases} x & \text{for } |x| \le 1, \\ (2-|x|)\operatorname{sgn} x & \text{for } 1 < |x| < 2, \\ 0 & \text{for } |x| \ge 2, \end{cases}$$

then equation (1.1) is non-oscillatory but not strongly non-oscillatory since along with the non-oscillatory solution $u(t) \equiv c$, where $|c| \ge 2$, it also has the oscillatory solution $u = c \sin t$, where $c \ne 0$ and $|c| \le 1$. Note that in this case for equation (1.1) the conditions of both Theorems 2.1 and 2.2 are satisfied.

3. Oscillation theorems. For an arbitrary $\rho \ge 0$, we suppose

$$f_*(t, \rho) = \inf\{|f(t, x, y)| : \; |x| \ge \rho, \; y \in \mathbf{R}\}.$$

Theorem 3.1. *Let condition* (1.2) *hold and there exist a number* $r > 0$ *and a continuous function* $g : [a, +\infty) \times [r, +\infty) \to [0, +\infty)$ *such that*

$$f(t, x, y) \operatorname{sgn} x \geq g(t, |x|) \quad \text{for } t \geq a, \ |x| \geq r, \ y \in \mathbf{R}. \tag{3.1}$$

Let, moreover,

$$\int_a^{+\infty} t f_*(t, \rho)\, dt = +\infty \quad \text{for } \rho > 0, \tag{3.2}$$

and the equation

$$v'' + g(t, v) = 0 \tag{3.3}$$

do not have a proper solution v *satisfying the inequality* $v(t) > r$ *in some neighborhood of* $+\infty$*. Then equation* (1.1) *is oscillatory.*

Proof. By contradiction, we assume that equation (1.1) has a non-oscillatory solution u on some interval $[a_0, +\infty) \subset [a, +\infty)$. Without loss of generality, let

$$u(t) > 0 \quad \text{for } t \geq a_0.$$

Then in view of (1.2) we have

$$u''(t) \leq 0 \quad \text{for } t \geq a_0.$$

According to Lemma 2.1, these two inequalities result in

$$u'(t) \geq 0, \qquad u(t) \geq \rho > 0 \quad \text{for } t \geq a_0,$$

where $\rho = u(a_0)$. Thus

$$u''(t) + f_*(t, \rho) \leq 0 \quad \text{for } t \geq a_0.$$

If we multiply this inequality by t and then integrate from a_0 to t, we obtain

$$tu'(t) + c_0 + \int_{a_0}^{t} s f_*(s, \rho)\, ds \leq u(t) \quad \text{for } t \geq a_0,$$

where $c_0 = u(a_0) - t_0 u'(a_0)$. Hence, in view of the non-negativeness of u' and condition (3.2), we get

$$\lim_{t\to+\infty} u(t) = +\infty.$$

Thus there exists a $t_0 \geq a_0$ such that

$$u(t) > r \quad \text{for } t \geq t_0. \tag{3.4}$$

Due to (3.1) we have

$$g(t,r) \geq 0, \qquad u''(t) + g(t,u(t)) \leq 0 \quad \text{for } t \geq t_0. \tag{3.5}$$

Applying Lemma 2.2, conditions (3.4) and (3.5) guarantee the existence of a solution v of equation (3.3) satisfying the inequalities

$$r \leq v(t) \leq u(t) \quad \text{for } t \geq t_0,$$

which is a contradiction. □

Let $m \in \mathbf{N}$. We introduce the function

$$\mathcal{K}_m(\tau;\varepsilon) = \frac{1}{4\tau^2}\sum_{k=0}^{m-1}\left(\prod_{i=0}^{k}\ln_i \tau\right)^{-2} + \frac{1+\varepsilon}{4\tau^2}\left(\prod_{i=1}^{m}\ln_i \tau\right)^{-2}$$
$$\text{for } \tau > e_m, \quad \varepsilon > 0.$$

If $m = 0$, we assume

$$\mathcal{K}_0(\tau;\varepsilon) = \frac{1+\varepsilon}{4\tau^2} \quad \text{for } \tau > 0,\ \varepsilon > 0.$$

Theorem 3.2. *Let conditions* (1.2) *and* (3.2) *hold, and*

$$\begin{gathered} f(t,x,y)\operatorname{sgn} x \geq p(t)|x| + q(t,|x|)|x| \\ \textit{for } t \geq a_0,\ |x| \geq r,\ y \in \mathbf{R}, \end{gathered} \tag{3.6}$$

where $a_0 \geq a$ *and* $r > 0$. *Let, moreover, equation* (1.8) *be non-oscillatory, let the function* q *be non-increasing in the second argument and for any* $c > 0$ *there exist* $\varepsilon > 0$, $t_0 \geq a_0$ *and* $m \in \mathbf{N} \cup \{0\}$ *such that*

$$w^4(t)q(t, cw_0(t)w(t)) \geq \mathcal{K}_m(w_0(t);\varepsilon) \ \textit{for } t \geq t_0, \tag{3.7}$$

where $w : [t_0, +\infty) \to (0, +\infty)$ *is a principal solution of equation* (1.8) *and* w_0 *is a function given by equality* (2.11). *Then equation* (1.1) *is oscillatory.*

Proof. Assume by contradiction that equation (1.1) is non-oscillatory. Then, by Theorem 3.1 and condition (3.6), without loss of generality we can assume that the equation

$$v'' + p(t)v + q(t, v)v = 0 \tag{3.8}$$

has a proper solution $v : [a_0, +\infty) \to (r, +\infty)$. Using the same argument as in the proof of Theorem 2.2 the function $z : [\tau_0, +\infty) \to (0, +\infty)$, given by equalities (2.15), is a solution of the equation

$$z'' + q_0(\tau)z = 0, \tag{3.9}$$

where

$$q_0(\tau) = w^4(t)q(t, w(t)z(\tau)), \quad t = \phi(\tau), \tag{3.10}$$

and ϕ is a function inverse to w_0. Moreover,

$$z''(\tau) \le 0 \quad \text{for } \tau \ge \tau_0.$$

Thus there exists a $c > 0$ such that

$$z(\tau) \le c\tau \quad \text{for } \tau \ge \tau_0,$$

i.e.,

$$z(\tau) \le cw_0(t) \quad \text{for } \tau = w_0(t) \ge \tau_0.$$

In view of the last inequality and condition (3.7), we have from (3.10)

$$q_0(\tau) \ge w^4(t)q(t, cw(t)w_0(t)) \ge \mathcal{K}_m(\tau; \varepsilon) \quad \text{for } \tau \ge \tau_1,$$

where $\tau_1 = w_0(t_0)$. Hence by the Kneser theorem (see [**3**, Chapter VI, Theorem 10]) it follows that equation (3.9) is oscillatory. This is a contradiction to the fact that this equation has the non-oscillatory solution. □

Corollary 3.1. *Let conditions* (1.2) *and* (3.2) *hold, and there exist* $\varepsilon > 0$, $n \in \mathbf{N}$, *and constants* $a_0 > a$ *and* $r > 0$ *such that*

$$(3.11)\quad f(t,x,y)\operatorname{sgn} x \geq \frac{|x|}{4t^2}\left[\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\ln_i t\right)^{-2} + (1+\varepsilon)\left(\prod_{i=1}^{n}\ln_i x^2\right)^{-2}\right]$$

for $t \geq a_0$, $|x| \geq r$ *and* $y \in \mathbf{R}$. *Then equation* (1.1) *is oscillatory.*

Proof. Without loss of generality we assume that $a_0 > e_n$ and $r > e_n$. Let

$$(3.12)\qquad p(t) = \frac{1}{4t^2}\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\ln_i t\right)^{-2} \quad \text{for } t \geq a_0$$

and

$$(3.13)\qquad q(t,x) = \frac{1+\varepsilon}{4t^2}\left(\prod_{i=1}^{n}\ln_i x^2\right)^{-2} \quad \text{for } t \geq a_0,\ |x| \geq r.$$

Then (3.11) takes the form (3.6).

Due to (3.12), equation (1.8) is non-oscillatory and the function

$$w(t) = \left(t\prod_{i=0}^{n-1}\ln_i t\right)^{1/2} > 1 \quad \text{for } t \geq a_0$$

is its principal solution. Moreover,

$$w_0(t) = \int_{a_0}^{t} w^{-2}(s)\,ds + \ln_n a_0 = \ln_n t \quad \text{for } t \geq a_0.$$

In view of (3.13), the function $q : [a_0,+\infty) \times [r,+\infty) \to [0,+\infty)$ decreases in the second argument and for any $c > 0$ and a sufficiently large $t_1 = t_1(c) > t_0$ satisfies the equality

$$w^4(t)q(t,cw(t)w_0(t)) = \gamma\mathcal{K}_0(w_0(t);\varepsilon_0)\left(\prod_{k=1}^{n}\ln_k t\right)^2\left(\prod_{k=1}^{n}\ln_k\left(c^2 t\prod_{i=1}^{n}\ln_i^2 t\ln_n t\right)\right)^{-2} \quad \text{for } t > t_1,$$

where

$$\varepsilon_0 = \varepsilon/2, \qquad \gamma = (1+\varepsilon)/(1+\varepsilon_0) > 1.$$

Obviously,

$$\lim_{t\to+\infty} \frac{\ln_k(c^2 t \prod_{i=1}^n \ln_i^2 t \ln_n t)}{\ln_k t} = 1 \quad (k = 1, \dots, n).$$

Therefore there exists a $t_0 > t_1$ such that

$$w^4(t)q(t, cw(t)w_0(t)) \ge \mathcal{K}_0(w_0(t); \varepsilon_0) \quad \text{for } t \ge t_0.$$

Applying Theorem 3.2, the conclusion follows. □

In the sequel, we will apply the following lemma.

Lemma 3.1. *Let $n \ge 2$. Then for any $c > 0$ and $\delta > 0$, there exists a $t_0 > 0$ such that we have for $t \ge t_0$*

$$\sum_{k=1}^{n-1}\left(\prod_{i=1}^{k} \ln_i(ct\ln^2 t)\right)^{-2} \ge \sum_{k=1}^{n-1}\left(\prod_{i=1}^{k} \ln_i t\right)^{-2} - \delta\left(\prod_{i=1}^{n} \ln_i t\right)^{-2}. \tag{3.14}$$

Proof. Let $x > 0$. Then

$$\frac{1}{(x+\ln t)^2} - \frac{1}{\ln^2 t} = -\frac{x(x+2\ln t)}{(x+\ln t)^2 \ln^2 t} > -\varepsilon_1(t,x)(\ln t)^{-5/2} \tag{3.15}$$

for $t > e_1$, whereby

$$\varepsilon_1(t,x) = 2x(\ln t)^{-1/2}.$$

Similarly,

$$\frac{1}{\ln^2(x+\ln t)} - \frac{1}{\ln^2(\ln t)} \ge -\varepsilon_2(t,x)(\ln_2 t)^{-5/2} \quad \text{for } t > e_2, \tag{3.16}$$

where

$$\varepsilon_2(t,x) = \ln\left(\frac{x}{\ln t} + 1\right) \ln(x + 2\ln t)(\ln_2 t)^{-3/2}.$$

Moreover, for any $i \geq 3$, according to the Lagrange theorem, we have

$$\frac{1}{\ln_{i-1}^2(x+\ln t)} - \frac{1}{\ln_{i-1}^2(\ln t)} = -\frac{2x}{\ln_{i-1}^3(s+\ln t)}\Bigg(\prod_{j=1}^{i-2}\ln_j(s+\ln t)\Bigg)^{-1} \quad \text{for } t > e_i,$$

where $s \in (0, x)$. Thus we have

$$\frac{1}{\ln_{i-1}^2(x+\ln t)} - \frac{1}{\ln_{i-1}^2(\ln t)} \geq -\varepsilon_i(t,x)(\ln_i t)^{-5/2} \quad \text{for } t > e_i, \tag{3.17}$$

where

$$\varepsilon_i(t,x) = \frac{2x}{\ln_2 t}(\ln_i t)^{-1/2}.$$

Let

$$t_1 > \max\left\{e_n, \exp\left(\frac{e_n}{c}\right)\right\}.$$

Then by virtue of inequalities (3.15)–(3.17), for $t > t_1$ we have

$$\frac{1}{\ln_1^2(ct\ln^2 t)} - \frac{1}{\ln_1^2 t} = \frac{1}{(\ln(c\ln^2 t)+\ln t)^2} - \frac{1}{\ln^2 t} \geq -\varepsilon_{10}(t)(\ln t)^{-5/2},$$

$$\frac{1}{\ln_i^2(ct\ln^2 t)} - \frac{1}{\ln_i^2 t} = \frac{1}{\ln_{i-1}^2(\ln(c\ln^2 t)+\ln t)} - \frac{1}{\ln_{i-1}^2(\ln t)} \geq -\varepsilon_{i0}(t)(\ln_i t)^{-5/2} \quad (i = 2, \dots, n-1),$$

where

$$\varepsilon_{i0}(t) = \varepsilon_i(t, \ln(c\ln^2 t)) \longrightarrow 0 \quad \text{as } t \to +\infty.$$

Thus

$$\Bigg(\prod_{i=1}^{k}\ln_i(ct\ln^2 t)\Bigg)^{-2} \geq \Bigg(\prod_{i=1}^{k}\ln_i t\Bigg)^{-2} - \varepsilon(t)\Bigg(\prod_{i=1}^{n}\ln_i t\Bigg)^{-2} \quad \text{for } t \geq t_1 \quad (k = 1, \dots, n-1), \tag{3.18}$$

where

$$\varepsilon(t) = \sum_{i=1}^{n-1}\varepsilon_i(t)(\ln_i t)^{-1/2}\Bigg(\prod_{j=i+1}^{n}\ln_j t\Bigg)^2 \longrightarrow 0 \quad \text{as } t \to +\infty.$$

Thus there exists a $t_0 > t_1$ such that

$$\varepsilon(t) < \delta/n \quad \text{for } t \ge t_0.$$

From here and(3.18), the inequality (3.14) follows. □

Corollary 3.2. *Let conditions* (1.2) *and* (3.2) *be fulfilled, and let there exist* $\varepsilon > 0$, $n \in \mathbf{N}$ *and constants* $a_0 > a$ *and* $r > 0$ *such that*

$$f(t,x,y)\operatorname{sgn} x \ge \frac{|x|}{4t^2}\left[\sum_{k=0}^{n-1}\left(\prod_{i=0}^{k}\ln_i x^2\right)^{-2} + (1+\varepsilon)\left(\prod_{i=1}^{n}\ln_i x^2\right)^{-2}\right] \tag{3.19}$$

for $t \ge a_0$, $|x| \ge r$ *and* $y \in \mathbf{R}$. *Then equation* (1.1) *is oscillatory.*

Proof. If $n = 1$, the corollary follows from Corollary 3.1. Let $n \ge 2$, and suppose

$$p(t) = \frac{1}{4t^2} \quad \text{for } t \ge a_0,$$

$$q(t,x) = \frac{1}{4t^2}\sum_{k=1}^{n-1}\left(\prod_{i=1}^{k}\ln_i x^2\right)^{-2} + \frac{1+\varepsilon}{4t^2}\left(\prod_{i=1}^{n}\ln_i x^2\right)^{-2} \tag{3.20}$$

for $t \ge a_0$ and $|x| \ge r$. Then (3.19) takes the form (3.6). On the other hand, equation (1.8) is non-oscillatory,

$$w(t) = t^{1/2}$$

is its principal solution and $w_0(t) = \ln t$. Thus (3.20) implies

$$\begin{aligned} w^4(t)q(t,cw_0(t)w(t)) = {} & \frac{1}{4}\sum_{k=1}^{n-1}\left(\prod_{i=1}^{k}\ln_i(c_0 t\ln^2 t)\right)^{-2} \\ & + \frac{1+\varepsilon}{4}\left(\prod_{i=1}^{n}\ln_i(c_0 t\ln^2 t)\right)^{-2} \quad \text{for } t \ge t_1, \end{aligned} \tag{3.21}$$

where $c_0 = c^2$, and $t_1 > a_0$ is a sufficiently large number.

Let

$$\varepsilon_0 = \frac{\varepsilon}{2}, \qquad \delta = \frac{\varepsilon_0}{2+\varepsilon}.$$

By Lemma 3.1, there exists a $t_0 > t_1$ such that

$$\sum_{k=1}^{n-1} \bigg(\prod_{i=1}^{k} \ln_i (c_0 t \ln^2 t) \bigg)^{-2} \geq \sum_{k=1}^{n-1} \bigg(\prod_{i=1}^{k} \ln_i t \bigg)^{-2} - \delta \bigg(\prod_{i=1}^{n} \ln_i t \bigg)^{-2} \quad \text{for } t > t_0.$$

Without loss of generality t_0 can be assumed to be so large that

$$\bigg(\prod_{i=1}^{n} \ln_i (c_0 t \ln^2 t) \bigg)^{-2} \geq (1-\delta) \bigg(\prod_{i=1}^{n} \ln_i t \bigg)^{-2} \quad \text{for } t \geq t_0.$$

Thus from (3.21) we have

$$\begin{aligned} w^4(t) q(t, c w_0(t) w(t)) &\geq \frac{1}{4} \sum_{k=1}^{n-1} \bigg(\prod_{i=1}^{k} \ln_i t \bigg)^{-2} \\ &\quad + \frac{1+\varepsilon_0}{4} \bigg(\prod_{i=1}^{n} \ln_i t \bigg)^{-2} \\ &= \mathcal{K}_{n-1}(w_0(t); \varepsilon_0) \quad \text{for } t \geq t_0. \end{aligned}$$

Consequently, all the conditions of Theorem 3.2 are satisfied which proves the validity of the corollary. □

Corollary 3.3. *Let conditions* (1.2) *and* (3.2) *be fulfilled, and let there exist* $\alpha \in (0, (1/2))$, $\beta > 0$, $\varepsilon > 0$, $n \in \mathbf{N}$ *and constants* $a_0 > a$ *and* $r > 0$ *such that*

$$f(t,x,y) \operatorname{sgn} x \geq \frac{|x|}{t^2} \bigg[\alpha(1-\alpha) + (1+\varepsilon) \gamma_n(\alpha,\beta) \bigg(\frac{\ln_n t}{\ln_n x^2} \bigg)^{\beta} \bigg] \tag{3.22}$$

for $t \geq a_0$, $|x| \geq r$ *and* $y \in \mathbf{R}$, *where* $\gamma_1(\alpha,\beta) = ((1/2)-\alpha)^2 (2-2\alpha)^{\beta}$ *and* $\gamma_n(\alpha,\beta) = ((1/2)-\alpha)^2$ *for* $n > 1$. *Then equation* (1.1) *is oscillatory.*

Proof. To prove the corollary, it is sufficient to show that the functions

$$p(t) = \frac{\alpha(1-\alpha)}{t^2} \quad \text{for } t \geq a_0, \tag{3.23}$$

$$q(t,x) = \frac{(1+\varepsilon)\gamma_n(\alpha,\beta)}{t^2}\left(\frac{\ln_n t}{\ln_n x^2}\right)^{\beta} \quad \text{for } t \geq a_0,\ |x| \geq r \tag{3.24}$$

satisfy the conditions of Theorem 3.2. Indeed, in view of (3.23) equation (1.8) is non-oscillatory and the function

$$w(t) = t^{\alpha} > 1 \quad \text{for } t \geq a_0$$

is its principal solution. Moreover,

$$w_0(t) = t^{1-2\alpha}\Big/(1-2\alpha).$$

On the other hand, due to (3.24), for any $c > 0$ and a sufficiently large $t_1 > a_0$ we have

$$w^4(t)q(t, cw_0(t)w(t)) = \frac{(1+\varepsilon)\gamma_n(\alpha,\beta)}{(1-2\alpha)^2}(w_0(t))^{-2}\left(\frac{\ln_n t}{\ln_n(c_0 t^{2-2\alpha})}\right)^{\beta} \quad \text{for } t \geq t_1, \tag{3.25}$$

where $c_0 = (1-2\alpha)^{-2}c^2$. Obviously,

$$\lim_{t\to+\infty}\frac{\ln_n(c_0 t^{2-2\alpha})}{\ln_n t} = \begin{cases} 2-2\alpha & \text{for } n = 1, \\ 1 & \text{for } n > 1. \end{cases}$$

Therefore, in view of (3.25), there exists $t_0 > t_1$ such that

$$w^4(t)q(t, cw_0(t)w(t)) \geq \frac{(1+\varepsilon_0)}{4w_0^2(t)} = \mathcal{K}_0(w_0(t);\varepsilon_0) \quad \text{for } t \geq t_0,$$

where $\varepsilon_0 = \varepsilon/2$. Applying Theorem 3.2, this inequality guarantees the oscillation of equation (1.1). □

Remark 3.1. Conditions (3.11), (3.19) and (3.22) in Corollaries 3.1–3.3 are optimal in the sense that it cannot be assumed $\varepsilon = 0$, as follows from Corollaries 2.1–2.3.

Theorem 3.3. *Let conditions* (1.2), (1.6) *and* (3.2) *be fulfilled, where* $a_0 > a$, $r > 0$ *and* $\lambda > 1$ *are constants, and let* $p : [a_0, +\infty) \to [0, +\infty)$ *and* $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *be continuous functions. Let, moreover,* (1.8) *be a non-oscillatory equation with a principal solution* $w : [a_0, +\infty) \to (0, +\infty)$, *let* w_0 *be a function defined by* (2.11) *and let the function* q *be non-decreasing in the second argument such that*

$$\int_{t_0}^{+\infty} w^{1+\lambda}(t) w_0(t) q(t, cw(t))\, dt = +\infty \tag{3.26}$$

for any $t_0 \geq a_0$ *and* $c > r/w(t_0)$. *Then equation* (1.1) *is oscillatory.*

Proof. Assume by contradiction that equation (1.1) is non-oscillatory. Then by Theorem 3.1 and condition (1.6), for some $t_0 \geq a_0$, equation (2.14) has a solution $v : [t_0, +\infty) \to (r, +\infty)$. On the other hand, the function $z : [\tau_1, +\infty) \to (0, +\infty)$, given by equalities (2.15), is a solution of the equation

$$z'' + q_0(\tau)|z|^\lambda \operatorname{sgn} z = 0, \tag{3.27}$$

where $\tau_1 = w(t_0)$,

$$q_0(\tau) = w^{3+\lambda}(t) q(t, w(t) z(\tau)), \quad t = \phi(\tau), \tag{3.28}$$

and $\phi : [\tau_1, +\infty) \to [t_0, +\infty)$ is a function inverse to w_0.

Due to Lemma 2.1,

$$z(\tau) \geq c \quad \text{for } \tau \geq \tau_1,$$

where

$$c = z(\tau_1) = \frac{v(t_0)}{w(t_0)} > \frac{r}{w(t_0)}.$$

From here, (3.26) and (3.28), we get

$$\begin{aligned} \int_{\tau_1}^{+\infty} \tau q_0(\tau)\, d\tau &= \int_{t_0}^{+\infty} w^{1+\lambda}(t) w_0(t) q(t, w(t) z(w_0(t)))\, dt \\ &\geq \int_{t_0}^{+\infty} w^{1+\lambda}(t) w_0(t) q(t, cw(t))\, dt = +\infty. \end{aligned}$$

By the Atkinson theorem [**2**], equation (3.27) is oscillatory, which is a contradiction. □

Theorems 2.2 and 3.3 imply the following corollary.

Corollary 3.4. *Let conditions* (1.2) *and* (3.2) *be fulfilled, and let there exist constants* $a_0 > a$, $\ell > 1$, $r > 0$, $\lambda > 1$ *and continuous functions* $p : [a_0, +\infty) \to [0, +\infty)$ *and* $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *such that*

$$q(t, |x|)|x|^\lambda \le \big(f(t, x, y) - p(t)x\big)\operatorname{sgn} x \le \ell q(t, |x|)|x|^\lambda \tag{3.29}$$

for $t \ge a_0$, $|x| \ge r$ *and* $y \in \mathbf{R}$. *Let, moreover,* (1.8) *be a non-oscillatory equation with a principal solution* $w : [a_0, +\infty) \to (0, +\infty)$, w_0 *be a function defined by* (2.11), *and let the function* q *be non-decreasing in the second argument.*

Then, equation (1.1) *is oscillatory if and only if* (3.26) *holds for any* $t_0 \ge a_0$ *and* $c > r/w(t_0)$.

Theorem 3.4. *Let conditions* (1.2), (1.6) *and* (3.2) *hold, where* $a_0 > a$, $r > 0$ *and* $\lambda < 1$ *are constants, and* $p : [a_0, +\infty) \to [0, +\infty)$ *and* $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *are continuous functions. Let, moreover,* (1.8) *be a non-oscillatory equation with a principal solution* $w : [a_0, +\infty) \to (0, +\infty)$, w_0 *be a function defined by* (2.13), *and the function* q *be non-increasing in the second argument such that*

$$\int_{t_0}^{+\infty} w^{1+\lambda}(t)w_0(t)q(t, cw(t)w_0(t))\,dt = +\infty \tag{3.30}$$

for any $c > 0$ *and sufficiently large* $t_0 \ge a_0$. *Then equation* (1.1) *is oscillatory.*

Proof. Assume by contradiction that equation (1.1) is non-oscillatory. Then by Theorem 3.1 and condition (1.6), there exists a $t_0 \ge a_0$ such that equation (2.14) has a solution $v : [t_0, +\infty) \to (r, +\infty)$. On the other hand, the function $z : [\tau_1, +\infty) \to (0, +\infty)$, given by equality

(2.15), is a solution of the equation

$$z'' + q_0(\tau)\frac{z}{(1+|z|)^{1-\lambda}} = 0, \tag{3.31}$$

where

$$q_0(\tau) = \left(\frac{1+z(\tau)}{z(\tau)}\right)^{1-\lambda} w^{3+\lambda}(t)q(t, w(t)z(\tau)), \quad t = \phi(\tau), \tag{3.32}$$

and $\phi : [\tau_1, +\infty) \to [t_0, +\infty)$ is a function inverse to w_0.

By Lemma 2.1, there exists a constant $c > 0$ such that

$$0 < z(\tau) \le c\tau \quad \text{for } \tau \ge \tau_1.$$

Moreover,

$$\left(\frac{1+z(\tau)}{z(\tau)}\right)^{1-\lambda} > 1 \quad \text{for } \tau \ge \tau_1.$$

Without loss of generality, it can be assumed that

$$cw_0(t)w(t) > r \quad \text{for } t \ge t_0.$$

Thus from (3.32) we find

$$q_0(\tau) \ge w^{3+\lambda}(t)q(t, cw_0(t)w(t)), \qquad \tau = w_0(t) \ge \tau_1.$$

Hence, due to (3.30), we have

$$\begin{aligned}\int_{\tau_1}^{+\infty} \tau^\lambda q_0(\tau)\, d\tau &= \int_{t_0}^{+\infty} w_0^\lambda(t) w^{-2}(t) q_0(w(t))\, dt \\ &\ge \int_{t_0}^{+\infty} w^{1+\lambda}(t) w_0^\lambda(t) q(t, cw_0(t)w(t))\, dt = +\infty.\end{aligned}$$

By the result of Kiguradze (see [**14**, Corollary 10.3]), the last inequality guarantees the oscillation of equation (3.31), which is a contradiction. □

Theorems 2.2 and 3.4 imply the following corollary.

Corollary 3.5. *Let conditions* (1.2), (3.2) *and* (3.29) *be fulfilled, where* $a_0 > a$, $\ell > 0$, $r > 0$ *and* $\lambda < 1$ *are constants, and* $p : [a_0, +\infty) \to [0, +\infty)$ *and* $q : [a_0, +\infty) \times [r, +\infty) \to [0, +\infty)$ *are continuous functions. Let, moreover,* (1.8) *be a non-oscillatory equation with a principal solution* $w : [a_0, +\infty) \to (0, +\infty)$, w_0 *be a function defined by* (2.11), *and let the function* q *be non-increasing in the second argument.*

Then, equation (1.1) *is oscillatory if and only if* (3.30) *holds for any* $c > 0$ *and sufficiently large* $t_0 \geq a_0$.

As an example, we consider the differential equations

$$u'' + \frac{1}{4t^2}u + g(t, u, u')(1 + |u|)^{\lambda-\mu}|u|^{\mu}\operatorname{sgn} u = 0 \tag{3.33}$$

and

$$u'' + \frac{\alpha(1-\alpha)}{t^2}u + g(t, u, u')(1 + |u|)^{\lambda-\mu}|u|^{\mu}\operatorname{sgn} u = 0, \tag{3.34}$$

where

$$\mu > 0, \qquad \lambda \neq 1, \qquad \alpha \in (0, 1/2),$$

and $g : [a, +\infty) \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function.

Let

$$\nu = \begin{cases} 1 & \text{for } \lambda > 1, \\ \lambda & \text{for } \lambda < 1. \end{cases}$$

Corollaries 3.4 and 3.5 yield the following result.

Corollary 3.6. *Let there exist a constant* $\ell > 1$ *and a continuous function* $q : [a, +\infty) \to [0, +\infty)$ *such that the inequalities*

$$q(t) \leq g(t, x, y) \leq \ell q(t)$$

hold on the set $[a, +\infty) \times \mathbf{R}^2$. *Then equation* (3.33) (*equation* (3.34)) *is oscillatory if and only if*

$$\int_a^{+\infty} t^{(1+\lambda)/2}(\ln t)^{\nu} q(t)\, dt = +\infty \quad \left(\int_a^{+\infty} t^{(1+\lambda-2\nu)\alpha+\nu} q(t)\, dt = +\infty \right).$$

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