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V. Kokilashvili, M. Mastyło and A. Meskhi

MULTILINEAR FRACTIONAL INTEGRALS IN WEIGHTED GRAND LEBESGUE SPACES

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Our goal is to present weighted inequalities for multilinear fractional integral operators in grand Lebesgue spaces. The theory of grand Lebesgue spaces introduced by T. Iwaniec and C. Sbordone [14] is one of the intensively developing directions of the modern analysis. It was realized the necessity for the study of these spaces because of their rather essential role and applications in various fields. These spaces naturally arise, for example, in the integrability problems of the Jacobian under minimal hypothesis (see [14] for the details).

Structural properties of grand Lebesgue spaces were investigated in the papers [4], [2]. In [5] the authors proved that for the boundedness of the Hardy–Littlewood maximal operator defined on $[0,1]$ in weighted grand Lebesgue spaces $L^{p)}_w([0,1])$ it is necessary and sufficient that the weight w satisfies the Muckenhoupt's A_p condition on the interval $[0,1]$.

The same phenomenon was noticed for the Hilbert transform in [22]. We refer also to [17], [16], [28] for one–weight results regarding maximal and singular integrals of various type in these spaces.

In [25] the boundedness criteria for fractional integral operators in weighted grand Lebesgue spaces from the one–weight viewpoint were established. In particular, in that paper the author determined values of the second parameter for grand Lebesgue spaces governing the boundedness of fractional integral operator in these spaces, and established criterion for which inequality (8) (see below) holds in the linear case (see also [23] for related topics).

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In [21] trace inequality criteria for fractional integrals in grand Lebesgue spaces defined on metric measure spaces were derived.

Multilinear fractional integrals were introduced and studied in the papers by L. Grafakos [6], C. Kenig and E. Stein [15], L. Grafakos and N. Kalton [8].

For the boundedness and other properties of multi(sub)linear fractional integrals in (weighted) Lebesgue spaces we refer, e.g., to [26], [27], [3], [29], [18], [19].

Recently, in [20] the authors of this paper presented the one-weight inequality for the multi(sub)linear Hardy–Littlewood maximal and Calderón-Zygmund operators defined on an SHT.

It should be stressed that the results of this paper are new even for Euclidean spaces. In the most cases the derived conditions are simultaneously necessary and sufficient for appropriate one-weight inequality in the linear case (see, e.g., [23], [22], [21], [25]).

In the sequel the following notation will be used:

$$\overrightarrow{p} := (p_1, \dots, p_m),$$

where $p_i \in (0, \infty)$ for each $1 \le i \le m$;

$$\overrightarrow{f} = (f_1, \dots, f_m),$$

where f_i are $\mu-$ measurable functions defined on X;

$$\frac{1}{p} := \sum_{i=1}^{m} \frac{1}{p_i} d\mu(\overrightarrow{y}), \quad d\overrightarrow{y} := d\mu(y_1) \cdots d\mu(y_m);$$

$$\nu_{\overrightarrow{w}} := \prod_{j=1}^{m} w_j^{p/p_j}, \quad \widetilde{\nu}_{\overrightarrow{w}} := \prod_{j=1}^{m} w_j^{q/q_j};$$

$$B_{xy} := \mu\big(B(x, d(x,y))\big).$$

Let $s \in [1, \infty]$. As usual we put $s' := \frac{s}{s-1}$ if $s \in (1, \infty)$ and $s' := \infty$ for $s = 1$ and $s' := 1$ for $s = \infty$;

$$A_{p,q,\alpha} := q(1/p - \alpha)$$

for $1 < p < q < \infty$ and $0 < \alpha < 1/p$;

Let (X, d, μ) be a quasi-metric measure space with quasi-metric d and measure μ. If μ satisfies the well-known doubling condition, then (X, d, μ) is called space of homogeneous type (SHT).

Let $1 \le r < \infty$. We denote by $L^r(X, \mu)$ the Lebesgue space on X with an exponent r.

If w is a weight (locally integrable, μ-a.e. positive function on X), then we denote the weighted Lebesgue spaces by $L^r_w(X, \mu)$, i.e., $f \in L^r_w(X, \mu)$ if $\|f\|_{L^r_w(X,\mu)} = \|f\|_{L^r(X, wd\mu)} < \infty$.

Let $\mu(X) < \infty$, $1 < p < \infty$ and let φ be a continuous positive function on $(0, p-1)$ such that it is non-decreasing on $(0, \sigma)$ for some small positive σ

and satisfies the condition $\lim\limits_{x\to 0+}\varphi(x)=0$. The generalized grand Lebesgue space $L^{p),\varphi}(X,\mu)$ is the class of those $f: X\to\mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi}(X,\mu)}=\sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\int\limits_X |f(x)|^{p-\varepsilon}\,d\mu(x)\right)^{1/(p-\varepsilon)}$$

is finite. If w is a weight on X, then the weighted grand Lebesgue space with weight w is denoted by $L^{p),\varphi}_w(X,\mu)$ and coincides with the class $L^{p),\varphi}(X,wd\mu)$. In this case we assume that $\|f\|_{L^{p),\varphi}_w(X,\mu)}=\|f\|_{L^{p),\varphi}(X,wd\mu)}$.

If $\varphi(x)=x^\theta$, where θ is a positive number, then we denote $L^{p),\varphi}(X,\mu)$ (resp., $L^{p),\varphi}_w(X,\mu)$) by $L^{p),\theta}(X,\mu)$ (resp. by $L^{p),\theta}_w(X,\mu)$).

The space $L^{p),\theta}(X,\mu)$ is a Banach space.

It is easy to check that the following continuous embeddings hold:

$$L^p(X,\mu)\hookrightarrow L^{p),\theta_1}(X,\mu)\hookrightarrow L^{p),\theta_2}(X,\mu)\hookrightarrow L^{p-\varepsilon}(X,\mu),$$

where $0<\varepsilon\le p-1$ and $\theta_1<\theta_2$.

It turns out that in the theory of PDEs the generalized grand Lebesgue spaces are appropriate to the solutions of existence and uniqueness, and, also, the regularity problems for various nonlinear differential equations. The space $L^{p),\theta}$ (defined on bounded domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in the paper [12], where the authors studied the nonhomogeneous n-harmonic equation div $A(x,\nabla u)=\mu$. If $\theta=1$, then $L^{p),\theta}(X,\mu)$ coincides with the Iwaniec–Sbordone space, which we denote by $L^{p)}(X,\mu)$. The grand Lebesgue space is non-reflexive, non-separable and, in general, is non-rearrangement invariant (see, e.g., [4]).

We define the class $\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)$ of vector functions $\overrightarrow{f}$ as follows: $\overrightarrow{f}\in\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)$ if

$$\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}=\sup_{1<r<p}\left\{\varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p}}\prod_{j=1}^m\|f_j\|_{L^{p_j/r}(X,\mu_j)}\right\}=$$
$$=\sup_{1<r<p}\left\{\prod_{j=1}^m\varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p_j}}\|f_j\|_{L^{p_j/r}(X,\mu_j)}\right\}<\infty.$$

The expression $\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}$ can be rewritten as follows:

$$\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}=$$
$$=\sup_{0<\eta<p-1}\left\{\prod_{j=1}^m\varphi(\eta)^{\frac{1}{p_j-\eta_j}}\|f_j\|_{L^{p_j-\eta_j}(X,\mu_j)}:\ \frac{p}{p-\eta}=\frac{p_j}{p_j-\eta_j},\ \ j=1,\dots,m\right\}.$$

It is easy to check that

$$\prod_{j=1}^{m} L^{p_j),\varphi}(X,\mu_j) \hookrightarrow \prod_{j=1}^{m} \mathcal{L}^{p_j),\varphi}(X,\mu_j),$$

in particular,

$$\|\overrightarrow{f}\|_{\prod_{j=1}^{m} \mathcal{L}^{p_j),\varphi}(X,\mu_j)} \le \|\overrightarrow{f}\|_{\prod_{j=1}^{m} L^{p_j),\varphi}(X,\mu_j)}.$$

This follows from the fact that if $\frac{p}{p-\eta} = \frac{p_j}{p_j-\eta_j}$, $j=1,\dots,m$, then $\eta \le \eta_j$ because $\frac{1}{\eta} = \sum_{j=1}^{m} \frac{1}{\eta_j}$.

When we deal with grand Lebesgue spaces we assume that $\mu(X) < \infty$.

Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt class $A_r(X)$ if

$$\|w\|_{A_r} := \sup_{B} \left(\frac{1}{\mu(B)} \int_{B} w\, d\mu \right) \left(\frac{1}{\mu(B)} \int_{B} w^{1-r'}\, d\mu \right)^{r-1} < \infty.$$

Let us recall the definition of the vector Muckenhoupt condition (see [24] for Euclidean spaces and [9] for metric measure spaces).

Definition A. Let $1 \le p_j < \infty$ for each $1 \le j \le m$, and $0 < p < \infty$. We say that $\overrightarrow{w}$ satisfies the $A_{\overrightarrow{p}}(X)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}} := \sup_{B\subset X} \left(\frac{1}{\mu(B)} \int_{B} \nu_{\overrightarrow{w}}(x) d\mu(x) \right) \times \\ \times \prod_{j=1}^{m} \left(\frac{1}{\mu(B)} \int_{B} w_j^{1-p'}(x)\, d\mu(x) \right)^{p/p'_j} < \infty,$$

where the supremum is taken over all balls B in X. For $p_j = 1$, the expression $\big(\frac{1}{\mu(B)} \int_{B} w_j^{1-p'}(x) d\mu(x)\big)^{1/p'_j}$ is understood as $(\inf_B w_j)^{-1}$.

The expression $\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}}$ is called $A_{\overrightarrow{p}}$ characteristic of $\overrightarrow{w}$.

It is known (see [24]) that if $\overrightarrow{w}$ satisfies the condition $A_{\overrightarrow{p}}(\mathbb{R}^n)$, then the boundedness of the multi(sub)linear Hardy–Littlewood and Calderón-Zygmund operators defined on $\mathbb{R}^n$ from $\prod_{j=1}^{m} L^{p_j}_{w_j}(\mathbb{R}^n)$ to $L^p_{\nu_{\overrightarrow{w}}}(\mathbb{R}^n)$ holds, where $1 < p_j < \infty$ for each $1 \le j \le m$, and $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$.

In the linear case ($m = 1$) the class $A_{\overrightarrow{p}}$ coincides with the well-known Muckenhoupt class A_p.

Definition B (vector Muckenhoupt-Wheeden condition). Let (X, d, μ) be a metric measure space, $1 \le p_i < \infty$ for $i = 1, \dots, m$. Suppose that $p < q < \infty$. Let $w_1, \dots, w_m$ be a weight functions on X. We say that

$\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies $A_{\overrightarrow{p},q}(X)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p},q}(X)$) if

$$\sup_B \left(\frac{1}{\mu B} \int_B \Big(\prod_{i=1}^m w_i \Big)^q d\mu \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{\mu B} \int_B w_i^{-p_i'} \, d\mu \right)^{1/p_i'} < \infty,$$

where the supremum is taken over all balls B in X. For $p_j = 1$, the expression $\big(\frac{1}{\mu(B)} \int_B w_j^{1-p'}(x) d\mu(x) \big)^{1/p_j'}$ is understood as $(\inf_B w_j)^{-1}$.

Theorem A ([26]). *Let $1 < p_1, \dots, p_m < \infty$, $0 < \alpha < mn$, $\frac{1}{m} < p < \frac{n}{\alpha}$. Suppose that q is an exponent satisfying the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that w_i are a.e. positive functions on $\mathbb{R}^n$ such that $w_i^{p_i}$ are weights. Then the inequality*

$$\left(\int_{\mathbb{R}^n} \Big(|\mathcal{J}_\alpha(\overrightarrow{f})(x)| \prod_{i=1}^m w_i(x) \Big)^q dx \right)^{1/q} \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \big(|f_i(y_i)| w_i \big)^{p_i} dy_i \right)^{1/p_i},$$

holds, where

$$\mathcal{J}_\alpha(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} \, d\overrightarrow{y},$$

holds, if and only if $\overrightarrow{w} \in A_{\overrightarrow{p},q}(\mathbb{R}^n)$.

The next statement characterizes those weights v on $\mathbb{R}^n$ for which the $\mathcal{I}_\alpha : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L_v^q(\mathbb{R}^n)$ holds, where $p < q < \infty$.

Theorem B ([18]). *Let $1 < p_i < \infty$ for each $1 \le i \le m$. Let $p < q$. Then $\mathcal{J}_\alpha$ is bounded from $\prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$ to $L_v^q(\mathbb{R}^n)$ if and only if*

$$\sup_Q \left(\int_Q v(x)(x) \, dx \right)^{1/q} |Q|^{\alpha - n/p} < \infty$$

is satisfied, where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

This statement remains valid if we replace $\mathbb{R}^n$ by an interval in $\mathbb{R}$, and $\mathcal{J}_\alpha$ by potential operator on an interval:

$$(J_\alpha f)(x) = \int_0^1 \frac{f(t)}{|x-t|^{1-\alpha}} \, dt, \quad 0 < \alpha < 1, \quad x \in [0,1]. \tag{1}$$

Let $1 < p < \infty$, $0 < \alpha < 1/p$ and q be the Hardy–Littlewood–Sobolev exponent, i.e., $q = \frac{p}{1-\alpha p}$. It is known (see [25]) that the operator J_α and, consequently, appropriate fractional maximal operator

$$(M_\alpha f)(x) = \sup_{I \ni x} \frac{1}{|I|^{1-\alpha}} \int_I |f(t)| \, dt, \quad 0 < \alpha < 1, \quad x \in [0,1]. \tag{2}$$

is bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$ if $\theta_2 \geq \frac{q\theta_1}{p}$. However, this boundedness fails if $\theta_2 < \frac{q\theta_1}{p}$. Moreover, it was shown that the one-weight inequality

$$\big\|T_\alpha(fw^\alpha)\big\|_{L_w^{q),\theta q/p}([0,1])} \leq C\|f\|_{L_w^{p),\theta}([0,1])},$$

where T_α is J_α or M_α, $1 < p < \frac{1}{\alpha}$, $q = \frac{p}{1-\alpha p}$, $\theta > 0$, holds if and only if $w \in A_{1+q/p'}([0,1])$.

The next statement gives D. Adams type (see [1]) trace inequality characterization for the fractional integrals and corresponding fractional maximal functions defined by

$$(\mathcal{T}_\alpha f)(x) = \int\limits_X \frac{f(y)}{\mu(B_{xy})^{1-\alpha}}\, d\mu(y), \quad x \in X, \quad 0 < \alpha < 1,$$

$$(\mathcal{M}_\alpha f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int\limits_B |f(y)|\, d\mu(y), \quad 0 < \alpha < 1,$$

where the supremum is taken over all balls $B \subset X$ containing x.

Theorem C ([21]). *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an SHT and ν is an another finite measure on X. Let $\theta > 0$. Then the following conditions are equivalent*:

(i) *the operator $\mathcal{T}_\alpha$ is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*

(ii) *the operator $\mathcal{M}_\alpha$ is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*

(iii) *there is a positive constant C such that for all balls B in X the inequality*

$$\nu(B) \leq C\big(\mu(B)\big)^{A_{p,q,\alpha}} \tag{3}$$

holds, where

$$A_{p,q,\alpha} := q\Big(\frac{1}{p} - \alpha\Big). \tag{4}$$

1. The Main Results

In this section we the main results.

1.1. **Unboundedness of Multilinear Fractional Integrals.** Let (X, d, μ) be an SHT and let

$$\mathcal{M}_\alpha(\overrightarrow{f})(x) = \sup_{B \ni x} \prod_{i=1}^m \frac{1}{\mu(B)^{1-\alpha/m}} \int\limits_B |f_i(y_i)|\, d\mu(y_i), \quad 0 \leq \alpha < m;$$

$$\mathcal{I}_\alpha(\overrightarrow{f})(x) = \int\limits_{X^m} \frac{f_1(y_1)\cdots f_m(y_m)}{\big(B_{xy_1} + \cdots + B_{xy_m}\big)^{m-\alpha}}\, d\mu(\overrightarrow{y}),$$

defined, generally speaking, on an STH in the classical Lebesgue spaces.

The following statement shows the range of the second parameter for which the boundedness of the operator $\mathcal{M}_\alpha$ (resp. $\mathcal{I}_\alpha$) from the product space to grand Lebesgue space fails (for linear fractional integrals on an interval see [25] and linear potentials on an SHT we refer to [21]).

Proposition 1. *Let (X, d, μ) be an SHT with $\mu(X) < \infty$. Suppose that $1 < p_j < \infty$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$ and $1 < p < q < \infty$. Let*

$$\liminf_{\mu(B)\to 0} \nu(B)\mu(B)^{A_{p,q,\alpha}} \neq 0,$$

where $A_{p,q,\alpha}$ is defined by (4). If $0 < \theta_2 < \frac{\theta_1 q}{p}$, then the operator $\mathcal{N}_\alpha$, where $\mathcal{N}_\alpha$ is either $\mathcal{M}_\alpha$ or $\mathcal{I}_\alpha$, is not bounded from $\prod_{j=1}^{m} \mathcal{L}_{L^{p_j),\theta_1}}(X, \mu)$ to $L^{q),\theta_2}(X, \nu)$.

Corollary 1. *Let (X, d, μ) be an SHT. Suppose that $0 < \alpha < 1$, $1 < p_j < \infty$ for each $1 \le j \le m$, $\frac{1}{p} = \sum_{j=1} \frac{1}{p_j}$ and $1/m < p < 1/\alpha$. We set $q = \frac{p}{1-\alpha p}$. Suppose that $0 < \theta_2 < \frac{\theta_1 q}{p}$. Then the operator $\mathcal{N}_\alpha$ is not bounded from $\prod_{j=1}^{m} \mathcal{L}^{p_j),\theta_1}(X, \mu)$ to $L^{q),\theta_2}(X, \mu)$, where $\mathcal{N}_\alpha$ is $\mathcal{M}_\alpha$ or $\mathcal{I}_\alpha$.*

Let (X, d, μ) be an SHT. To formulate the next statement we need to introduce the class $M_{\overrightarrow{p},q}(X, \nu, \mu_1, \dots, \mu_m)$ ($p_j, q > 1$, $1 \le j \le m$) of $m+1$-tuple of finite measures $(\nu, \mu_1, \dots, \mu_m)$ defined on X.

Definition C. Let (X, d, μ) be an SHT and let $\mu_1, \dots, \mu_m, \nu$ be measures on X. A multilinear operator T belongs to the class $M_{\overrightarrow{p},q}(X, \mu_1, \dots, \mu_m, \nu)$ if T is bounded from $\prod_{j=1}^{m} L^{p_j}(X, \mu_j)$ to $L^q(X, \nu)$.

If $d\mu_j = w_j d\mu$ for every $1 \le j \le m$, $d\nu = v d\mu$ for some weight functions $w_1, \dots, w_m, v$ then we denote $M_{\overrightarrow{p},q}(X, \mu_1, \dots, \mu_m, \nu)$ by $M_{\overrightarrow{p},q}(X, w_1, \dots, w_m, v)$.

Let $1 < q < \infty$, $\varepsilon_0 \in (0, q-1)$ and $\eta_0 \in (0, a)$, where a is sufficiently small positive number. Ne denote

$$g(x) := \frac{q\varepsilon_0(p-\eta_0)x}{\eta_0(q-\varepsilon_0)(p-x) + x\varepsilon_0(p-x)}, \tag{5}$$

$$\Psi(x) := \Phi(g(x))^{\frac{p-x}{q-g(x)}}, \tag{6}$$

with $\Phi \in R(0, \sigma)$, where $R(0, \sigma)$ is the class of those increasing functions ϕ an interval $(0, \sigma)$, with small positive σ, such that $\lim_{x\to 0} \phi(x) = 0$.

Theorem 1. *Let (X, d, μ) be an SHT. Let $1 < p_j < \infty$ for each $1 \le j \le m$. Let $1 < q < \infty$. We set $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$. Let $\Psi \in R(0, \sigma)$ and Ψ is defined by (6). Suppose that a multilinear operator T satisfies the condition $T \in M_{\overrightarrow{p},q}(X, \mu_1, \dots, \mu_m, \nu) \cap M_{\overrightarrow{p}/r,q/s}(X, \mu_1, \dots, \mu_m, \nu)$ for some $r, s > 1$. Then T is bounded from $\prod_{j=1}^{m} \mathcal{L}^{p_j),\Psi}(X, \mu_j)$ to $L^{q),\Phi}(X, \nu)$.*

Now we reformulate the one-weight result for $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$, where $X = [0,1]$ and $d\mu = dx$ is the Lebesgue measure (cf. Theorem A).

Let w_j are weights on $[0,1]$ for $1 \le j \le m$. In what follows we assume that

$$\widetilde{\mathcal{N}}_{\alpha,\overrightarrow{w}}\overrightarrow{f} := \mathcal{N}_\alpha(f_1 w_1^{\alpha_1}, \dots, f_m w_m^{\alpha_m}),$$

where $\mathcal{N}_\alpha$ is $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$. We put $\alpha_j = \frac{1}{p_j} - \frac{1}{q_j}$ for each $1 \le j \le m$ and

$$\alpha = \frac{1}{p} - \frac{1}{q}, \quad \frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}, \quad \frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}. \tag{7}$$

Taking the version of Theorem B for bounded interval into account we find that the next statement holds:

Proposition B. *Let $1 < p_j < \infty$ for each $1 \le j \le m$, and $\frac{1}{m} < p < \frac{1}{\alpha}$, where $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$. We set $q = \frac{p}{1-\alpha p}$. Let for weight functions w_j, $1 \le j \le m$,*

$$\widetilde{\nu}_{\overrightarrow{w}} := \prod_{j=1}^m w_j^{q/q_j}.$$

Then the inequality

$$\|\widetilde{\mathcal{N}}_{\alpha,\overrightarrow{w}}\overrightarrow{f}\|_{L^q_{\widetilde{\nu}_{\overrightarrow{w}}}([0,1])} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}([0,1])}$$

holds if and only if $\overrightarrow{w} \in A_{\overrightarrow{l(p,q)}}([0,1])$, where $\overrightarrow{l(p,q)} := (1 + q_1/p_1', \dots, 1 + q_m/p_m')$, i.e.

$$\sup_I \left(\frac{1}{|I|}\int_I \widetilde{\nu}_{\overrightarrow{w}}(x)\,dx\right)^{1/q} \prod_{j=1}^m \left(\frac{1}{|I|}\int_I w_j^{-p_j'/q}(x)\,dx\right)^{1/p_j'} < \infty,$$

where the supremum is taken over all subintervals I of $[0,1]$.

Theorem 2. *Let $1/m < p < \infty$, $p_i = mp$ for each $1 \le i \le m$. We set $q = \frac{p}{1-\alpha p}$. Let $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m} \ge 0$. Suppose that $\theta > 0$. Then the condition $\overrightarrow{w} \in A_{\overrightarrow{l(p,q)}}([0,1])$, where $\overrightarrow{l(p,q)} := (1 + q_1/p_1', \dots, 1 + q_m/p_m')$ guarantees the following one-weight inequality*

$$\|\mathcal{N}_\alpha(f_1 w_j^{\alpha_j}, \dots, f_m w_m^{\alpha_m})\|_{L^{q),\theta q/p}_{\widetilde{\nu}_{\overrightarrow{w}}}([0,1])} \le C\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\theta}_{w_j}([0,1])}, \tag{8}$$

where $\mathcal{N}_\alpha$ is $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$, and α_j are defined by (7), $j = 1, \dots, m$.

Now we formulate another type of one-weight inequality which is new even in the linear ($m = 1$) case.

Theorem 3. *Let* $1/m < p < \infty$, $p_j = mp$ *for each* $1 \le i \le m$. *We set* $q = \frac{p}{1-\alpha p}$. *Let* $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m} > 0$. *Suppose that* $\theta > 0$. *Let for weight functions* w_j, $1 \le j \le m$,

$$\widetilde{\nu}_{\overrightarrow{w}} := \prod_{j=1}^{m} w_j^{q/q_j}.$$

Then the condition $\overrightarrow{w} \in A_{\overrightarrow{p},q}([0,1])$ *implies the one-weight inequality*

$$\big\|(\mathcal{N}_\alpha \overrightarrow{f})\widetilde{\nu}_{\overrightarrow{w}}\big\|_{L^{q),\theta q/p}([0,1])} \le C \prod_{j=1}^{m} \big\|f_j w_j\big\|_{L^{p_j),\theta}([0,1])},$$

where $\mathcal{N}_\alpha$ *is* $\mathcal{I}_\alpha$ *or* $\mathcal{M}_\alpha$ *and the positive constant* C *is independent of* f_j, $1 \le j \le m$.

In the linear case the latter statement is formulated as follows:

Corollary 2. *Let* $m = 1$ *and let* $1 < p < \infty$. *We set* $q = \frac{p}{1-\alpha p}$. *Suppose that* $\theta > 0$. *Let* J_α *be the fractional integral operator defined by* (1). *If the condition* $w \in A_{\overrightarrow{p},q}([0,1])$ *is satisfied, then the following one-weight inequality holds*

$$\big\|(J_\alpha f)w\big\|_{L^{q),\theta q/p}([0,1])} \le C\big\|fw\big\|_{L^{p),\theta}([0,1])}$$

with the positive constant C *independent of* f.

1.2. **Trace type inequality.** Now we give necessary and sufficient condition governing the boundedness of $\mathcal{N}_\alpha$ from $\prod_{j=1}^m \mathcal{L}^{p_j),\theta}([0,1])$ to $L^{q),\theta q/p}([0,1],\nu)$, where T_α is J_α or M_α and ν is another measure on $[0,1]$. Here J_α or M_α are defined by (1), (2) respectively.

Theorem 4. *Let* $1 < p_j < \infty$ *for every* $1 \le j \le m$ *and let* $\theta > 0$. *Let* $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$. *Suppose that* $0 < \alpha < \frac{1}{p}$ *and* $p < q < \infty$. *Then the following conditions are equivalent*:

(i) the operator J_α is bounded from $\prod_{j=1}^m \mathcal{L}^{p_j),\theta}([0,1])$ to $L_v^{q),\theta q/p}([0,1])$;

(ii) the operator J_α is bounded from $\prod_{j=1}^m \mathcal{L}^{p_j),\theta}([0,1])$ to $L_v^{q),\theta q/p}([0,1])$;

(iii) there is a positive constant C such that

$$v(I) \le C|I|^{A_{\alpha,p,q}}. \tag{9}$$

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Authors' addresses:

V. Kokilashvili and A. Meskhi

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili st., Tbilisi 0177, Georgia
E-mail: kokil@rmi.ge; meskhi@rmi.ge

M. Mastylo

Faculty of Mathematics and Computer Science
Adam Mickiewicz University Poznań
and Institute of Mathematics,
Polish Academy of Sciences (Poznań branch)
Umultowska 87,61-614 Poznań, Poland
E-mail: mastylo@amu.edu.pl