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POTENTIALS WITH PRODUCT KERNELS IN GRAND LEBESGUE SPACES

Let R_0 be a fixed bounded parallelepiped in $\mathbb{R}^n$ with sides parallel to the coordinate axes. In this note we discuss the boundedness of the product kernel potential operator

$$(T_\alpha f)(x) = \int\limits_{R_0} \frac{f(t_1, \dots, t_n)}{\prod_{i=1}^n |x_i - t_i|^{1-\alpha}} dt_1 \dots dt_n, \quad x = (x_1, \dots, x_n) \in R_0,$$

and the appropriate strong fractional integral operator

$$(M_\alpha f)(x) = \sup_{R \ni x} \frac{1}{|R|^{1-\alpha}} \int\limits_R |f(t)| dt, \quad x \in R_0,$$

(here the supremum is taken over all subintervals $R \subset R_0$ containing x with sides parallel to the coordinate axes containing) in grand (weighted) Lebesgue spaces.

Let Ω be bounded subset of $\mathbb{R}^n$ and let w be almost everywhere positive, integrable function on Ω (i.e. a weight). The weighted generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ $(1 < p < \infty)$ is the class of those $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\theta}_w(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{|\Omega|} \int\limits_\Omega |f(t)|^{p-\varepsilon} w(t) dt \right)^{1/(p-\varepsilon)}$$

is finite.

If $w \equiv$ const, then we use the notation $L^{p),\theta}(\Omega) := L^{p),\theta}_w(\Omega)$.

The grand Lebesgue spaces $L^{p),1}(\Omega) =: L^{p)}(\Omega)$ first appeared in the paper by T. Iwaniec and C. Sbordone [4] when they investigated the integrability problem of the Jacobian, while the generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ was introduced by E. Greco, T. Iwaniec and C. Sbordone [3] regarding the study of the nonhomogeneous $n-$ harmonic equation div $A(x, \nabla u) = \mu$.

The space $L^{p),\theta}_w(\Omega)$ is not rearrangement invariant unless $w \equiv$ const.

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Hölder's inequality and simple estimates yield the following embeddings (see also [2], [3]):

$$L^p_w(\Omega) \subset L^{p),\theta_1}_w(\Omega) \subset L^{p),\theta_2}_w(\Omega) \subset L^{p-\varepsilon}_w(\Omega),$$

where $0 < \varepsilon < p-1$ and $\theta_1 < \theta_2$.

In the classical weighted Lebesgue spaces L^p_w the equality

$$\|f\|_{L^p_w} = \|w^{1/p} f\|_{L^p}$$

holds but this property fails in the case of grand Lebesgue spaces. In particular, there is $f \in L^{p)}_w$ such that $w^{1/p} f \notin L^{p)}$ (see also [2] for the details). For structural properties of grand Lebesgue spaces we refer to the paper [1].

We begin with the unboundedness result:

Theorem 1. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, θ_1 and θ_2 be positive numbers such that $\theta_2 < \theta_1 q/p$, where $q = \frac{p}{1-\alpha p}$. Then the operator K_α is not bounded from $L^{p),\theta_1}(R_0)$ to $L^{q),\theta_2}(R_0)$, where K_α is T_α or M_α.*

The next statement gives the boundedness of the operators T_α or M_α in grand Lebesgue spaces:

Theorem 2. *Let $1 < p < \infty$, $0 < \alpha < 1/p$. Suppose that $p \le r \le q < \infty$, where $q = \frac{p}{1-\alpha p}$. Let $\theta > 0$. Then the operator K_α is bounded from $L^{p),\theta}(R_0)$ to $L^{r),\frac{r}{p}\theta}(R_0)$, where K_α is T_α or M_α.*

To formulate our next result let us recall the definition of the Muckenhoupt class of weights defined with respect to parallelepipeds.

Definition. Let $1 < r < \infty$. We say that a weight function w belongs to the class $\mathcal{A}_r(R_0)$ ($w \in \mathcal{A}_r(R_0)$) if

$$\sup_{R \subset R_0} \left(\frac{1}{|R|} \int\limits_R w \right)^{1/r} \left(\frac{1}{|R|} \int\limits_R w^{1-r'} \right)^{1/r'} < \infty,$$

where the supremum is taken over all $n-$ dimensional subintervals $R \subset R_0$ with sides parallel to the coordinate axes.

It is known (see [5]) that the one–weight inequality

$$\|w(K_\alpha f)\|_{L^q(R_0)} \le c\|wf\|_{L^p(R_0)}, \quad q = \frac{p}{1-\alpha p},$$

where K_α is T_α or M_α, holds if and only if $w^q \in \mathcal{A}_{1+q/p'}(R_0)$.

Our weighted result reads as follows:

Theorem 3. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that $\theta > 0$. We set $q = \frac{p}{1-\alpha p}$. Then the following conditions are equivalent:*

(i)

$$\|T_\alpha(f w^\alpha)\|_{L^{q),\theta q/p}_w(R_0)} \le c\|f\|_{L^{p),\theta}_w(R_0)}$$

(ii)

$$\|M_\alpha(fw^\alpha)\|_{L_w^{q),\theta q/p}(R_0)} \le c\|f\|_{L_w^{p),\theta}(R_0)}$$

(iii) $w \in \mathcal{A}_{1+q/p'}(R_0)$.

Finally we point out that weighted boundedness criteria for various integral operators were established in the papers [6], [7], [8], [9], [11]. For weighted criteria of integral operators with product kernels we refer also to the recent monograph [10].

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