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The multisublinear maximal type operators in Banach function lattices

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ABSTRACT

The main aim of this paper is to study a general multisublinear operators generated by quasi-concave functions between weighted Banach function lattices. These operators, in particular, generalize the Hardy–Littlewood and fractional maximal functions playing an important role in harmonic analysis. We prove that under some general geometrical assumptions on Banach function lattices two-weight weak type and also strong type estimates for these operators are true. To derive the main results of this paper we characterize the strong type estimate for a variant of multilinear averaging operators. As special cases we provide boundedness results for fractional maximal operators in concrete function spaces.

1. Introduction

Recently, much attention has been paid to the study of the boundedness of various types of operators between weighted L^p-spaces playing an important role in analysis, in particular, in harmonic analysis and its applications in partial differential equations (PDE). For this purpose the Hardy–Littlewood maximal function defined for any $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$\mathcal{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|\, dy, \quad x \in \mathbb{R}^n,$$

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where the supremum is taken over all cubes with sides parallel to the coordinate axes, has proved to be a tool of great importance. One of the important related operators is the so-called fractional maximal function $\mathcal{M}_\lambda$ $(0 < \lambda < 1)$ defined by

$$\mathcal{M}_\lambda f(x) = \sup_{Q \ni x} \frac{1}{|Q|^\lambda} \int\limits_Q |f|\, dy, \quad x \in \mathbb{R}^n$$

for any $f \in L^1_{loc}(\mathbb{R}^n)$. It is well-known that $\mathcal{M}_\lambda$ is deeply connected to the Riesz potential operator I_α $(0 < \alpha < n)$, given by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n$$

(with $\alpha = n(1-\lambda)$), which play an important role in the theory of Sobolev's embeddings (see, e.g., [10]).

Multisublinear maximal operators appeared naturally in connection with multilinear Calderón–Zygmund theory. A multisublinear maximal operator that acts on the product of m-Lebesgue spaces and is smaller than the m-fold product of the Hardy–Littlewood maximal function was studied in [8]. It was used to obtain a precise control on multilinear singular integral operators of Calderón–Zygmund type and to build a theory of weights adapted to the multilinear setting. For the boundedness and other properties of multisublinear fractional maximal operators in (weighted) Lebesgue spaces we refer to [11,12].

The main aim of this paper is to study more abstract multisublinear maximal operators between weighted Banach function lattices which, in particular, generalize fractional maximal functions. We believe our results will find further important applications in the study of multilinear Riesz potential operators, in the way fractional maximal function did in the study of the Riesz potential operators.

We use standard definitions and notation from the theory of Banach lattices (see, e.g., [7,9]). Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L^0(\mu) = L^0(\Omega, \mu)$ denote the space of all equivalence classes of μ-a.e. finite real-valued measurable functions on Ω with the topology of convergence in measure on μ-finite sets. Let $\widetilde{L}^0(\mu, \Omega)$ denote the space of extended real-valued measurable functions on Ω. A linear subspace E of $L^0(\mu)$ is said to be an *ideal* if $f \in E$ and $|g| \le |f|$ in $L^0(\mu)$, then $g \in E$. By E_+ we denote the collection of non-negative functions in E.

Let $E \subset L^0(\Omega_1, \mu_1)$ be an ideal and let $T\colon E \to L^0(\Omega_2, \mu_2)$ be a positive operator. Suppose that there exists a map $T^\times\colon L^0(\Omega_2, \mu_2) \to \widetilde{L}^0(\Omega_1, \mu_1)_+$ such that for all $f \in E_+$ and all $g \in L^0(\Omega_2, \mu_2)_+$ we have

$$\int\limits_{\Omega_2} (Tf) g\, d\mu_2 = \int\limits_{\Omega_1} f\big(T^\times g\big)\, d\mu_1,$$

then the linear map T' acting from $F := \{g \in L^0(\Omega_2, \mu_2);\ T^\times(|g|) \in L^0(\Omega_1, \mu_1)\}$ to $L^0(\Omega_1, \mu_1)$ by $T'g = T^\times g^+ - T^\times g^-$ for $g \in F$ is called an *adjoint* of T. It is easy to see that T' is a positive linear map such that

$$\int\limits_{\Omega_2} (Tf) g\, d\mu_2 = \int\limits_{\Omega_1} f\big(T'g\big)\, d\mu_1, \quad (f, g) \in E_+ \times F_+.$$

Notice that this definition is motivated by the well-known fact which we will use later: if $K \in L^0(\Omega_2 \times \Omega_1, \mu_2 \times \mu_1)_+$ and $E = \{f \in L^0(\Omega_1, \mu_1);\ \int_{\Omega_1} K(\cdot, t)|f(t)|\, d\mu_1 < \infty\ \mu_1\text{-a.e.}\}$, then it follows by Tonelli's theorem that for the integral operator defined by

$$Tf(s) := \int\limits_{\Omega_1} K(s,t) f(t)\, d\mu_1 \quad f \in E,\ s \in \Omega_2,$$

we have $F = \{g \in L^0(\Omega_2, \mu_2);\ \int_{\Omega_2} K(s,t)|g(s)|\,d\mu_2 < \infty\ \mu_1\text{-a.e.}\}$ and the adjoint T' of T is the integral operator $T' : F \to L^0(\Omega_1, \mu_1)$ given by the formula $T'g(t) = \int_{\Omega_2} k(s,t)g(s)\,d\mu_2$ for all $g \in F$, $t \in \Omega_1$.

A Banach (function) lattice $(X, \|\cdot\|_X)$ on (Ω, Σ, μ) is an ideal of $L^0(\mu)$ which is complete with respect to the norm $\|\cdot\|_X$. We also assume that the support of the space X is Ω ($\mathrm{supp}(X) = \Omega$), that is, there is an element $u \in X$ with $u > 0$ μ-a.e. on Ω.

Let X be a Banach lattice. X is called *minimal* if the closed linear span $\{\chi_A;\ \mu(A) < \infty\}$ is dense in X, where χ_A is the characteristic function of a set A. It is said that X has the *Fatou property* (or X is maximal) if for any $f \in L^0$, $f_n \in X_+$ such that $f_n \uparrow f$ a.e. and $\sup \|f_n\|_X < \infty$, we have that $f \in X$ and $\|f_n\|_X \to \|f\|_X$. We say that X has the *weak Fatou property* whenever if f_n, $f \in X_+$, $f_n \uparrow f$ a.e., then $\|f_n\|_X \to \|f\|_X$.

The *Köthe dual* space X' of a Banach lattice X on (Ω, Σ, μ) is the space of all $f \in L^0(\mu)$ such that $\int_\Omega |fg|\,d\mu < \infty$ for every $g \in X$. It is a Banach lattice on (Ω, Σ, μ) when equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_\Omega |fg|\,d\mu, \quad f \in X'.$$

Let us remark that the Köthe dual X' of X is a maximal Banach lattice on (Ω, μ), as for a number of classical spaces such as Lebesgue spaces L_p, $1 \le p \le \infty$, Orlicz spaces or more general Musielak–Orlicz spaces. It is well known that a Banach lattice X is maximal if and only if $X = X'' := (X')'$ with equality of norms (see, e.g., [7]).

In what follows we will use the following well-known fact that the Köthe dual X' identified in a natural way with a subspace of the Banach dual X^* is a norming subspace, i.e.,

$$\|f\|_X = \sup_{\|g\|_{X'} \le 1} \left| \int_\Omega fg\,d\mu \right|, \quad f \in X,$$

if and only if X has the weak Fatou property (see [7]).

If X is a Banach lattice on (Ω, Σ, μ) and $w \in L^0(\mu)$ is strictly positive a.e., then we define $X(w)$ to be the Banach lattice of all $f \in L^0(\mu)$ such that $fw \in X$, equipped with the norm $\|f\|_{X(w)} = \|fw\|_X$. In what follows we will use the following easily verified formula, which holds with equality of norms

$$X(w)' = X'\left(w^{-1}\right).$$

If $T : X \to Y$ is a bounded operator between Banach spaces, then we say that T is of *strong type* (or has strong type). Let X be a Banach space and Y be a Banach lattice on (Ω, μ). Then a map $T : X \to L^0(\mu)$ is said to be of *weak type* (X, Y) (or has weak type (X, Y)) if there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$\|\chi_{\{\omega \in \Omega;\ |Tx(\omega)| > \lambda\}}\|_Y \le C\lambda^{-1}\|x\|_X, \quad x \in X.$$

Throughout the paper, we consider $\mathbb{R}^n$ equipped with the Lebesgue measure denoted by μ. The family of all cubes in $\mathbb{R}^n$ with edges parallel to the coordinate axes is denoted by $\mathcal{B}$. We denote by $\mathcal{P}$ the set of all increasing functions $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$. The maximal function $\mathcal{M}_\varphi$ generated by $\varphi \in \mathcal{P}$ is defined by

$$\mathcal{M}_\varphi f(x) := \sup_{Q \ni x} \frac{1}{\varphi(|Q|)} \int_Q |f|\,d\mu, \quad x \in \mathbb{R}^n$$

for any $f \in L^1_{loc} := L^1_{loc}(\mathbb{R}^n)$, where the supremum is taken over all cubes $Q \in \mathcal{B}$. Here, as usual, $|Q| := \mu(Q)$. Notice that in the case when $\varphi(t) = t$ for all $t \geq 0$, we obtain the classical Hardy–Littlewood maximal operator $\mathcal{M}$.

We denote by $\bar{Q} := \{Q_i\}$ a countable subfamily of $\mathcal{B}$ satisfying the condition $Q_i^\circ \cap Q_j^\circ = \emptyset$ for $i \neq j$, where Q° denotes the interior of a cube Q.

Given a function $\varphi \in \mathcal{P}$, for any subfamily $\bar{Q}$ in $\mathcal{B}$, we define the averaging operator $T_{\bar{Q}}$ relative to $\bar{Q}$ and φ by

$$T_{\bar{Q}} f = \sum_i \left(\frac{1}{\varphi(|Q_i|)} \int_{Q_i} f \, d\mu \right) \chi_{Q_i}, \quad f \in L^1_{loc}.$$

In the case when $\bar{Q}$ contains only one cube Q, we write T_Q instead of $T_{\bar{Q}}$.

In what follows if X is a Banach space, Y is a Banach lattice on $(\mathbb{R}^n, \mu)$ and S is a map from a subspace E of X to Y, then we put $\|S\|_{X \to Y} := \sup\{\|Sx\|_Y;\ x \in X \cap E,\ \|x\|_X \leq 1\}$. If $\|S\|_{X \to Y} < \infty$ and there is no misunderstanding, we say for short that S is a bounded operator from X to Y. Note that in the paper we consider the case $E = \prod_{k=1}^m L^1_{loc}$ and $X = \prod_{k=1}^m X_k$ equipped with the norm $\|(x_1, \ldots, x_m)\|_X := \max_{1 \leq k \leq m} \|x_k\|_{X_k}$, where $X_1, \ldots, X_m$ are Banach latices on $(\mathbb{R}^n, \mu)$ and $S : E \to L^0(\mathbb{R}^n, \mu)$ is a multi(sub)linear operator.

The following statement is well known (see [5]).

Proposition 1.1. *Let $(X(w), Y(v))$ be a pair of weighted Banach lattices on $(\mathbb{R}^n, \mu)$. Then for any $\varphi \in \mathcal{P}$ the following statement is true:*

$$A := \sup\big\{\|T_Q\|_{X(w) \to Y(v)};\ Q \in \mathcal{B}\big\} < \infty$$

if and only if $(w, v) \in A_\varphi(X, Y)$, i.e.,

$$C_0(\varphi, w, v) := \sup_{Q \in \mathcal{B}} \frac{1}{\varphi(|Q|)} \|v\chi_Q\|_Y \big\|w^{-1}\chi_Q\big\|_{X'} < \infty.$$

Moreover we have $A = C_0(\varphi, v, w)$.

We need the following definition: A pair (X, Y) of Banach lattices on $(\mathbb{R}^n, \mu)$ is said to have the property $G(\mathcal{B})$ ($(X, Y) \in G(\mathcal{B})$ for short) if there is a constant $C_1 = C_1(\mathcal{B}, X, Y)$ such that

$$\sum_i \|x\chi_{Q_i}\|_X \|y\chi_{Q_i}\|_{Y'} \leq C_1 \|x\|_X \|y\|_{Y'}, \quad (x, y) \in X \times Y$$

for any family $\{Q_i;\ Q_i \in \mathcal{B}\}$ of disjoint cubes. If the above inequality holds for any family $\{Q_i\}$ of pairwise disjoint Lebesgue measurable sets, then we write $(X, Y) \in G$.

The following result is due to Berezhnoi [1].

Theorem 1.1. *Let w and v be weights on $\mathbb{R}^n$ and let (X, Y) be a pair of maximal Banach lattices in $G(\mathcal{B})$. If $\varphi \in \mathcal{P}$, then $\sup_{\bar{Q}} \|T_{\bar{Q}}\|_{X(w) \to Y(v)} < \infty$ if and only if $(w, v) \in A_\varphi(X, Y)$. Moreover we have*

$$\sup_{\bar{Q}} \|T_{\bar{Q}}\|_{X(w) \to Y(v)} \leq C_0(\varphi, w, v) C_1(\mathcal{B}, X, Y).$$

2. Multilinear case

Motivated by the results mentioned in the previous section, we aim to study the boundedness of the natural variants of multilinear averaging and maximal operators from the product of weighted Banach lattices to a weighted Banach lattice.

We begin with a brief discussion and definitions. For an m tuple $\vec{\varphi} := (\varphi_1, \dots, \varphi_m) \in \mathcal{P}^m$ and subfamily $\bar{Q} = \{Q_i\}$ in $\mathcal{B}$, we define the multilinear averaging operator $T_{\bar{Q}}$ and the maximal operator $\mathcal{M}_{\vec{\varphi}}$, respectively, by

$$T_{\bar{Q}}\vec{f} = \sum_i \left(\prod_{k=1}^m \frac{1}{\varphi_k(|Q_i|)} \int_{Q_i} f_k \, d\mu \right) \chi_{Q_i},$$

respectively,

$$\mathcal{M}_{\vec{\varphi}}\vec{f}(x) = \sup_{Q \ni x} \prod_{k=1}^m \frac{1}{\varphi_k(|Q|)} \int_Q f_k \, d\mu, \quad x \in \mathbb{R}^n$$

for all $\vec{f} = (f_1, \dots, f_m) \in \prod_{k=1}^m L^1_{loc}$. Note that if $\varphi_j(t) = t$ for every $t \geq 0$ and each $1 \leq j \leq m$, we obtain the multisublinear Hardy–Littlewood maximal operator studied in [8].

We need to define also a mutlilinear variant of $G(\mathcal{B})$-property. Let $X_1, \dots, X_m$, Y be Banach lattices on $(\mathbb{R}^n, \mu)$. We write $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathcal{B})$ if there exists a constant $C_0 = C_0(\mathcal{B}, X_1, \dots, X_m, Y)$ such that for any family $\{Q_i;\ Q_i \in \mathcal{B}\}$ of disjoint cubes,

$$\sum_i \|x_1 \chi_{Q_i}\|_{X_1} \cdots \|x_m \chi Q_i\|_{X_m} \|y \chi_{Q_i}\|_{Y'} \leq C \|x_1\|_{X_1} \cdots \|x_m\|_{X_m} \|y\|_{Y'}$$

holds for all $x_j \in X_j$ $(1 \leq j \leq m)$ and $y \in Y'$.

If the above estimate holds for any family $\{Q_i\}$ of pairwise disjoint Lebesgue measurable sets, then we write $(X_1, \dots, X_m, Y) \in G^{(m)}$. For example, if $X_1 = L^{p_1}, \dots, X_m = L^{p_m}$ and $Y = L^r$ with $1 \leq p_1, \dots, p_m, r < \infty$, then $(X_1, \dots, X_m, Y) \in G^{(m)}$ provided that $1/p_1 + \cdots + 1/p_m + 1/r' \geq 1$, where $1/r + 1/r' = 1$.

It is easy to see that if $X_1, \dots, X_m$ and Y are Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_{k_1}, \dots, X_{k_n}, Y) \in G^{(n)}(\mathcal{B})$ with $1 \leq k_j < m$ for $1 \leq j \leq n$, then $(X_1, \dots, X_m, Y) \in G^{(m)}(\mathcal{B})$.

In what follows we will work with a variant of Morrey spaces. For a given $\varphi \in \mathcal{P}$ we denote by M_φ the space of all $f \in L^0(\mathbb{R}^n, \mu)$ such that

$$\sup_{Q \in \mathcal{B}} \frac{1}{\varphi(|Q|)} \int_Q |f| \, d\mu < \infty.$$

It is easy to verify that M_φ is a Banach lattice on $(\mathbb{R}^n, \mu)$ with the Fatou property when equipped with the norm

$$\|f\|_{M_\varphi} = \sup_{Q \in \mathcal{B}} \frac{1}{\varphi(|Q|)} \int_Q |f| \, d\mu.$$

Proposition 2.1. *Let $\vec{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{P} \times \mathcal{P}$ be such that $t \mapsto \varphi_2(t)/t$ is a non-increasing function. Assume that $X(w)$ and $Y(v)$ are weighted Banach lattices on $(\mathbb{R}^n, \mu)$. If the maximal operator $\mathcal{M}_{\vec{\varphi}}$ is bounded from $X(w) \times M_{\varphi_2}$ to $Y(v)$, then $(w, v) \in A_{\varphi_1}(X, Y)$.*

Proof. Our hypothesis on the boundedness of $\mathcal{M}_{\vec{\varphi}}$ implies that there exists a constant $C > 0$ such that

$$I := \left\| \mathcal{M}_{\vec{\varphi}}(f, g) \right\|_{Y(v)} \le C \|f\|_{X(w)} \|g\|_{M_{\varphi_2}}, \quad (f, g) \in X(w) \times M_{\varphi_2}.$$

Fix $Q \in \mathcal{B}$ and take $g = \frac{\varphi_2(|Q|)}{|Q|} \chi_Q$. Since $t \mapsto \frac{\varphi_2(t)}{t}$ is a non-increasing function, it is easily to see that $g \in M_{\varphi_2}$ with $\|g\|_{M_{\varphi_2}} \le 1$. Hence, we conclude that for all f in the unit ball of $X(w)$ we have

$$\begin{aligned} I &\ge \left(\frac{1}{\varphi_1(|Q|)} \int\limits_Q |f| \, d\mu \right) \left(\frac{1}{\varphi_2(|Q|)} \int\limits_Q |g| \, d\mu \right) \|v \chi_Q\|_Y \\ &= \left(\frac{1}{\varphi_1(|Q|)} \int\limits_Q |f| \, d\mu \right) \|v \chi_Q\|_Y, \end{aligned}$$

and so this gives the desired statement that $(w, v) \in A_{\varphi_1}(X, Y)$. $\square$

Now under some conditions we give a characterization of the boundedness of the multilinear averaging operator $T_{\bar{Q}}$ from the product of weighted Banach lattices to weighted Banach lattices.

Theorem 2.1. *Let $X_1(w_1), \ldots, X_m(w_m)$, $Y(v)$ be weighted Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_1, \ldots, X_m, Y) \in G^{(m)}(\mathcal{B})$. Suppose that Y has the weak Fatou property. Then the multilinear averaging operator $T_{\bar{Q}}$ generated by $\vec{\varphi} = (\varphi_1, \ldots, \varphi_m) \in \mathcal{P}^m$ is uniformly bounded with respect to a subfamily $\bar{Q} = \{Q_i\}$ of $\mathcal{B}$ from $X_1(w_1) \times \cdots \times X_m(w_m)$ to $Y(v)$, i.e., the inequality*

$$\sup_{\bar{Q}} \|T_{\bar{Q}}\|_{X_1(w_1) \times \cdots \times X_m(w_m) \to Y(v)} < \infty$$

holds if and only if $(w_1, \ldots, w_m, v) \in A_{\vec{\varphi}}(X_1, \ldots, X_m, Y)$, i.e.,

$$C_1 := \sup_{Q \in \mathcal{B}} \|v \chi_Q\|_Y \prod_{k=1}^m \frac{1}{\varphi_k(|Q|)} \left\| w_k^{-1} \chi_Q \right\|_{X'_k} < \infty.$$

Proof. Necessity follows in the same way as in the linear case by using the obvious inequality $\sup_{\bar{Q}} \|T_{\bar{Q}}\| \ge \sup_{Q \in \mathcal{B}} \|T_Q\|$ and choosing appropriate test functions.

To prove sufficiency assume that $C_1 < \infty$. Fix $g \in Y(v)'$ with $\|g v^{-1}\|_{Y'} \le 1$. Applying Hölder's inequality and $G^{(m)}$ property for $(X_1, \ldots, X_m, Y)$ we conclude that for any $\vec{f} = (f_1, \ldots, f_m) \in \prod_{k=1}^m X_k(w_k)$,

$$\begin{aligned} \int\limits_{\mathbb{R}^n} T_{\bar{Q}}(\vec{f}) g \, d\mu &= \sum_i \left(\prod_{k=1}^m \frac{1}{\varphi_k(|Q_i|)} \int\limits_{Q_i} (f_k w_k) w_k^{-1} \, d\mu \right) \left(\int\limits_{Q_i} (g v^{-1}) v \, d\mu \right) \\ &\le \sum_i \|v \chi_{Q_i}\|_Y \left(\prod_{k=1}^m \frac{1}{\varphi_k(|Q_i|)} \left\| w_k^{-1} \chi_{Q_i} \right\|_{X'_k} \right) \\ &\quad \times \left(\prod_{k=1}^m \|f_k w_k \chi_{Q_i}\|_{X_k} \right) \left\| g v^{-1} \chi_{Q_i} \right\|_{Y'} \\ &\le C_1 C(\mathcal{B}, X_1, \ldots, X_m, Y) \prod_{k=1}^m \|f_k\|_{X_k(w_k)}. \end{aligned}$$

Since g was arbitrary, it follows from the weak Fatou property of $Y(v)$ that $T_{\bar{Q}}$ is uniformly bounded as an operator from the product $X_1(w_1) \times \cdots \times X_m(w_m)$ to $Y(v)$ with

$$\sup_{\bar{Q}} \|T_{\bar{Q}}\| \le C_1 C(\mathcal{B}, X_1, \ldots, X_m, Y). \qquad \square$$

Following [1], we show general examples of Banach lattices $X_1, \ldots, X_m$, Y such that $(X_1, \ldots, X_m, Y) \in G^{(m)}(\mathcal{B})$. To do this we recall that a Banach lattice X on (Ω, μ) is said to be *p-convex* $(1 < p \le \infty)$, respectively, *q-concave* $(1 \le q < \infty)$, if there exists a constant $C > 0$ such that

$$\left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_X \le C \left(\sum_{k=1}^{n} \|x_k\|_X^p \right)^{1/p},$$

respectively,

$$\left(\sum_{k=1}^{n} \|x_k\|_X^q \right)^{1/q} \le C \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|_X,$$

for any choice of elements $x_1, \ldots, x_n$ in X and $n \in \mathbb{N}$. If in the above definitions elements $x_1, \ldots, x_n$ are pairwise disjoint, then X is said to satisfy an upper p-estimate and lower q-estimate, respectively. Clearly, p-convexity implies upper p-estimate, and q-concavity implies lower q-estimate of a Banach lattice X. More properties may be found in the book [9].

It is easy to check that if X satisfies a lower p-estimate, then the Köthe dual X' satisfies an upper p'-estimate. This immediately gives the following observation: if $X_1, \ldots, X_m$, Y are Banach lattices on $(\mathbb{R}^n, \mu)$ such that X_k satisfies a lower p_k for each $1 \le k \le m$ and Y satisfies an upper q-estimate with $1/p_1 + \cdots + 1/p_m + 1/q' \ge 1$, then $(X_1, \ldots, X_m, Y) \in G^{(m)}(\mathcal{B})$.

Applying the well-known results on p-convex and q-concave Orlicz spaces (see, e.g., [9]) or the Lorentz spaces (see [6]), based on the above remark we obtain concrete general examples of Banach lattices for which we have $(X_1, \ldots, X_m, Y) \in G^{(m)}(\mathcal{B})$.

2.1. Weak type inequality

Below we state and prove a theorem which gives a characterization of the generalized weak type inequality for the maximal multisublinear operator $\mathcal{M}_{\vec{\varphi}}$ from the product of weighted Banach lattices to the weighted Banach lattice satisfying the $G^{(m)}$ property. In what follows if $E_1, \ldots, E_m$ are Banach spaces and F is a Banach lattice on (Ω, ν), then a mapping $T : E_1 \times \cdots \times E_m \to L^0(\mu)$ is said to be of weak type $(E_1, \ldots, E_m, F)$ if

$$\sup_{\lambda > 0} \lambda \|\chi_{\{\omega \in \Omega;\ |T(x_1, \ldots, x_n)(\omega)| > \lambda\}}\|_F \le \|x_1\|_{E_1} \cdots \|x_m\|_{E_m}$$

for all $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$.

Theorem 2.2. *Let $X_1(w_1), \ldots, X_m(w_m)$, $Y(v)$ be weighted Banach lattices on $(\mathbb{R}^n, \mu)$ such that $(X_1, \ldots, X_m, Y) \in G^{(m)}(\mathcal{B})$. Then the multisublinear operator $\mathcal{M}_{\vec{\varphi}}$ generated by $\vec{\varphi} = (\varphi_1, \ldots, \varphi_m) \in \mathcal{P}^m$ is of weak type $(X_1(w_1), \ldots, X_m(w_m), Y(v))$ if and only if $(w_1, \ldots, w_m, v) \in A_{\vec{\varphi}}\ (X_1, \ldots, X_m, Y)$.*

Proof. Necessity is a direct consequence of Theorem 2.1, the pointwise estimate $\mathcal{M}_{\vec{\varphi}} \vec{f} \ge T_{\bar{Q}} \vec{f}$ which holds for any m-tuple $\vec{f} = (f_1, \ldots, f_m) \ge 0$ of locally integrable functions on $\mathbb{R}^n$ and subfamily $\overline{Q}$ of $\mathcal{B}$.

To prove sufficiency we use Besicovitch covering lemma (see [2, pp. 2–3]). Thus for a fixed $0 \le \vec{f} = (f_1, \dots, f_m) \in X_1(w_1) \times \dots \times X_m(w_m)$ we can assume that

$$\left\{x \in \mathbb{R}^n;\ \mathcal{M}_{\vec{\varphi}} \vec{f}(x) > \lambda\right\} \subset \bigcup Q_{ji},$$

where the first index $j \in \{1, \dots, 4^n + 1\}$, $Q^\circ_{jk} \cap Q^\circ_{ji} = \emptyset$ for each $i \ne k$ and

$$\prod_{k=1}^{n} \frac{1}{\varphi_k(|Q_{ji}|)} \int_{Q_{ji}} f_k \, d\mu > \lambda.$$

This implies

$$\left(\prod_{k=1}^{n} \frac{1}{\varphi_k(|Q_{ji}|)} \int_{Q_{ji}} f_k \, d\mu\right) \chi_{Q_{ji}} > \lambda \chi_{Q_{ji}},$$

and hence we obtain the following estimates with $\bar{Q}_j := \{Q_{ij}\}_i$

$$\begin{aligned}
\|\chi_{\{x \in \mathbb{R}^n;\ \mathcal{M}_{\vec{\varphi}} \vec{f}(x) > \lambda\}}\|_{Y(v)} \le \|\chi_{\bigcup Q_{ji}}\|_{Y(v)} &\le \left\|\sum_{i,j} \chi_{Q_{ji}}\right\|_{Y(v)} \\
&\le \frac{1}{\lambda} \left\|\sum_{i,j} \left(\prod_{k=1}^{n} \frac{1}{\varphi_k(|Q_{ji}|)} \int_{Q_{ji}} f_k \, d\mu\right) \chi_{Q_{ji}}\right\|_{Y(v)} \\
&\le \frac{1}{\lambda} \sum_{j=1}^{4^n+1} \left\|\sum_{i} \left(\prod_{k=1}^{n} \frac{1}{\varphi_k(|Q_{ji}|)} \int_{Q_{ji}} f_k \, d\mu\right) \chi_{Q_{ji}}\right\|_{Y(v)} \\
&= \frac{1}{\lambda} \sum_{j=1}^{4^n+1} \|T_{\bar{Q}_j} \vec{f}\|_{Y(v)} \le \frac{C}{\lambda} \prod_{k=1}^{m} \|f_k\|_{X_k(w_k)},
\end{aligned}$$

which completes the proof. □

2.2. Strong type estimate

In the remaining part of the paper, we investigate the boundedness of a bisublinear maximal operator $\mathcal{M}_{\vec{\varphi}}$. We need some definitions. If $\varphi \in \mathcal{P}$ is such that there exists $C \ge 1$ with

$$\varphi(s + t) \le C\left(\varphi(s) + \varphi(t)\right), \quad s, t > 0, \tag{2.1}$$

then we write $\varphi \in \widetilde{\mathcal{P}}$. Note that the condition (2.1) implies that $\varphi(t)/t \le C\varphi(s)/s$ for all $0 < s < t$. Since φ is non-decreasing, the function $\widetilde{\varphi}$ given by $\widetilde{\varphi}(t) := \inf_{s>0}(1 + t/s)\varphi(s)$ for $t > 0$ and $\widetilde{\varphi}(0) = 0$ is concave on $[0, \infty)$ and satisfies $C^{-1}\varphi(t) \le \widetilde{\varphi}(t) \le 2\varphi(t)$ for all $t \ge 0$ and so, in particular, $\widetilde{\varphi}$ is a quasi-concave function on $[0, \infty)$, i.e., $\widetilde{\varphi} \in \mathcal{P}$ and $t \mapsto t/\widetilde{\varphi}(t)$ is a non-decreasing function on $(0, \infty)$.

In what follows we will use the following simple observation (see [1]): for any $\varphi \in \widetilde{\mathcal{P}}$, then there exist $\gamma, \alpha \in (0, 1)$ such that for all $s, t > 0$

$$\frac{\varphi(s)}{\varphi(t)} \le \gamma \quad \text{implies} \quad \frac{s}{t} \le \alpha. \tag{2.2}$$

Theorem 2.3. *Let $\vec{\varphi} = (\varphi_1, \varphi_2) \in \widetilde{\mathcal{P}} \times \widetilde{\mathcal{P}}$ and let X_1 and Y be minimal Banach lattices on $(\mathbb{R}^n, \mu)$, where Y has the Fatou property. Let $(X_1, Y) \in G(\mathcal{B})$. Suppose that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded in the weighted Banach lattice $X_1(w_1)$. Then the $\mathcal{M}_{\vec{\varphi}}$ is bounded from $X_1(w_1) \times M_{\varphi_2}$ to $Y(v)$ if and only if $(w_1, v) \in A_{\varphi_1}(X_1, Y)$.*

Proof. Necessity is a consequence of Proposition 2.1. Now we prove sufficiency. As we noticed, $\widetilde{\varphi}$ is equivalent to φ for any $\varphi \in \widetilde{\mathcal{P}}$. Now since $\widetilde{\varphi}$ is quasi-concave, without loss of generality we can assume that both φ_1 and φ_2 are quasi-concave functions on $[0, \infty)$ by the relation $\mathcal{M}_{(\widetilde{\varphi}_1, \widetilde{\varphi}_2)} f \sim \mathcal{M}_{(\varphi_1, \varphi_2)} f$ for all $f \in L^1_{loc}$. Then $\varphi := \varphi_1 \varphi_2$ is also quasi-concave and so it satisfies the inequality (2.1) with $C = 1$. We fix $t > 0$ so that

$$\gamma > \frac{\rho(2^n)}{t}, \tag{2.3}$$

where γ is given in described fact for φ_1 (see (2.2)) and $\rho \in \mathcal{P}$ is a submuliplicatve function (i.e., $\rho(st) \leq \rho(s)\rho(t)$ for all $s, t > 0$) defined by

$$\rho(s) := \sup\{\varphi(st)/\varphi(t);\ t > 0\}, \quad s > 0.$$

Let $\mathcal{B} = \mathcal{D}$, i.e., let it be the family of all dyadic cubes in $\mathbb{R}^n$ with the side length less than or equal to 2^{k_0}. Let $\vec{f} = (f_1, \dots, f_m) \geq 0$ and let

$$\Omega_k = \{s \in \mathbb{R}^n;\ \mathcal{M}_{\vec{\varphi}} \vec{f}(s) > t^k\}, \quad k \in \mathbb{Z}.$$

It is easy to see similarly as in the case $\mathcal{M}$ that Ω_k is the union of certain family $S_k := \{Q_{ki};\ i \in I_k\}$ of cubes in $\mathcal{D}$ (see [1] for details).

For each $k \in \mathbb{Z}$, in the family S_k we choose a maximal subfamily $\bar{S}_k := \{Q_{ki};\ i \in \bar{I}_k\}$ so that every cube in $\bar{S}_k$ is not contained in any other cube of S_k. Since the side length of dyadic cubes in $\mathcal{D}$ is bounded by 2^{k_0}, we have that

$$Q^\circ_{ki} \cap Q^\circ_{kj} = \emptyset, \quad i, j \in \bar{I}_k,\ i \neq j,$$
$$\bigcup_{i \in I_k} Q_{ki} = \bigcup_{i \in \bar{I}_k} Q_{ki}.$$

For each $k \in \mathbb{Z}$ and $i \in \bar{I}_k$ we let $E_{ki} := Q_{ki} \setminus \Omega_{k+1}$. We claim that

$$|E_{ki}| > (1 - \alpha)|Q_{ki}|, \tag{2.4}$$

where α is defined by (2.2). Indeed, if $J_k := \{j \in \bar{I}_{k+1};\ Q_{k+1j} \subset Q_{ki}\}$ then by the maximality of the family $\bar{S}_k$ we have

$$\begin{aligned} t^k \varphi_1(2^n|Q_{ki}|)\varphi_2(2^n|Q_{ki}|) &\geq \Big(\int_{Q_{ki}} f_1\, d\mu\Big)\Big(\int_{Q_{ki}} f_2\, d\mu\Big) \\ &\geq \sum_{j \in J_k} \frac{\varphi_1(|Q_{k+1j}|)}{\varphi_1(|Q_{k+1j}|)} \frac{\varphi_2(|Q_{k+1j}|)}{\varphi_2(|Q_{k+1j}|)} \Big(\int_{Q_{k+1j}} f_1\, d\mu\Big)\Big(\int_{Q_{k+1j}} f_2\, d\mu\Big) \\ &\geq t^{k+1} \sum_{j \in J_k} (\varphi_1\varphi_2)(|Q_{k+1j}|) \geq t^{k+1}(\varphi_1\varphi_2)\Big(\sum_{j \in J_k} |Q_{k+1j}|\Big). \end{aligned}$$

This gives

$$(\varphi_1\varphi_2)\big(2^n|Q_{ki}|\big) \ge t(\varphi_1\varphi_2)\big(|Q_{ki} \cap \Omega_{k+1}|\big)$$

and so

$$t^{-1}\rho\big(2^n\big) \ge \frac{(\varphi_1\varphi_2)(|Q_{ki} \cap \Omega_{k+1}|)}{(\varphi_1\varphi_2)(|Q_{ki}|)}.$$

Combining (2.2) and (2.3) we conclude that (2.4) holds.

Now we prove that $\mathcal{M}_{\vec{\varphi}}(f_1, f_2) \in Y(v)$,

$$\begin{aligned}
\big\|\mathcal{M}_{\vec{\varphi}}(f_1, f_2)\big\|_{Y(v)} &\le \Big\|\sum_{k\in\mathbb{Z}} t^{k+1}\chi_{\Omega_k\setminus\Omega_{k+1}}\Big\|_{Y(v)} \\
&\le t\Big\|\sum_{k\in\mathbb{Z},\, j\in\bar{I}_k}\Big(\frac{1}{\varphi_1(|Q_{kj}|)}\int_{Q_{kj}} f_1\,d\mu\Big)\Big(\frac{1}{\varphi_2(|Q_{kj}|)}\int_{Q_{kj}} f_2\,d\mu\Big)\chi_{E_{kj}}\Big\|_{Y(v)} \\
&= t\big\|T_0(f_1, f_2)\big\|_{Y(v)},
\end{aligned}$$

where

$$T_0(f_1, f_2) := \sum_{k\in\mathbb{Z},\, j\in\bar{I}_k}\Big(\frac{1}{\varphi_1(|Q_{kj}|)}\int_{Q_{kj}} f_1\,d\mu\Big)\Big(\frac{1}{\varphi_2(|Q_{kj}|)}\int_{Q_{kj}} f_2\,d\mu\Big)\chi_{E_{kj}}.$$

Here we have used the obvious estimate for all $x \in \Omega_k \setminus \Omega_{k+1}$, $k \in \mathbb{Z}$

$$\sup_{Q\in\mathcal{B},\, Q\ni x}\Big(\frac{1}{\varphi_1(|Q|)}\int_Q f_1\,d\mu\Big)\Big(\frac{1}{\varphi_2(|Q|)}\int_Q f_2\,d\mu\Big) \le T_0(f_1, f_2)(x).$$

Now we prove that T_0 is bounded as an operator from $X_1(w) \times X_2$ to $Y(v)$. For this we will need to show that the operator $T_0^{(\varphi_1)}$ defined by

$$T_0^{(\varphi_1)}f = \sum_{k\in\mathbb{Z},\, j\in\bar{I}_k}\Big(\frac{1}{\varphi_1(|Q_{kj}|)}\int_{Q_{kj}} f\,d\mu\Big)\chi_{E_{kj}}$$

is bounded from $X_1(w)$ to $Y(v)$ whenever $(w_1, v) \in A_{\varphi_1}(X, Y)$. Following [1] let us define on L^1_{loc} two operators $T_1^{(\varphi_1)}$, $T_2^{(\varphi_1)}$ by

$$T_1^{(\varphi_1)}f = \sum_{k\in\mathbb{Z}\, j\in\bar{I}_k}\Big(\frac{1}{|Q_{kj}|}\int_{Q_{kj}} f\,d\mu\Big)\chi_{E_{kj}},$$

$$T_2^{(\varphi_1)}f = \sum_{k\in\mathbb{Z}\, j\in\bar{I}_k}\Big(\frac{1}{\varphi_1(|Q_{kj}|)}\int_{E_{kj}} f\,d\mu\Big)\chi_{E_{kj}}, \quad f \in L^1_{loc}.$$

It is easy to see that the adjoints of these operators are given by

$$\big(T_0^{(\varphi_1)}\big)'g = \sum_{k\in\mathbb{Z},\, j\in\bar{I}_k}\Big(\frac{1}{\varphi_1(|Q_{kj}|)}\int_{E_{ki}} g\,d\mu\Big)\chi_{Q_{kj}},$$

$$\big(T_1^{(\varphi_1)}\big)'g = \sum_{k\in\mathbb{Z}\ j\in\bar{I}_k} \bigg(\frac{1}{|Q_{kj}|}\int\limits_{E_{kj}} g\,d\mu\bigg)\chi_{Q_{kj}},$$

$$\big(T_2^{(\varphi_1)}\big)' = T_2^{(\varphi_1)}.$$

Since $(T_0^{(\varphi_1)})'$ is a positive linear operator, to prove the boundedness of $T_0^{(\varphi_1)}$ from $X_1(w_1)$ to $Y(v)$ it is enough to show that $(T_0^{(\varphi_1)})'$ maps $Y(v)'$ to $X(w)'$.

Let $0 \le g$. Then applying the estimate $|E_{ki}| > (1-\alpha)|Q_{ki}|$ for each $k \in \mathbb{Z}$ and $i \in \bar{I}_k$ proved above, we obtain

$$\begin{aligned}
\big(T_0^{(\varphi_1)}\big)'f &= \sum_{k\in\mathbb{Z},\ j\in\bar{I}_k} \bigg(\frac{1}{\varphi_1(|Q_{kj}|)}\int\limits_{E_{kj}} g\,d\mu\bigg)\chi_{Q_{kj}} \\
&= \sum_{k\in\mathbb{Z},\ j\in\bar{I}_k} \bigg(\frac{|Q_{ki}|}{|E_{kj}|\varphi_1(|Q_{kj}|)|Q_{kj}|}\int\limits_{E_{kj}}\bigg(\int\limits_{E_{kj}} g\,d\mu\bigg)\,d\mu\bigg)\chi_{Q_{kj}} \\
&\le \frac{1}{1-\alpha}\sum_{k\in\mathbb{Z},\ j\in\bar{I}_k} \bigg(\frac{1}{|Q_{kj}|}\int\limits_{E_{kj}}\bigg(\sum_{m\in\mathbb{Z},\ n\in\bar{I}_k}\frac{1}{\varphi_1(|Q_{mn}|)}\int\limits_{E_{mn}} g\,d\mu\bigg)\,\chi_{E_{mn}}\,d\mu\bigg)\chi_{Q_{kj}} \\
&= \frac{1}{1-\alpha}\big(T_1^{(\varphi_1)}\big)'T_2^{(\varphi_1)}g.
\end{aligned}$$

Observe that $|T_1^{(\varphi_1)}| \le \mathcal{M}$, where $\mathcal{M}$ is the Hardy–Littlewood maximal operator. Since $\mathcal{M}$ is bounded in $X_1(w_1)$, $(T_1^{(\varphi_1)})'$ is bounded in $X(w_1)'$ and so

$$\big\|\big(T_0^{(\varphi_1)}\big)'g\big\|_{X_1(w_1)'} \le (1-\alpha)^{-1}\,\big\|T_1^{(\varphi_1)}\big\|_{X_1(w_1)\to X_1(w_1)}\,\big\|T_2^{(\varphi_1)}g\big\|_{X_1(w_1)'}.$$

Since $T_2^{(\varphi_1)}$ is the averaging operator generated by the family $\{E_{kj}\}$, we conclude that (see also [1]) $T_2^{(\varphi_1)}$ is bounded from $X_1(w_1)$ to $Y(v)$ if $(w_1, v) \in A_{\varphi_1}(X_1, Y)$ and $(X_1, Y) \in G(\mathcal{B})$. Hence

$$\big\|\big(T_0^{(\varphi_1)}\big)'\big\|_{Y(v)'\to X(w)'} \le (1-\alpha)^{-1}\,C_0(\varphi_1, w_1, v)\|\mathcal{M}\|_{X_1(w_1)\to X_1(w_1)}.$$

We also have

$$\begin{aligned}
\big\|T_0(f_1,f_2)\big\|_{Y(v)} &\le \bigg\|\sum_{k\in\mathbb{Z},\ j\in\bar{I}_k}\bigg(\frac{1}{\varphi_1(|Q_{kj}|)}\int\limits_{Q_{kj}} f_1\,d\mu\bigg)\bigg(\frac{1}{\varphi_2(|Q_{kj}|)}\int\limits_{Q_{kj}} f_2\,d\mu\bigg)\,\chi_{E_{kj}}\bigg\|_{Y(v)} \\
&\le \big\|T_0^{(\varphi_1)}f_1\big\|_{Y(v)}\|f_2\|_{X_2} \\
&\le (1-\alpha)^{-1}C_0(\varphi_1, w_1, v)\|\mathcal{M}\|_{X_1(w_1)\to X_1(w_1)}\|f_1\|_{X_1(w_1)}\|f_2\|_{X_1}.
\end{aligned}$$

Combining with the previously shown inequality

$$\big\|\mathcal{M}_{\vec{\varphi}}(f_1,f_2)\big\|_{Y(v)} \le \big\|T_0(f_1,f_2)\big\|_{Y(v)},$$

we complete the proof for the dyadic maximal operator.

Now let $\mathcal{M}^{(k_0,t)}_{(\varphi_1,\varphi_2)}\vec{f}$ be the bisublinear maximal function of $\vec{f} = (f_1, f_2)$ constructed with respect to cubes with side length less than or equal to 2^{k_0} and dyadic cubes $Q - t$. Then it is easy to see that the theorem is true for $\mathcal{M}^{(k_0,t)}_{(\varphi_1,\varphi_2)}\vec{f}$. Taking now into account that $\mathcal{M}^{(k_0,t)}_{(\varphi_1,\varphi_2)}\vec{f} \uparrow \mathcal{M}^{(\infty,t)}_{(\varphi_1,\varphi_2)}\vec{f}$ (as $k_0 \to \infty$), the assumption that X_1 and Y are minimal and that Y satisfies the Fatou property, we see that the theorem is true for the operator $\mathcal{M}^{(\infty,t)}_{(\varphi_1,\varphi_2)}\vec{f}$ with bound independent of t. Since $(\varphi_1\varphi_2)(2^n t) \leq C(\varphi_1\varphi_2)(t)$ for all $t > 0$, we can conclude that the theorem holds for the maximal operator defined with respect to any cubes. $\square$

We finish the paper with some corollaries. First we introduce the definition: if $1 \leq p < \infty$ and $\varphi \in \mathcal{P}$, then M^p_φ denotes the Banach lattice of all $f \in L^0(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{M^p_\varphi} = \sup_{Q\in\mathcal{B}} \frac{|Q|^{1/p'}}{\varphi(|Q|)} \biggl(\int\limits_Q |f|^p \, d\mu \biggr)^{1/p}.$$

The following corollary is a consequence of Theorem 2.3, $(X_1, Y) \in G(\mathcal{B})$ and the continuous inclusion $M^p_\varphi \hookrightarrow M_\varphi$ with norm equal to 1, which follows from Hölder's inequality.

Corollary 2.1. *Let $\vec{\varphi} = (\varphi_1, \varphi_2) \in \widetilde{\mathcal{P}} \times \widetilde{\mathcal{P}}$. Suppose that $X_1 = L^{p_1}(\mathbb{R}^n)$, $X_2 = M^{p_2}_{\varphi_2}$ and $Y = L^q(\mathbb{R}^n)$, where $1 < p_1 \leq q < \infty$, $1 < p_2 < \infty$. Then the inequality*

$$\bigl\|v\mathcal{M}_{\vec{\varphi}}(f_1, f_2)\bigr\|_{L^q} \leq \|w_1 f_1\|_{L^{p_2}} \|f_2\|_{M^{p_2}_{\varphi_2}}, \quad (f_1, f_2) \in X_1 \times X_2$$

holds if

$$\sup_{Q\in\mathcal{B}} \frac{1}{\varphi_1(|Q|)} \|v\chi_Q\|_{L^q} \bigl\|w_1^{-1}\chi_Q\bigr\|_{L^{p_1'}} < \infty,$$

where $1/p_1 + 1/p_1' = 1$.

In the case when $\varphi_1(t) = t^\alpha$ for all $t \geq 0$, we obtain the following corollary:

Corollary 2.2. *Let $1 < p_1 \leq q < \infty$, $r, p_2 \in (1, \infty)$. Suppose that $\varphi_1(t) = t^\alpha$, $\alpha \in (0, 1)$, and $\varphi_2(t) = t^{1/p_2'}$ for all $t \geq 0$, where $1/p_2 + 1/p_2'$. If*

$$\sup_{Q\in\mathcal{B}} \frac{1}{|Q|^\alpha} \|v\chi_Q\|_{L^p} \bigl\|w^{-1}\chi_Q\bigr\|_{L^{p_1'}} < \infty,$$

then the operator $\mathcal{M}_{(\varphi_1,\varphi_2)}$ is bounded from $L^{p_1}(w_1) \times M^r_{\varphi_2}$ to $L^q(v)$.

We conclude the paper with the following remark that some weighted estimates for multilinear fractional integrals in Morrey spaces were derived in the papers [3,4].

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