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ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR TWO-DIMENSIONAL NONLINEAR SINGULAR DIFFERENTIAL SYSTEMS

Dedicated to the blessed memory of Professor Levan Magnaradze

Abstract. For two-dimensional nonlinear differential systems with strong singularities with respect to a time variable, unimprovable sufficient conditions for solvability and well-posedness of the Nicoletti type nonlocal boundary value problem are established.

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Let $-\infty < a < b < +\infty$, $C(]a,b[)$ be the space of continuous functions $u :]a,b[\to R$ with finite right and left limits $u(a+)$ and $u(b-)$ at the points a and b and with the norm $\|u\|_C = \sup\{|u(t)| : \ a < t < b\}$, and let $L^2(]a,b[)$ be the space of square integrable functions $u :]a,b[\to R$ with the norm

$$\|u\|_{L^2} = \bigg(\int_a^b u^2(t)\,dt\bigg)^{1/2}.$$

By $C_0^{1,2}(]a,b[\,;R^2)$ we denote the space of vector functions $(u_1,u_2) :]a,b[\to R^2$ with continuously differentiable components u_1 and u_2, satisfying the conditions

$$u_1(a+) = 0, \quad \int_a^b \big({u_1'}^2(t) + u_2^2(t)\big)\,dt < +\infty.$$

We consider the nonlinear differential system

$$\frac{du_1}{dt} = f_1(t,u_2), \quad \frac{du_2}{dt} = f_2(t,u_1) \tag{1}$$

with the Nicoletti type nonlocal boundary conditions

$$u_1(a+) = 0, \quad u_2(b-) = \varphi(u_1,u_2). \tag{2}$$

Here $f_1 :]a,b[\times R \to R$ and $f_2 :]a,b] \times R \to R$ are continuous functions, and $\varphi : C(]a,b[) \times L^2(]a,b[) \to R$ is a continuous functional.

A vector function $(u_1,u_2) :]a,b[\to R^2$ is said to be a solution of the problem (1), (2) if:

(i) u_1 and u_2 are continuously differentiable and satisfy the system (1) at every point of the interval $]a,b[$;

(ii) $u_1 \in C(]a,b[)$, $u_2 \in L^2(]a,b[)$, and the equalities (2) are satisfied.

In the present paper, unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability of (1), (2) in the space $C_0^{1,2}(]a,b[\,;R^2)$ and the stability of its solution with respect to small perturbations of right-hand sides of (1) and the functional φ. In contrast to the results from [2]–[6], concerning the solvability and well-posedness of the Nicoletti type problems, the theorems below cover the case, where the system (1) with respect to a time variable has a strong singularity at the point a in the Agarwal–Kiguradze sense [1], i.e., the case, where

$$\int_a^b (t-a)\Big(|f_2(t,x)| - f_2(t,x)\operatorname{sgn}(x)\Big)\,dt = +\infty \quad \text{for} \quad x \neq 0.$$

Along with the problem (1), (2) we consider the auxiliary problem

$$\frac{du_1}{dt} = \lambda f_1(t,u_2), \quad \frac{du_2}{dt} = \lambda\delta(t) f_2(t,u_1), \tag{3}$$

$$u_1(a+) = 0, \quad u_2(b-) = \lambda\varphi(u_1,u_2), \tag{4}$$

dependent on a parameter $\lambda \in [0,1]$ and an arbitrary continuous function $\delta : [a,b] \to [0,1]$.

Theorem 1 (A principle of a priori boundedness). *Let there exist a nonnegative function $g \in L^2(]a,b[)$ and a positive constant ρ such that*

$$|f_1(t,x)| \leq g(t)(1+|x|) \quad \text{for} \quad a < t < b, \quad x \in R,$$

and for any number $\lambda \in [0,1]$ and an arbitrary continuous function $\delta : [a,b] \to [0,1]$ every solution (u_1,u_2) of the problem (3), (4) admits the estimate

$$\|u_1'\|_{L^2} + \|u_2\|_{L^2} < \rho.$$

Then the problem (1), (2) has at least one solution in the space $C_0^{1,2}(]a,b[;R^2)$.

Consider now the case, where

$$\varphi(u_1,u_2)\operatorname{sgn}(u_1(b-)) \leq$$
$$\leq \alpha_0 + \alpha_1\|u_1'\|_{L^2} + \alpha_2\|u_2\|_{L^2} \quad \text{for} \quad (u_1,u_2) \in C_0^{1,2}(]a,b[;R^2), \tag{5}$$

and in the domain $]a,b[\times R$ the inequalities

$$\ell_0|x| \leq \big[f_1(t,x) - f_1(t,0)\big]\operatorname{sgn}(x) \leq \ell_1|x|, \tag{6}$$

$$\big[f_2(t,x) - f_2(t,0)\big]\operatorname{sgn}(x) \geq -\frac{\ell}{(t-a)^2}\,|x| \tag{7}$$

are fulfilled.

On the basis of Theorem 1, the following theorem can be proved.

Theorem 2. *Let*

$$\int_a^b f_1^2(t,0)\,dt < +\infty, \quad \int_a^b (t-a)^{1/2}|f_2(t,0)|\,dt < +\infty, \tag{8}$$

and let the conditions (5)–(7) *hold, where* $\alpha_i \geq 0$ $(i = 0,1,2)$, $\ell_k > 0$ $(k = 0,1)$, *and* $\ell \geq 0$ *are constants such that*

$$(b-a)^{1/2}\,(\alpha_1\ell_1 + \alpha_2)\,\ell_1 + 4\ell_1^2\ell < \ell_0. \tag{9}$$

Then the problem (1), (2) *has at least one solution in the space* $C_0^{1,2}(]a,b[;R^2)$.

Particular case of the boundary conditions (2) are the multi-point boundary conditions

$$u_1(a+) = 0, \quad u_2(b-) = \sum_{k=1}^{n-1} \beta_k u_1(t_k) + \beta_n u_1(b-) + \beta_0, \tag{10}$$

where $\beta_k \in R$ $(k = 0, \dots, n)$.

Suppose

$$[\beta_n]_+ = \frac{1}{2}\big(|\beta_n| + \beta_n\big).$$

From Theorem 2 it follows

Corollary 1. *Let the conditions* (6)–(8) *be satisfied, where* $\ell_k > 0$ $(k = 0,1)$ *and* $\ell \geq 0$ *are constants such that*

$$\frac{2}{\pi}\,(b-a)^{1/2}\left(\sum_{k=1}^{n-1} |\beta_k|(t_k-a)^{1/2} + [\beta_n]_+(b-a)^{1/2}\right)\ell_1^2 + 4\ell_1^2\ell < \ell_0. \tag{11}$$

Then the problem (1), (10) *has at least one solution in the space* $C_0^{1,2}(]a,b[;R^2)$.

Now we consider the perturbed problem

$$\frac{dv_1}{dt} = f_1(t,v_2) + q_1(t), \quad \frac{dv_2}{dt} = f_2(t,v_1) + q_2(t), \tag{12}$$

$$v_1(a+) = 0, \quad v_2(b-) = \varphi(v_1,v_2) + \alpha, \tag{13}$$

and we introduce the following

Definition. The problem (1), (2) is said to be **well-posed** in the space $C_0^{1,2}(]a,b[\,;R^2)$ if it has a unique solution (u_1,u_2) in that space and there exists a positive constant r such that for any continuous functions $q_i\,]a,b[\,\to R$ $(i = 1,2)$, satisfying the condition

$$\nu(q_1,q_2) = \left(\int_a^b q_1^2(t)\,dt\right)^{1/2} + \int_a^b (t-a)^{1/2}|q_2(t)|\,dt < +\infty,$$

and for any real number α, the problem (12), (13) has at least one solution $(v_1,v_2) \in C_0^{1,2}(]a,b[\,;R^2)$, and every such solution admits the estimate

$$\|v_1' - u_1'\|_{L^2} + \|v_2 - u_2\|_{L^2} \leq r\big(\nu(q_1,q_2) + |\alpha|\big).$$

Theorem 3. *Let*

$$\varphi(u_1,u_2)\,\mathrm{sgn}(u_1(b-))\leq$$
$$\leq \alpha_1\|u_1'\|_{L^2}+\alpha_2\|u_2\|_{L^2}\ \text{ for }\ (u_1,u_2)\in C_0^{1,2}(]a,b[\,;R^2),$$

and let in the domain $]a,b[\times R$ *the conditions*

$$\ell_0|x|\leq f_1(t,x)\,\mathrm{sgn}(x)\leq \ell_1|x|, \tag{14}$$

$$f_2(t,x)\,\mathrm{sgn}(x)\geq -\frac{\ell}{(t-a)^2}\,|x| \tag{15}$$

be fulfilled, where $\alpha_i\geq 0$ $(i=1,2)$, $\ell_k>0$ $(k=0,1)$, *and* $\ell\geq 0$ *are constants, satisfying the inequality* (9). *Then the problem* (1), (2) *is well-posed in the space* $C_0^{1,2}(]a,b[\,;R^2)$.

In the case, where the boundary conditions (2) have the form

$$u_1(a+)=0,\quad u_2(b-)=\sum_{k=1}^{n-1}\beta_k u_1(t_k)+\beta_n u_1(b-), \tag{16}$$

Theorem 3 yields

Corollary 2. *Let in the domain* $]a,b[\times R$ *the conditions* (14) *and* (15) *be satisfied, where* $\ell_k>0$ $(k=0,1)$ *and* $\ell\geq 0$ *are constants, satisfying the inequality* (11). *Then the problem* (1), (16) *is well-posed in the space* $C_0^{1,2}(]a,b[\,;R^2)$.

As an example, we consider the differential system

$$\frac{du_1}{dt}=p_1(t,u_2)u_2,\quad \frac{du_2}{dt}=\frac{p_2(t,u_1)}{(t-a)^2}\,u_1, \tag{17}$$

where $p_1:]a,b[\times R\to R$ and $p_2:]a,b]\times R\to R$ are continuous functions. For this system from Corollary 2 it follows

Corollary 3. *Let in the domain* $]a,b[\times R$ *the conditions*

$$\ell_0\leq p_1(t,x)\leq \ell_1,\quad p_2(t,x)\geq -\ell$$

be satisfied, where $\ell_i>0$ $(i=0,1)$ *and* $\ell\geq 0$ *are constants, satisfying the inequality* (11). *Then the problem* (17), (16) *is well-posed.*

Remark 1. If the conditions of Corollary 3 are satisfied and in the domain $]a,b[\times R$ the inequality

$$p_2(t,x)\leq -\overline{\ell}$$

holds, where $\overline{\ell}$ is a positive constant, then the system (17) with respect to a time variable has a strong singularity at the point a in the Agarwal–Kiguradze sense.

Remark 2. The condition (9) in Theorems 2 and 3 is unimprovable and it cannot be replaced by the condition

$$(b-a)^{1/2}(\alpha_1\ell_1+\alpha_2)\ell_1+4\ell_1^2\ell\leq \ell_0.$$

Also, the strict inequality (11) in Corollaries 1–3 cannot be replaced by the non-strict one

$$\frac{2}{\pi}\,(b-a)^{1/2}\bigg(\sum_{k=1}^{n-1}|\beta_k|(t_k-a)^{1/2}+[\beta_n]_+(b-a)^{1/2}\bigg)\ell_1^2+4\ell_1^2\ell\leq \ell_0.$$

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